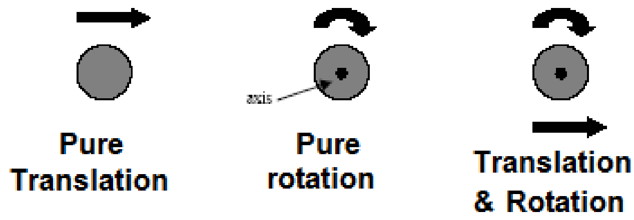
# Chapter 11
# Rolling, Torque, and Angular Momentum

## 11-1 ROLLING AS TRANSLATION AND ROTATION COMBINED

**Translation vs. Rotation**




**General Rolling Motion**

- General rolling motion consists of both translation and rotation.
- Although analysis of the general rotary motion of a rigid body in space may be quite complicated, it is made easier by a few simplifying constraints.
- Initially we will consider only objects with an **extremely high degree of symmetry** about a rotational axis, e.g., hoops, cylinders, spheres.

Consider a uniform cylinder of radius *R* rolling on a rough (no slipping) horizontal surface.

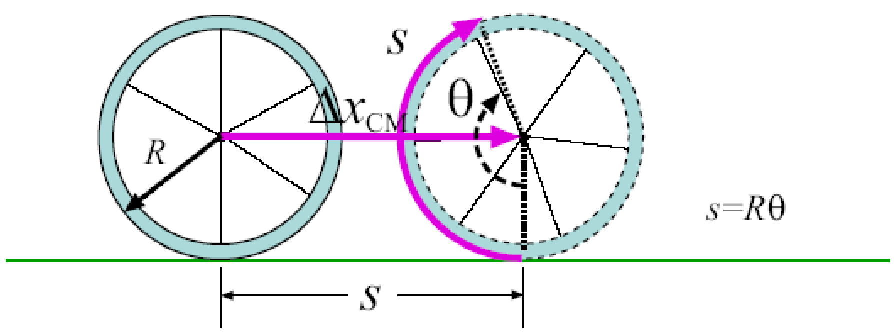
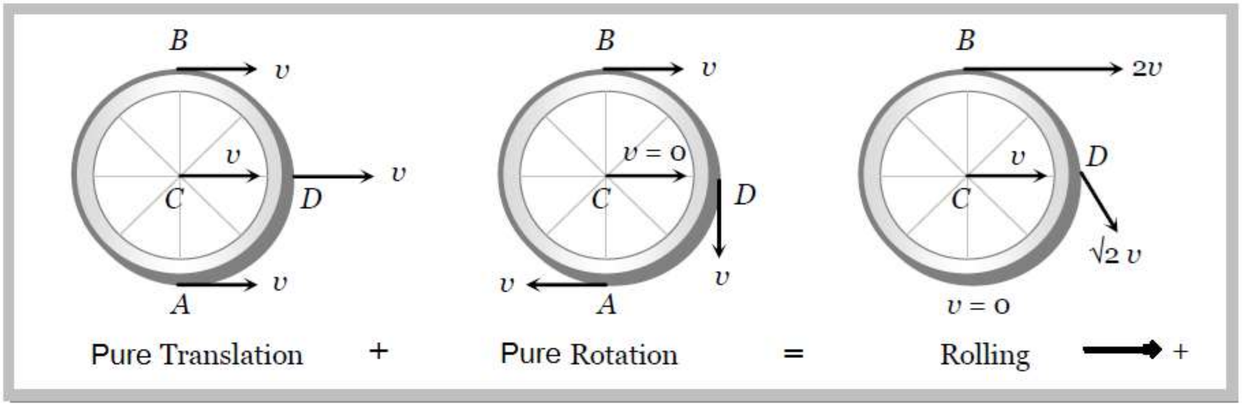



Figure 2.1: Illustration of the relation between $\Delta x$, $s$, $R$ and $\theta$ for a rolling object.

- As the cylinder rotates through an angular displacement θ, it's center of mass (com) moves through distance $s_{cm} = s = R\theta$, or the same distance as the arc length.
  - $s_{cm} = R\theta$
  - $v_{cm} = \dfrac{\Delta s}{\Delta t} = R\dfrac{\Delta\theta}{\Delta t} = R\omega$
  - $a_{cm} = \dfrac{\Delta v_{cm}}{\Delta t} = R\dfrac{\Delta\omega}{\Delta t} = R\alpha$
- If one looks at the velocity of a point on the surface of the cylinder in linear terms the situation is quite complicated. The total linear velocity is composed of two components: the tangential component, due strictly to rotation, and the translational component.

**Linear velocity of different points in rolling:** In case of rolling, all points of a rigid body have same angular speed but different linear speed. Let *A*, *B*, *C* and *D* are four points then their velocities are shown in the following figure. In the figure we used $v_{cm} = v$.




**In Pure rotation**

$$V_B = v_{cm};$$
$$V_A = -v_{cm};$$
$$V_C = 0$$

**In Pure translation**

$$V_B = v_{cm};$$
$$V_A = v_{cm};$$
$$V_C = v_{cm}$$

**In rolling**

$$V_B = 2v_{cm};$$
$$V_A = 0; \qquad \text{surprise}$$
$$V_C = v_{cm}$$

- It may be easily shown that the total linear velocity of a point at the very top of the cylinder, point *B*, relative to the surface across which it rolls, is $2v_{cm} = 2\omega R$, and that the linear velocity of a point at the bottom of the cylinder (in contact with the surface, point *A*) is zero, relative to the surface. The linear velocity of the axis around which the cylinder rotates is, of course, $v_{cm}$.

**Note:** In rolling, we will consider the system is rotating about the point $A$ (the point of contact).

Linear quantities should be used judiciously in problems that involve rotation or a combination of translation and rotation.

**11-2 FORCES AND KINETIC ENERGY OF ROLLING**

For a rolling symmetric object, of mass $M$ and angular speed $\omega$, one can calculate the total kinetic energy as:

$$K_{roll} = \tfrac{1}{2} I_A \omega^2; \qquad I_A = I_{cm} + MR^2$$

Where $I_A$ is the rotational inertia of the object about the axis through point $A$. $I_{cm}$ is the rotational inertia of the object about an axis through its center of mass. $R$ is the radius of the object.

$$K_{roll} = \tfrac{1}{2} I_A \omega^2 = \tfrac{1}{2}\left(I_{cm} + MR^2\right)\omega^2 = \tfrac{1}{2} I_{cm}\omega^2 + \frac{1}{2} MR^2\omega^2$$

$$= \tfrac{1}{2} I_{cm}\omega^2 + \tfrac{1}{2} M v_{cm}^2, \qquad \omega = v_{cm}/R$$

$\frac{1}{2} I_{cm}\omega^2$ = the kinetic energy associated with the rotational of the object about an axis through its center of mass. It represents the rotational kinetic energy of the object about its symmetry axis. $\frac{1}{2} M v_{cm}^2$ = the kinetic energy associated with the translational motion of the object's center of mass. It represents the kinetic energy the object would have if it moved along with speed $v_{CM}$ without rotating (i.e. just translational motion).

We can remember this relation simply as:

$$\boxed{K_{roll} = K_{rot} + K_{trans}}.$$

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**Example:** A bowling ball has a mass of $M = 4.0$ kg, a M.I. $I_{cm} = 1.6\times 10^{-2}$ kg $\cdot$m$^2$ and a radius $R = 0.10$ m. If it rolls straight in +x-direction without slipping with a linear speed of $v_{cm} = 4.0$ m/s, what is its total energy?

**Answer:** The total (kinetic) energy of an object which rolls without slipping is given by $K_{roll} = \frac{1}{2} I_{cm}\omega^2 + \frac{1}{2} M v_{cm}^2$. To use this equation we have everything we need, except the angular speed of the ball. From $v_{cm} = R\omega$ the angular speed is:

$$\omega = \frac{v_{cm}}{R} = \frac{4.0}{0.1} = 40.0 \text{ rad/s}$$

and then the kinetic energy is

$$K_{roll} = \frac{1}{2} I_{cm}\omega^2 + \frac{1}{2} M v_{cm}^2 = \frac{1}{2}\left(1.6\times 10^{-2}\right)\left(40.0\right)^2 + \frac{1}{2}\left(4.0\right)\left(4.0\right)^2 = 44.8 \text{ J.}$$

The total kinetic energy of the ball is 44.8 J.

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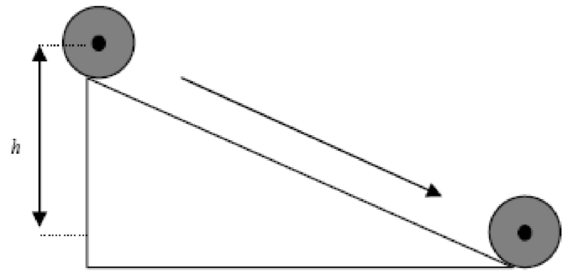
**Example:** Consider a solid cylinder of radius *R* that rolls without slipping down an incline from some initial height *h*. Calculate the linear velocity, $v_{cm}$, of the cylinder at the bottom of the incline and the angular velocity $\omega$.




**Answer:**

- If the cylinder starts from rest, all of its subsequent kinetic energy comes from gravitational potential energy $PE_i = mgh$.
- Because the cylinder is both translating and rotating as it moves down the plane, some of this initial energy goes into rotation and some goes into translation.
- This means that the linear velocity of the cylinder at the bottom of the plane is slower than it would be if the cylinder slid down the plane without rotating. Energy is still conserved, but the initial potential energy is now converted into two types of kinetic energy.

$$PE_i = KE_{f_{rot}} + KE_{f_{trans}}$$
$$mgh = \frac{1}{2}I\omega^2 + \frac{1}{2}mv_{cm}{}^2$$

For pure rolling motion (i.e. no slipping) $v_{cm} = R\omega$

$$mgh = \frac{1}{2}I\left(\frac{v_{cm}}{R^2}\right)^2 + \frac{1}{2}mv_{cm}^2 \quad \Rightarrow \quad v_{cm} = \sqrt{\frac{2mgh}{\left(m + \frac{I}{R^2}\right)}}$$

For a solid cylinder rotating about a symmetry axis down the length of the cylinder, $I = \frac{1}{2}mR^2$.

Inserting this into the equation above yields:

$$v_{cm} = \sqrt{\frac{4}{3}gh}$$

We can also solve for angular velocity using the equation $\omega = \dfrac{v_{cm}}{R} = \sqrt{\dfrac{4gh}{3R^2}}$,

Or we can do it another way, such as:

$$mgh = \frac{1}{2}\left(\frac{1}{2}mR^2\right)\omega^2 + \frac{1}{2}m(R\omega)^2$$

$$gh = \frac{1}{2}\left(\frac{1}{2}R^2\right)\omega^2 + \frac{1}{2}(R\omega)^2 \rightarrow gh = \frac{1}{4}R^2\omega^2 + \frac{1}{2}R^2w^2 \therefore gh = \frac{3}{4}R^2\omega^2 \therefore \omega = \sqrt{\frac{4}{3}\frac{gh}{R^2}}$$

What is the ratio of rotational to translational energy?

$$\frac{KE_r}{KE_t} = \frac{\left(\frac{1}{4}mR^2\right)\left(\frac{v_{cm}}{R}\right)^2}{\frac{1}{2}mv_{cm}^{\ 2}} = \frac{\frac{1}{4}}{\frac{1}{2}} = 50\% \text{ (rotation has half the energy of translation)}$$

What percentage of the total kinetic energy goes into rotational?

$$\frac{KE_r}{KE_{total}} = \frac{\left(\frac{1}{4}mR^2\right)\left(\frac{v_{cm}}{R}\right)^2}{\frac{1}{2}mv_{cm}^{\ 2} + \left(\frac{1}{4}mR^2\right)\left(\frac{v_{cm}}{R}\right)^2} = \frac{\frac{1}{4}}{\frac{3}{4}} = 33\%$$

- You should perform the same analysis for a both a hoop and a sphere of the same mass *m* and the same radius *R*. Based on your calculations, which reaches the bottom first in a three-way race, a hoop, a solid sphere, or a solid cylinder?

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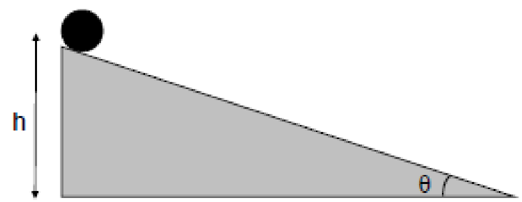
**Example**: Two objects (a solid disk and a solid sphere) are rolling down without slipping an incline from some initial height h. Both objects start from rest and from the same height. Which object reaches the bottom of the ramp first?




The object with the largest linear velocity (*v*) at the bottom of the ramp will win the race.
**Answer:** Apply the conservation of mechanical energy

| | |
|---|---|
| $E_i = E_f$<br>$U_i + K_i = U_f + K_f$<br>$mgh + 0 = 0 + \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv^2 + \frac{1}{2}I\left(\frac{v}{R}\right)^2$<br>$mgh = \frac{1}{2}\left(m + \frac{I}{R^2}\right)v^2$ | Solving for $v$ $\Rightarrow$ $v = \sqrt{\frac{2mgh}{\left(m + \frac{I}{R^2}\right)}}$ |

The moments of inertia are: $I_{disk} = \frac{1}{2}mR^2$, $I_{sphere} = \frac{2}{5}mR^2$

For the disk: $v_{disk} = \sqrt{\frac{4}{3}gh}$

For the sphere: $v_{sphere} = \sqrt{\frac{10}{7}gh}$

Since $V_{sphere} > V_{disk}$ the sphere wins the race.

Compare these with point mass (box), $I = 0$, sliding down the ramp $v_{box} = \sqrt{2gh}$

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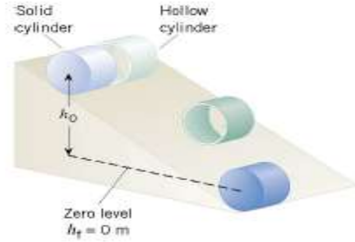
**H.W.** A thin-walled hollow cylinder (mass = *m*, radius = *r*) and a solid cylinder (also, mass = *m*, radius = *r*) start from rest at the top of an incline. Determine which cylinder has the greatest translational speed upon reaching the bottom.

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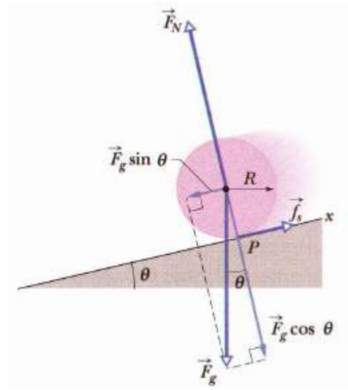
**H.W.** The figure shows a round uniform body of mass $M$ and radius $R$ rolling smoothly down a ramp at angle $\theta$, along an $x$ axis. What is its linear acceleration?




**Answer:**

**We have to apply Newton's second law in linear and rotation motion.**

**1-** Write Newton's second law for components along the $x$ axis in Fig. ($F_{net,x} = m\,a_x$) as

$$f_s - Mg\sin\theta = Ma_{com,x}. \qquad (11\text{-}7)$$

This equation contains two unknowns, $f_s$ and $a_{com,x}$. (We should *not* assume that $f_s$ is at its maximum value $f_{s,max}$. All we know is that the value of $f_s$ is just right for the body to roll smoothly down the ramp, without sliding.)

**2-** Apply Newton's second law in angular form to the body's rotation about its center of mass.

$$Rf_s = I_{com}\alpha. \qquad (11\text{-}8)$$

Because the body is rolling smoothly, we can use Eq. 11-6 ($a_{com} = \alpha R$) to relate the unknowns $a_{com,x}$ and $\alpha$. But we must be cautious because here $a_{com,x}$ is negative (in the negative direction of the $x$ axis) and $\alpha$ is positive (counterclockwise). Thus we substitute $-a_{com,x}/R$ for $\alpha$ in Eq. 11-8. Then, solving for $f_s$, we obtain

$$f_s = -I_{com}\frac{a_{com,x}}{R^2}. \qquad (11\text{-}9)$$

Substituting the right side of Eq. 11-9 for $f_s$ in Eq. 11-7, we then find

$$a_{com,x} = -\frac{g\sin\theta}{1 + I_{com}/MR^2}. \qquad (11\text{-}10)$$

We can use this equation to find the linear acceleration $a_{com,x}$ of any body rolling along an incline of angle $\theta$ with the horizontal.

**Notes:**

Note that the pull by the gravitational force causes the body to come down the ramp, but it is the frictional force that causes the body to rotate and thus roll. If you eliminate the friction (by, say, making the ramp slick with ice or grease) or arrange for $Mg\sin\theta$ to exceed $f_{s,max}$, then you eliminate the smooth rolling and the body slides down the ramp.

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## Extra Problems:

### 11-3 THE YO-YO

A yo-yo, which travels vertically up or down a string, can be treated as a wheel rolling along an inclined plane at angle $\theta = 90^{\circ}$.

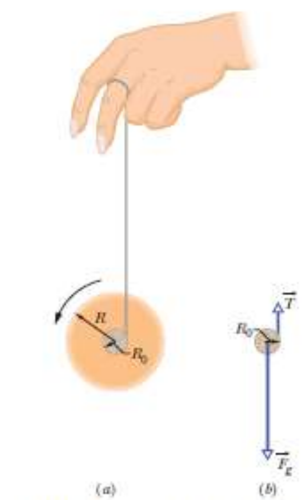



**Figure 11-9** (*a*) A yo-yo, shown in cross section. The string, of assumed negligible thickness, is wound around an axle of radius $R_0$. (*b*) A free-body diagram for the falling yo-yo. Only the axle is shown.

#### The Yo-Yo

A yo-yo is a physics lab that you can fit in your pocket. If a yo-yo rolls down its string for a distance $h$, it loses potential energy in amount $mgh$ but gains kinetic energy in both translational ($\frac{1}{2}Mv_{com}^2$) and rotational ($\frac{1}{2}I_{com}\omega^2$) forms. As it climbs back up, it loses kinetic energy and regains potential energy.

In a modern yo-yo, the string is not tied to the axle but is looped around it. When the yo-yo "hits" the bottom of its string, an upward force on the axle from the string stops the descent. The yo-yo then spins, axle inside loop, with only rotational kinetic energy. The yo-yo keeps spinning ("sleeping") until you "wake it" by jerking on the string, causing the string to catch on the axle and the yo-yo to climb back up. The rotational kinetic energy of the yo-yo at the bottom of its string (and thus the sleeping time) can be considerably increased by throwing the yo-yo downward so that it starts down the string with initial speeds $v_{com}$ and $\omega$ instead of rolling down from rest.

To find an expression for the linear acceleration $a_{com}$ of a yo-yo rolling down a string, we could use Newton's second law (in linear and angular forms) just as we did for the body rolling down a ramp in Fig. 11-8. The analysis is the same except for the following:

1. Instead of rolling down a ramp at angle $\theta$ with the horizontal, the yo-yo rolls down a string at angle $\theta = 90^{\circ}$ with the horizontal.
2. Instead of rolling on its outer surface at radius $R$, the yo-yo rolls on an axle of radius $R_0$ (Fig. 11-9*a*).
3. Instead of being slowed by frictional force $\vec{f}_s$, the yo-yo is slowed by the force $\vec{T}$ on it from the string (Fig. 11-9*b*).

The analysis would again lead us to Eq. 11-10. Therefore, let us just change the notation in Eq. 11-10 and set $\theta = 90^{\circ}$ to write the linear acceleration as

$$a_{com} = -\frac{g}{1 + I_{com}/MR_0^2}, \qquad (11\text{-}13)$$

where $I_{com}$ is the yo-yo's rotational inertia about its center and $M$ is its mass. A yo-yo has the same downward acceleration when it is climbing back up.

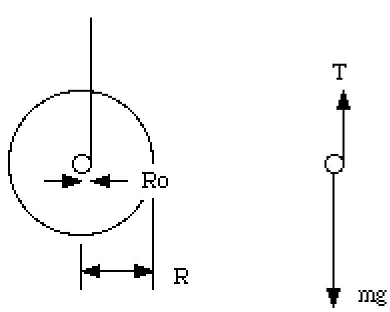



Figure. The yo-yo.

Figure shows a schematic drawing of a yo-yo. What is its linear acceleration?

There are two forces acting on the yo-yo: an upward force equal to the tension in the cord, and the gravitational force. The acceleration of the system depends on these two forces:

$$\sum F = mg - T = ma$$

The rotational motion of the yo-yo is determined by the torque exerted by the tension T (the torque due to the gravitational force is zero)

$$\sum \tau = I\alpha = TR_0$$

The rotational acceleration “a” is related to the linear acceleration a:

$$a = R_0\alpha$$

We can now write down the following equations for the tension T

$$T = \frac{I\alpha}{R_0} = \frac{Ia}{R_0^{\,2}}$$

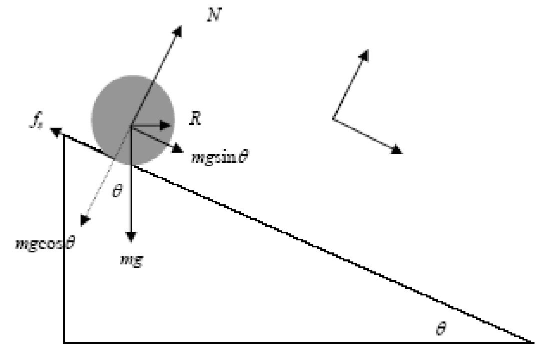
$$T = mg - ma$$

The linear acceleration a can now be calculated

$$a = g\frac{1}{1 + \frac{I}{mR_0^{\,2}}}$$

Thus, the yo-yo rolls down the string with a constant acceleration. The acceleration can be made smaller by increasing the rotational inertia and by decreasing the radius of the axle.

**Example:** A uniform cylinder rolls down a ramp inclined at an angle of θ to the horizontal. What is the linear acceleration of the cylinder at the bottom of the ramp? Remember that: The friction force is used to rotate the object.




$$\sum F_x = mg\sin\theta - f_s = ma_{cm} \qquad (1)$$

$$Rf_s = I\alpha \qquad (2)$$

Note: $I\alpha = \frac{1}{2}mR^2\alpha$

Note: $R\alpha = a_{cm}$

$$f_s = \frac{mRa_{cm}}{2R} = \frac{1}{2}ma_{cm}$$

$(2) \rightarrow (1)$

$$\sum F_x = mg\sin\theta - \frac{1}{2}ma_{cm} = ma_{cm}$$

$$g\sin\theta = \frac{3}{2}a_{cm}$$

$$a_{cm} = \frac{2g\sin\theta}{3}$$

To find $v_{cm}$, you can use the equation $v_f^2 = v_i^2 + 2a_{cm}s$, where $h = s\sin\theta$, to get $v_{cm} = \sqrt{\frac{4gh}{3}}$.

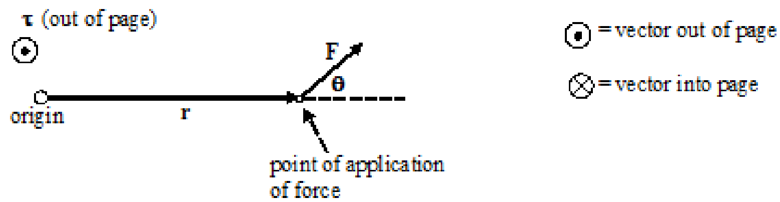
# Chapter 11
# Rolling, Torque, and Angular Momentum

**Please go to the following sight to see the flywheel demos.**
**http://www.wfu.edu/physics/demolabs/demos/avimov/byalpha/abvideos.html**
**http://www.wfu.edu/physics/demolabs/demos/1/1q/1Q4010.html**

## 11-4 TORQUE REVISITED (done with Chapter 10)

In chapter 10 the torque $\vec{\tau}$, i.e. "**to twist**", is defined as a force that causes a rotational acceleration of a rigid body about an axis or motion of a single particle relative to some fixed point. The Torque

1- is a vector
2- is positive when the body rotate counterclockwise
3- is negative when the body rotate clockwise




Symbolically, if we suppose the force $F$ (whose direction lies in the plane of rotation) is applied at a point $r$ (relative to the rotation axis which is the pivot). Suppose that the (smallest) angle between r and $F$ is $\theta$, Then the magnitude of the torque exerted on the object by this force is

$$\left|\vec{\tau}\right| = r(F\sin\theta) = r\,F_{\perp} = \underbrace{r_{\perp}}_{\text{moment arm of } \vec{F}} F$$

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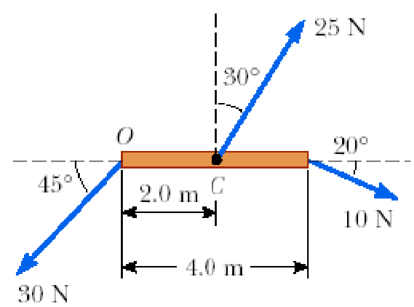
**Example**: Calculate the net torque (magnitude and direction) on the beam in the figure about the O- and C- axes.




**Answer:** We will choose clockwise as our positive direction and apply the formula for a torque:

$$\vec{\tau}_{net} = \sum_i \vec{r}_i \times \vec{F}_i = \sum_i r_{i\perp} F_i \sin\theta_i = \sum_i r_i F_{i\perp} \sin\theta_i$$

a) About the O-axis: One way is:

$$\tau_o = -25\times 2\times\cos 30^{\circ} + 10\times 4\times\sin 20^{\circ} + 0 = -29.6 \text{ N.m}$$

And the other is:

$$\tau_o == -25\times 2\times\sin 60^{\circ} + 10\times 4\times\sin 20^{\circ} + 0 = -29.6 \text{ N.m}$$

This net torque is **counterclockwise**

b) About the C-axis:

$$\tau_C = 0 + 10\times 2\times\sin 20^{\circ} - 30\times 2\times\sin 45^{\circ} = -35.6 \text{ N.m}$$

This net torque is again **counterclockwise**

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We can also assign direction to torque with cross product as:

$$\hat{\tau} = \vec{r} \times \vec{F}$$

The vector torque **τ** is defined with respect to an origin (which is usually, but not always, the axis of rotation). So, if you change the origin, you change the torque (since changing the origin changes the position vector **r**).

With these definitions of vector angular acceleration and vector torque, the fixed-axis equation $\tau = I\alpha$ becomes

$$\vec{\tau} = I\vec{\alpha} = I\frac{d\vec{\omega}}{dt} \qquad (\text{ like } \vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt})$$

To the calculation, we can use the following expression:

$$\vec{\tau} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ (x - x_o) & (y - y_o) & (z - z_o) \\ F_x & F_y & F_z \end{vmatrix}$$

Where the coordinates of the origin point is $(x_o, y_o, z_o)$

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**Example**: A force $\vec{F} = (2.0\,\hat{i} + 3.0\,\hat{j})$ N is applied to an object that is pivoted about a fixed axis aligned along the *z*-axis. If the force is applied at the point of coordinates (4.0, 5.0, 0.0) m, what is the applied torque (in N.m) about the z axis?

**Answer:**

$$\vec{\tau} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ (4-0) & (5-0) & (0-0) \\ 2 & 3 & 0 \end{vmatrix}$$

$$= \underline{(2\,\hat{k})\ \text{N}\cdot\text{m}}$$

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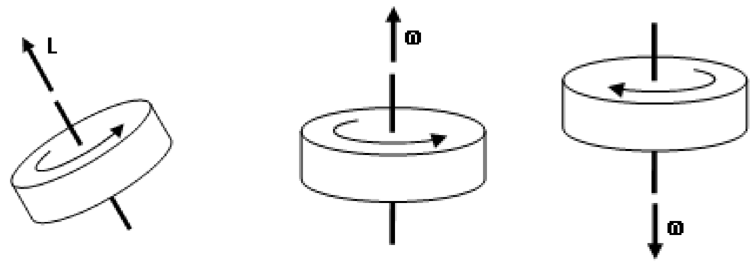
**Example**: At an instant, a particle of mass 2.0 kg has a position of $\vec{r} = (9.0\,\hat{i} + 15.0\,\hat{j})$ m and acceleration of $\vec{a} = (-3.0\,\hat{i} + 3.0\,\hat{j})$ m/s$^2$. What is the net torque on the particle at this instant about the point having the position vector: $\vec{r}_o = (9.0\,\hat{i})$ m ?

**Answer:**

$$\vec{\tau} = \vec{r} \times \vec{F} = m(\vec{r} \times \vec{a}) = m\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ (x - x_o) & (y - y_o) & (z - z_o) \\ a_x & a_y & a_z \end{vmatrix} = 2\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ (9-9) & (15-0) & (0-0) \\ -3 & 3 & 0 \end{vmatrix}$$

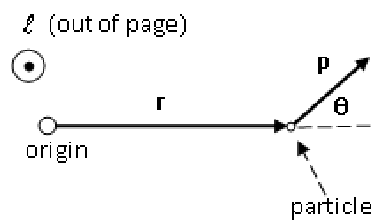
$$= \underline{(90\,\hat{k})\ \ \text{N}\cdot\text{m}}$$

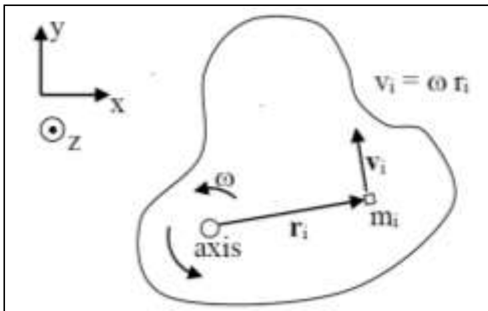
## 11-5 ANGULAR MOMENTUM (Important)




Now, a new concept: angular momentum $\vec{\ell}$ = "spin". Angular momentum, a vector, is the rotational analogue of linear momentum. So, based on our analogy between translation and rotation, we expect $\vec{\ell} = I\vec{\omega}$ (like $\vec{p} = m\vec{v}$). Note that this equation implies that the direction of **L** is the direction of **ω**.

**Definition of angular momentum** of a particle with momentum **p** = m**v** at position **r** relative to an origin is $\vec{\ell} \equiv \vec{r} \times \vec{p}$. Like torque **τ**, the angular momentum $\vec{\ell}$ is defined w.r.t. an origin, often the axis of rotation. We now show that the <u>total</u> angular momentum of a object spinning about a fixed axis is $\vec{\ell} = I\vec{\omega}$. Consider an object spinning about an axis pointing along the +z direction. We place the origin at the axis.




$$\vec{\ell}_{tot} = \sum_i \vec{\ell}_i = \sum_i \vec{r}_i \times \vec{p}_i = \sum_i \vec{r}_i \times (m_i \vec{v}_i)$$

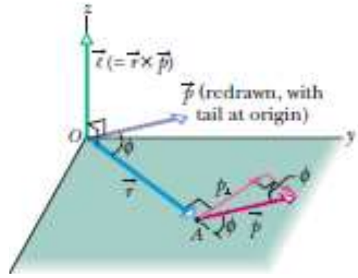
$$= \hat{z} \sum_i r_i\, m_i\, v_i = \hat{z} \sum_i m_i\, r_i^2\, \omega = \hat{z}\, I\omega$$

$$\vec{\ell}_{tot} = I\vec{\omega}$$

If something has a big moment of inertia I and is spinning fast (big ω), then it has a big "spin", big angular momentum. Angular momentum is a very useful concept, because angular momentum is conserved.

**Important fact:** the angular momentum of a object spinning about an axis that passes through the center of mass is given by $\vec{\ell} = I_{CM}\,\vec{\omega}$, <u>independent</u> of the location of the origin; that is, even if the origin is chosen to be outside the spinning object, the angular momentum has the same value as if the origin was chosen to be at the axis. (Proof not given here).

Just as the moment of inertia "$I$" is the rotational analog to mass "m", and torque "$\vec{\tau}$"is the rotational analog to force "$\vec{F}$", angular momentum "$\vec{\ell}$" is the rotational analog to linear momentum "$\vec{p}$".




- The angular momentum of a particle, $\vec{\ell}$, with respect to the origin O is:

$$\vec{\ell} = \vec{r}\times\vec{p} = m\left(\vec{r}\times\vec{v}\right) = m\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ (x-x_o) & (y-y_o) & (z-z_o) \\ v_x & v_y & v_z \end{vmatrix} = m\,r\,v\sin\phi$$

- The product $\vec{r}\times\vec{p}$ is in a plane perpendicular to the plane containing $\vec{r}$ and $\vec{p}$ and in this case is out of the plane of the page.
- Angular momentum is a vector and its direction is determined from the right hand rule. The magnitude of the angular momentum vector is $r\,p\sin\phi$.
- Notice that a particle does not have to rotate about O in order to have angular momentum with respect to O.
- Notice that just as Newton's second law may be written in terms of linear momentum:

$$\vec{F}_{net} = \frac{d\vec{p}}{dt}$$

  it may also be written in terms of angular momentum (see next section):

$$\vec{\tau}_{net} = \frac{d\vec{L}}{dt}$$

- Angular momentum may be written in terms of moment of inertia and angular velocity for a rigid body and a fixed axis:

$$\vec{\ell} = I\,\vec{\omega}$$

Angular momentum is an enormously useful quantity in physics for several reasons, such as:

1. Angular momentum is conserved, which means that in the absence of any external torques the angular momentum of a system remains constant.

$$\vec{\ell}_i = \vec{\ell}_f \quad \Rightarrow I_i\omega_i = I_f\omega_f$$
$$\Rightarrow mvr_i = mvr_f$$

2. Angular momentum may be computed in a wide variety of situations that, at first glance, don't involve rotational motion.
3. All that it really necessary to compute angular momentum is to show motion with respect to any coordinate that one may compute angular momentum with respect to.
4. In the case of instantaneous values this is normally an easy calculation.

-----------------------------------------------------------------------

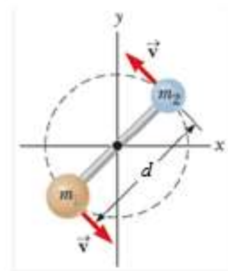
**Example**: A stone attached to a string is whirled at 3.0 rev/s around a horizontal circle of radius 0.75 m. The mass of the stone is 0.15 kg. The magnitude of the angular momentum of the stone relative to the center of the circle is:

**Answer:**

$$\ell = mvr = mr^2\omega = 0.15\times(0.75)^2\times(3\times 2\pi) = \underline{1.6\text{ kg.m}^2/\text{s}}$$

--------------------------------------------------------------------------------

**Example**: A light, rigid rod of length $d$ = 1.00 m joins two particles, with masses $m_1$ = 4.00 kg and $m_2$ = 3.00 kg, at its ends. The combination rotates in the xy plane about a pivot through the center of the rod (see figure). Determine the angular momentum of the system about the origin when the speed of each particle is 2.00 m/s.



**Answer:** Angular momentum of the system:

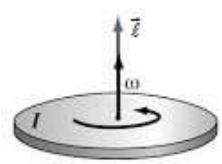
$$\vec{\ell} = \vec{\ell}_1 + \vec{\ell}_2 = \vec{r}\times\vec{p}_1 + \vec{r}\times\vec{p}_2 = m_1(\vec{r}\times\vec{v}) + m_2(\vec{r}\times\vec{v}) = (m_1+m_2)\left(\frac{d}{2}\right)v\,\hat{z}$$

$$= (4+3)\left(\frac{1}{2}\right)\times 2\ \hat{z} = 7\ \hat{z}\ \text{ kg.m}^2/\text{s}$$

Angular momentum is on the $\vec{z}$ direction. N.B. The right hand rule is of great help to visualize the torque (and any cross product) direction. In this case $\vec{r}$ and $\vec{v}$ are in the plane of the figure, the torque cross product must be oriented perpendicular to the plane.

-----------------------------------------------------------------

**Example**: A uniform solid disk of mass m = 2.94 kg and radius $r$ = 0.200 m rotates about a fixed axis perpendicular to its face with angular frequency 6.02 rad/s.



(a) Calculate the magnitude of the angular momentum of the disk when the axis of rotation passes through its center of mass. $\left[I_{\text{CM}} = \frac{1}{2}mr^2\right]$

**(b)** What is the magnitude of the angular momentum when the axis of rotation passes through a point midway between the center and the rim?

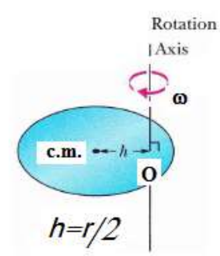
**(c)** What is the magnitude of the angular momentum when the axis of rotation passes through a point at the rim?

**Answer:**

**(a)**

$$\left|\vec{\ell}_{\text{CM}}\right| = \left|I_{\text{CM}}\vec{\omega}\right| = \frac{1}{2}mr^2\omega = \frac{1}{2}\times 2.94\times(0.2)^2\times 6.02 = \underline{0.354\text{ kg.m}^2/\text{s}}$$

(b) If the rotation axis is shifted to a point midway the center and the rim, the moment of inertia will change from $I_{\text{O}} = \frac{1}{2}mr^2 + m\left(\frac{r}{2}\right)^2 = \frac{3}{4}mr^2$ . The angular momentum will change to:




$$\left|\vec{\ell}_{\text{O}}\right| = \left|I_{\text{O}}\vec{\omega}\right| = \frac{3}{4}mr^2\omega = \frac{3}{4}\times 2.94\times(0.2)^2\times 6.02 = \underline{0.531\text{ Kg.m}^2/\text{s}}$$

(c) If the rotation axis is shifted to a point at the rim, the moment of inertia will change from $I_{O'} = \frac{1}{2}mr^2 + m(r)^2 = \frac{3}{2}mr^2$. The angular momentum will change to:

$$\left|\vec{\ell}_O\right| = \left|I_O\vec{\omega}\right| = \frac{3}{2}mr^2\omega = \frac{3}{2}\times 2.94\times(0.2)^2\times 6.02 = \underline{1.06\ \text{kg.m}^2/\text{s}}$$

------------------------------------------------------------------------

**11-6 NEWTON'S SECOND LAW IN ANGULAR FORM**

Newton's second law written in the form

$$\vec{F}_{net} = \frac{d\vec{p}}{dt} \quad \text{(single particle)} \tag{11-22}$$

expresses the close relation between force and linear momentum for a single particle. We have seen enough of the parallelism between linear and angular quantities to be pretty sure that there is also a close relation between torque and angular momentum. Guided by Eq. 11-22, we can even guess that it must be

$$\vec{\tau}_{net} = \frac{d\vec{\ell}}{dt} \quad \text{(single particle).} \tag{11-23}$$

Equation 11-23 is indeed an angular form of Newton's second law for a single particle:

The (vector) sum of all the torques acting on a particle is equal to the time rate of change of the angular momentum of that particle.

Equation 11-23 has no meaning unless the torques $\vec{\tau}$ and the angular momentum $\vec{\ell}$ are defined with respect to the same point, usually the origin of the coordinate system being used.

------------------------------------------------------------

**Example.** The angular momentum of a flywheel decreases from 3.00 to 2.00 kg.m$^2$/s in 2.00 seconds. Its moment of inertia is 0.125 kg.m$^2$. Assuming a uniform angular acceleration, calculate the angle through which the flywheel has turned in this time.

**Answer:** Compute the torque:

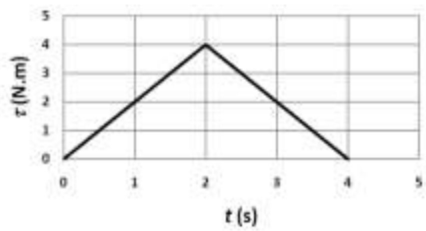
$$\vec{\tau} = \frac{d\vec{\ell}}{dt} = \frac{\ell_f - \ell_i}{dt} = \frac{2-3}{2} = -0.5\ \text{N.m}$$

$$\because \vec{\tau} = I\,\vec{\alpha} \Rightarrow \vec{\alpha} = -\frac{0.5}{0.125} = -4.0\ \text{rad/s}^2$$

$$\therefore \theta = \theta_o + \underset{\frac{\ell_o}{I}}{\omega_o}\, t + \tfrac{1}{2}\alpha t^2 = 0 + \frac{3}{0.125}\times 2 + \tfrac{1}{2}\times(-4.0)2^2 = \underline{40\ \text{rad}}$$

---------------------------------------------------------------------

**Example:** The Figure below shows a graph of a torque applied to a rotating object as a function of time. Assuming the object was initially at rest, what is the angular momentum, in units of kg·m$^2$/s, of the object at t = 4.0 s?




**Solution**

$$\tau = \frac{dL}{dt} \Rightarrow \Delta L = \int \tau dt$$

$$\therefore\ L_f - L_i = Area\ under\ the\ curve$$

$$L_f = Area = \frac{1}{2}(4)(4) = 8.0\ kg \cdot m^2/s$$

---

**Q:** At time *t*, the vector $\vec{r}(t) = 4.0\,t^2\,\hat{\text{i}} - \left(2.0\,t\ + 6.0\,t^2\right)\hat{\text{j}}$ gives the position of a 2.0 kg relative to the origin of an *xy* coordinate system ($\vec{r}$ is in meters and *t* is in seconds).

(a) Find an expression for the torque acting on the particle relative to the origin.

(b) Is the magnitude of the particle's angular momentum relative to the origin increasing, decreasing, or unchanging?

**Answer:**

(a) We note that $\vec{v} = \dfrac{d\vec{r}}{dt} = \left[8.0\,t\,\hat{\text{i}} - (2.0 + 12\,t)\,\hat{\text{j}}\right]$

with SI units understood. From Eq. 11-18 (for the angular momentum) and Eq. 3-30, we find the particle's angular momentum

$$\begin{aligned}\vec{L}(t) &= m\left(\vec{r}\times\vec{v}\right) = 2.0\left[4.0\,t^2\,\hat{\text{i}} - \left(2.0\,t\ + 6.0\,t^2\right)\hat{\text{j}}\right]\times\left[8.0\,t\,\hat{\text{i}} - (2.0 + 12\,t)\,\hat{\text{j}}\right]\\ &= 2.0\left[\left(4.0\,t^2\right)\left\{-(2.0+12\,t)\right\} - \left\{-\left(2.0\,t\ + 6.0\,t^2\right)\right\}(8.0\,t)\right]\hat{\text{k}}\\ &= 16.0\,t^2\hat{\text{k}}\end{aligned}$$

Using Eq. 11-23 (relating its time-derivative to the (single) torque) then yields

$$\vec{\tau}(t) = \frac{d\vec{L}(t)}{dt} = \frac{d\left(16.0\,t^2\right)}{dt}\hat{\text{k}} = \left(32.0\,t\,\hat{\text{k}}\ \right)\text{N.m}$$

(b) The results in (a) indicate the $\vec{L} \propto t^2$ and $\vec{\tau} \propto t$

---

A 2.00-kg particle-like object moves in a plane with velocity components $v_x$ = 15.0 m/s and $v_y$ = 12.0 m/s as it passes through the point with (x, y) coordinates of (4.00, −5.00) m. At that instant, what is the angular momentum of the object about the origin (in units of kg m$^2$/s)?

Ans:

$\vec{r} = 4\hat{i} - 5\hat{j}$ (m)

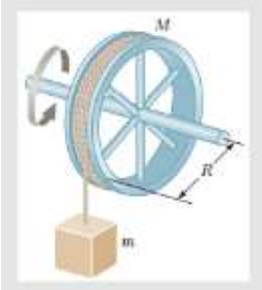
$\vec{v} = 15\hat{i} + 12\hat{j}$ (m/s)

$\vec{r} \times \vec{v} = 48\,\hat{k} + 75\,\hat{k} = 123\,\hat{k}$

$\vec{l} = \vec{r} \times \vec{p} = \vec{r} \times (m\vec{v}) = m\,(\vec{r} \times \vec{v}) = 246\,\hat{k}$

## Extra problems

**Example**: A counterweight of mass $m$ = 4.40 kg is attached to a light cord that is wound around a pulley as shown in the figure below. The pulley is a thin hoop of radius $R$ = 9.00 cm and mass $M$ = 2.50 kg. The spokes have negligible mass.




a) What is the net torque on the system about the axle of the pulley?

b) When the counterweight has a speed $v$, the pulley has an angular speed = v/R. Determine the magnitude of the total angular momentum of the system about the axle of the pulley.

c) Using your result from (b) and $\vec{\tau}_{net} = d\vec{L}/dt$, calculate the acceleration of the counterweight. (Enter the magnitude of the acceleration.)

**Answer:**

a) The system about the axle of the pulley is under the torque applied by the cord. At rest, the tension in the cord is balanced by the counterweight T = mg. If we choose the rotation axle towards a certain $\vec{z}$, we should have:

$$\vec{\tau}_{net} = \vec{R} \times \vec{T} = Rmg\vec{z} = 0.09 \times 4.40 \times 9.8\vec{z} = 3.88\,\vec{z}$$

The net torque has a magnitude of τ = 3.88N.m and its direction is along the rotation axis towards the right in the figure.

b) Taking into account rotation of the pulley and translation of the counterweight, the total angular momentum of the system is:

$$\begin{aligned} \vec{L} &= \vec{R} \times m\vec{v} + I\vec{\omega} \\ &= mRv + MR\frac{v}{R} = (m+M)Rv = (4.40+2.50) \times 0.09 = 0.621 \text{ Kg.m} \end{aligned}$$

c)

$$\begin{aligned} \tau &= \frac{dL}{dt} \\ mgR &= (M+m)R\frac{dv}{dt} = (M+m)Ra \\ a &= \frac{mg}{m+M} = \frac{4.40 \times 9.8}{6.90} = 6.25 \text{ m/s}^2 \end{aligned}$$

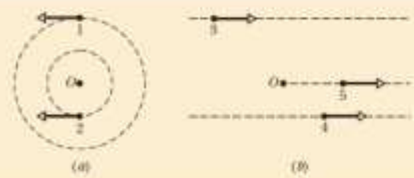
---------------------------------------------------------------------

# Rolling, Torque, and Angular Momentum

## 11-7 ANGULAR MOMENTUM OF A RIGID BODY

**Checkpoint 4**

In part *a* of the figure, particles 1 and 2 move around point *O* in circles with radii 2 m and 4 m. In part *b*, particles 3 and 4 travel along straight lines at perpendicular distances of 4 m and 2 m from point *O*. Particle 5 moves directly away from *O*. All five particles have the same mass and the same constant speed. (a) Rank the particles according to the magnitudes of their angular momentum about point *O*, greatest first. (b) Which particles have negative angular momentum about point *O*?




### The Angular Momentum of a System of Particles

Now we turn our attention to the angular momentum of a system of particles with respect to an origin. The total angular momentum $\vec{L}$ of the system is the (vector) sum of the angular momenta $\vec{\ell}$ of the individual particles (here with label $i$):

$$\vec{L} = \vec{\ell}_1 + \vec{\ell}_2 + \vec{\ell}_3 + \cdots + \vec{\ell}_n = \sum_{i=1}^{n} \vec{\ell}_i. \qquad (11\text{-}26)$$

With time, the angular momenta of individual particles may change because of interactions between the particles or with the outside. We can find the resulting change in $\vec{L}$ by taking the time derivative of Eq. 11-26. Thus,

$$\frac{d\vec{L}}{dt} = \sum_{i=1}^{n} \frac{d\vec{\ell}_i}{dt}. \qquad (11\text{-}27)$$

From Eq. 11-23, we see that $d\vec{\ell}_i/dt$ is equal to the net torque $\vec{\tau}_{\text{net},i}$ on the $i$th particle. We can rewrite Eq. 11-27 as

$$\frac{d\vec{L}}{dt} = \sum_{i=1}^{n} \vec{\tau}_{\text{net},i}. \qquad (11\text{-}28)$$

That is, the rate of change of the system's angular momentum $\vec{L}$ is equal to the vector sum of the torques on its individual particles. Those torques include *internal torques* (due to forces between the particles) and *external torques* (due to forces on the particles from bodies external to the system). However, the forces between the particles always come in third-law force pairs so their torques sum to zero. Thus, the only torques that can change the total angular momentum $\vec{L}$ of the system are the external torques acting on the system.

***Net External Torque.*** Let $\vec{\tau}_{\text{net}}$ represent the net external torque, the vector sum of all external torques on all particles in the system. Then we can write Eq. 11-28 as

$$\vec{\tau}_{\text{net}} = \frac{d\vec{L}}{dt} \quad \text{(system of particles),} \qquad (11\text{-}29)$$

- The net external torque $\vec{\tau}_{net}$ acting on a system of particles is equal to the time rate of change of the system's total angular momentum $\vec{L}$.

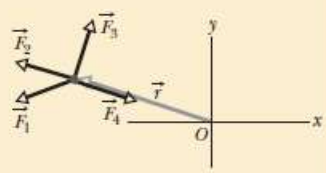
**Table 11-1 More Corresponding Variables and Relations for Translational and Rotational Motion**[a]

| Translational | | Rotational | |
|---|---|---|---|
| Force | $\vec{F}$ | Torque | $\vec{\tau}\ (=\vec{r}\times\vec{F})$ |
| Linear momentum | $\vec{p}$ | Angular momentum | $\vec{\ell}\ (=\vec{r}\times\vec{p})$ |
| Linear momentum[b] | $\vec{P}\ (=\Sigma\vec{p}_i)$ | Angular momentum[b] | $\vec{L}\ (=\Sigma\vec{\ell}_i)$ |
| Linear momentum[b] | $\vec{P}=M\vec{v}_{com}$ | Angular momentum[c] | $L=I\omega$ |
| Newton's second law[b] | $\vec{F}_{net}=\dfrac{d\vec{P}}{dt}$ | Newton's second law[b] | $\vec{\tau}_{net}=\dfrac{d\vec{L}}{dt}$ |
| Conservation law[d] | $\vec{P}$ = a constant | Conservation law[d] | $\vec{L}$ = a constant |

[a] See also Table 10-3.
[b] For systems of particles, including rigid bodies.
[c] For a rigid body about a fixed axis, with $L$ being the component along that axis.
[d] For a closed, isolated system.

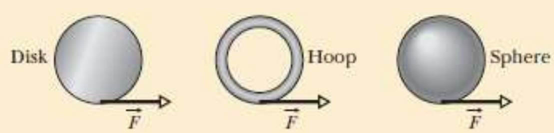
### Checkpoint 5

The figure shows the position vector $\vec{r}$ of a particle at a certain instant, and four choices for the direction of a force that is to accelerate the particle. All four choices lie in the $xy$ plane. (a) Rank the choices according to the magnitude of the time rate of change ($d\vec{\ell}/dt$) they produce in the angular momentum of the particle about point $O$, greatest first. (b) Which choice results in a negative rate of change about $O$?

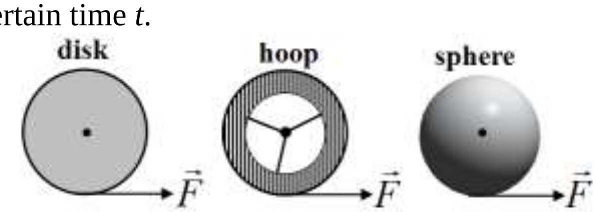



### Checkpoint 6

In the figure, a disk, a hoop, and a solid sphere are made to spin about fixed central axes (like a top) by means of strings wrapped around them, with the strings producing the same constant tangential force $\vec{F}$ on all three objects. The three objects have the same mass and radius, and they are initially stationary. Rank the objects according to (a) their angular momentum about their central axes and (b) their angular speed, greatest first, when the strings have been pulled for a certain time $t$.




**Check point:** In the figure, a disk, a hoop, and a solid sphere are made to spin about fixed central axes (like a top) by means of strings wrapped around them, with the strings producing the same constant tangential force on all three objects. The three objects have the same mass and radius, and they are initially stationary. Rank the objects according to (a) their angular momentum about their central axes and (b) their angular speed, greatest first, when the strings have been pulled for a certain time *t*.




(a) all tie (same t, same $\vec{\tau}$, thus same $\vec{L}$); $\tau = \rightarrow RF$ (same for all).

$$\tau = \frac{\Delta L}{\Delta t} \Rightarrow \Delta L = \tau \cdot \Delta t$$

$$L = \tau \cdot t \longrightarrow \text{same for all}$$

**(b)** sphere, disk, hoop (reverse order of *I*, $\omega = L/I \propto 1/I$)

## 11-8 CONSERVATION OF ANGULAR MOMENTUM

If a system is isolated from external torques, then its total angular momentum L is constant.

$$\tau_{ext} = 0 \quad \Rightarrow \quad L_{tot} = I\omega \ = \ \text{constant} \quad (\text{ like } F_{ext} = 0 \ \Rightarrow \ p_{tot} = \text{constant} \ )$$

Here is a proof (not needed) of conservation of angular momentum:

First, we argue that $\vec{\tau}_{net} = \dfrac{d\vec{L}}{dt}$ (this is like $\vec{F}_{net} = \dfrac{d\vec{p}}{dt}$ ):

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}\left(\sum_i \vec{r}_i \times \vec{p}_i\right) = \sum_i \left(\frac{d\vec{r}}{dt}_i \times \vec{p}_i + \vec{r}_i \times \frac{d\vec{p}_i}{dt}\right).$$

Now, the first term in the last expression is zero:

$\displaystyle\sum_i \left(\frac{d\vec{r}}{dt}_i \times \vec{p}_i\right) = \sum_i (\vec{v}_i \times m_i \vec{v}_i) = \sum_i m_i (\vec{v}_i \times \vec{v}_i) = 0$, since any vector crossed into itself is zero. So, we have $\displaystyle\frac{d\vec{L}}{dt} = \sum_i \left(\vec{r}_i \times \frac{d\vec{p}_i}{dt}\right) = \sum_i \left(\vec{r}_i \times \vec{F}_i\right)$ (since $\vec{F}_{net} = \dfrac{d\vec{p}}{dt}$). Finally, $\displaystyle\sum_i \left(\vec{r}_i \times \vec{F}_i\right) = \sum_i \vec{\tau}_i = \vec{\tau}_{net}$, so we have $\dfrac{d\vec{L}}{dt} = \vec{\tau}_{net}$.

So now we have, $\vec{\tau}_{net} = \dfrac{\Delta\vec{L}}{\Delta t} \Rightarrow$ if $\vec{\tau}_{net} = 0$, then $\dfrac{\Delta\vec{L}}{\Delta t} = 0 \Rightarrow \vec{L} = \text{constant}$. Done.

It turns out that only 4 things are conserved:

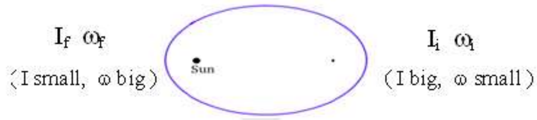
- Energy
- Linear momentum p
- Angular momentum L
- Charge q

Let's review the correspondence between translational and rotational motion

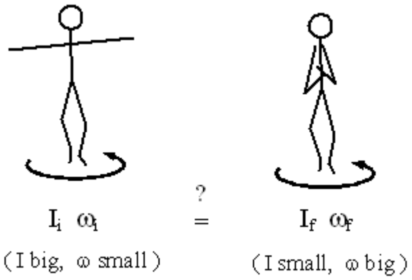
| **Translation** | $\leftrightarrow$ | **Rotation** |
|---|---|---|
| $x$ | $\leftrightarrow$ | $\theta$ |
| $v = \dfrac{\Delta x}{\Delta t}$ | $\leftrightarrow$ | $\omega = \dfrac{\Delta \theta}{\Delta t}$ |
| $a = \dfrac{\Delta v}{\Delta t}$ | $\leftrightarrow$ | $\alpha = \dfrac{\Delta \omega}{\Delta t}$ |
| $F$ | $\leftrightarrow$ | $\tau = r F_{\perp}$ |
| $M$ | $\leftrightarrow$ | $I = \sum m r^2$ |
| $F_{net} = M a$ | $\leftrightarrow$ | $\tau_{net} = I \alpha$ |
| $KE_{trans} = (1/2) M v^2$ | $\leftrightarrow$ | $KE_{rot} = (1/2) I \omega^2$ |
| $p = m v$ | $\leftrightarrow$ | $L = I \omega$ |
| $F_{net} = dp / dt$ | $\leftrightarrow$ | $\tau_{net} = dL / dt$ |
| If $F_{ext} = 0$, $p_{tot} =$ constant | $\leftrightarrow$ | If $\tau_{ext} = 0$, $L_{tot} =$ constant |

**Examples of law of conservation of angular momentum:**

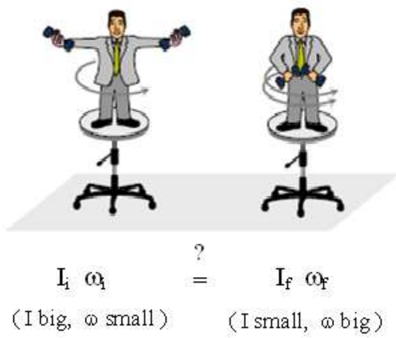
(1) The angular velocity of revolution of a planet around the sun in an elliptical orbit increases when the planet comes closer to the sun and vice-versa because when planet comes closer to the sun, it's moment of inertia $I$ decreases therefore $\omega$ increases. (Note: $I = mr^2$)



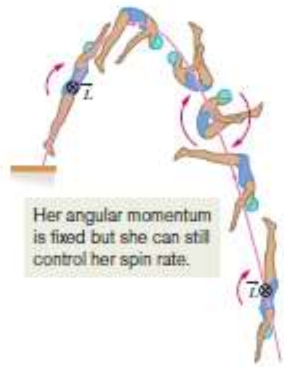

(2) A spinning skater performs feats involving spin by bringing his arms closer to his body or vice-versa. On bringing the arms closer to body, his moment of inertia *I* decreases, hence $\omega$ increases.




(3) A person-carrying heavy weight in his hands and standing on a rotating platform can change the speed of platform. When the person suddenly folds his arms, its moment of inertia decreases and in accordance the angular speed increases.



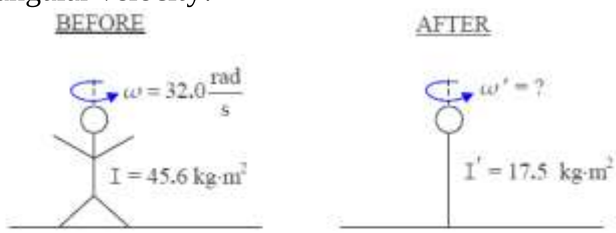

**(4)** A diver performs somersaults by Jumping from a high diving board keeping his legs and arms out stretched first and then curling his body.



**Checkpoint 7**

A rhinoceros beetle rides the rim of a small disk that rotates like a merry-go-round. If the beetle crawls toward the center of the disk, do the following (each relative to the central axis) increase, decrease, or remain the same for the beetle–disk system: (a) rotational inertia, (b) angular momentum, and (c) angular speed?

**Example:** A skater is spinning at 32.0 rad/s with his arms and legs extended outward. In this position his moment of inertia with respect to the vertical axis about which he is spinning is 45.6 kg·m$^2$. He pulls her arms and legs in close to her body changing his moment of inertia to 17.5 kg·m$^2$. What is his new angular velocity?




**Answer:**

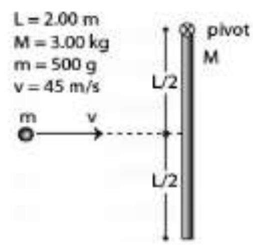
$$\overbrace{L_i = I\omega \quad (1) \qquad\qquad L_f = I'\omega' \quad (2)}^{L}$$

Equating (1) and (2), one finds

$$\omega' = \frac{I}{I'}\omega = \frac{45.6\text{ kg.m}^2}{17.5\text{ kg.m}^2}32.0\text{ rad/s} = 83.4\text{ rad/s}$$

---------------------------------------------------------------------------

**Example**: A thin uniform rod of mass $M$ = 3.0 kg and length $L$ = 2.0 m is suspended vertically from a frictionless pivot at its upper end. An object of mass $m$ = 500 $g$, traveling horizontally with a speed $v$ = 45 $m/s$ strikes the rod at its center of mass and sticks there (See Figure). What is the angular velocity of the system just after the collision?




**Answer:**

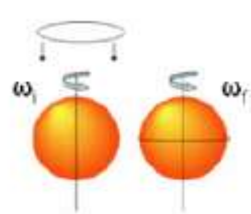
$$\overbrace{L_i = mv\left(\frac{L}{2}\right) \quad (1) \qquad = \qquad L_f = m\left(\frac{L}{2}\right)^2\omega + \frac{1}{3}ML^2\omega \quad (2)}^{L}$$

Equating (1) and (2), one finds

$$\Rightarrow \omega = \underline{5\text{ rad/s}}$$

-----------------------------------------------------------------

**Example**: A solid sphere of mass $M$ =1.0 kg and radius $R$=10 cm rotates about a frictionless axis at 4.0 rad/s (see Figure). A hoop of mass m=0.10 kg and radius $R$ =10 cm falls onto the ball and sticks to it in the middle exactly. Calculate the angular speed of the whole system about the axis just after the hoop sticks to the sphere.




**Answer:**

$$\overbrace{L_i = I_s\omega_i + I_r \times 0 \quad (1) \quad = \quad L_f = (I_s + I_r)\omega_f \quad (2)}^{L}$$

Equating (1) and (2), one finds

$$\omega_f = \frac{I_s}{I_s + I_r}\omega_i = \frac{\frac{2}{5}MR^2}{\frac{2}{5}MR^2 + mR^2}\omega_i = \frac{2M}{2M + 5m}\omega_i = \frac{2\times 1}{2\times 1 + 5\times 0.1}\times 4 = \underline{3.2 \text{ rad/s}}$$

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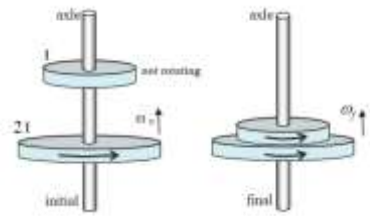
**Q20:** A disk (rotational inertia = 2*I*) rotates with angular velocity $\omega_o$ about a vertical, frictionless axle. A second disk (rotational inertia = *I*) and initially not rotating, drops onto the first disk (see figure). The two disks stick together and rotate with an angular velocity $\omega_f$. Find $\omega_f$.




**Answer: Note:** *watch for the directions of in both disks.*

$$\overbrace{L_i = 2I \times \omega_o + I \times 0 \quad (1) \quad = \quad L_f = (2I + I)\omega_f \quad (2)}^{L}$$

Equating (1) and (2), one finds

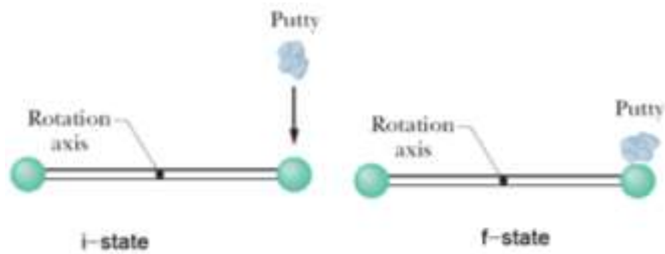
$$\omega_f = \underline{2\omega_o / 3}$$

-----------------------------------------------

## Extra problems

**Q:** In the **Figure**, two $M$ = 2.00 kg balls are attached to the ends of a thin and massless rod of length $d$ = 50.0 cm. The rod is free to rotate in a vertical plane without friction about a horizontal axis through its center. With the rod initially horizontal, a 50.0 g piece of putty (clay) drops onto one of the balls, hitting it with speed of 3.00 m/s and sticking to it. Find the angular speed of the system just after the putty hits.

**Answer:**




For initial state:

$$\ell_i = \sum_i m_i v_i r_i = M(0)\times(d/2) + M(0)\times(d/2) + 0.05\times 3\times(d/2) = 0.15(d/2) \tag{1}$$

For final state: All masses rotates with the same $\omega$.

$$\ell_f = \left(I_{masses} + I_{putty}\right)\omega, \tag{2}$$

where

$$I_{masses} = 2M\left(d/2\right)^2 = 2\times 2\times\left(d/2\right)^2 = 4\left(d/2\right)^2;$$

$$I_{putty} = m\left(d/2\right)^2 = 0.05\times\left(d/2\right)^2 = 0.05\left(d/2\right)^2$$

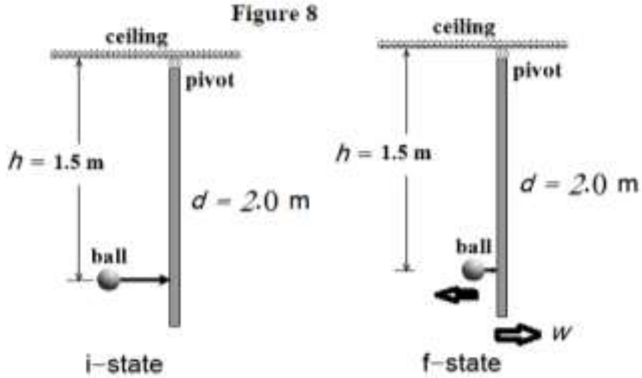
Equating (1) and (2), we have

$$0.15(d/2) = (4+0.05)(d/2)^2\omega \quad\Rightarrow\quad \omega = \frac{0.15}{4.05(0.5/2)} = \underline{0.148 \text{ rad/s}}$$

-----------------------------------------------------------------

**Q:** A thin, uniform metal rod, of length $d$ = 2.0 m, is hanging vertically from the ceiling by a frictionless pivot, as shown in **Figure 8**. Its rotational inertia about the pivot is 4.0 kg.m$^2$. It is struck at $h$ = 1.5 m below the ceiling by a small 0.050 kg ball, initially travelling horizontally at 10 m/s. The ball rebounds in the opposite direction with a speed of 5.0 m/s. Find the angular speed of the rod just after the collision.

Figure 8



**Answer:** use *b* for the ball, and *r* for the rod.

$$\vec{l}_{ib} = (mhv)\ \hat{\text{k}} = 0.05\ \times 1.5 \times 10\,\hat{\text{k}} = 0.75\,\hat{\text{k}}$$

$$\vec{l}_{fb} = -0.05\ \times 1.5 \times 5\,\hat{\text{k}} = -0.375\,\hat{\text{k}}$$

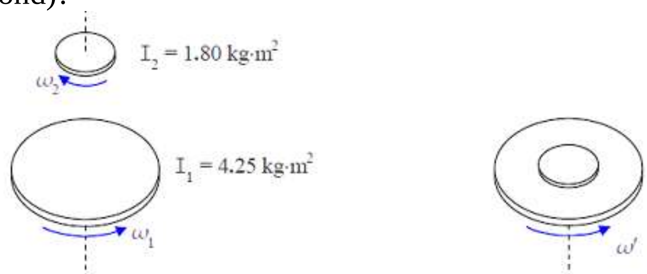
$$\vec{l}_{fr} = I\omega\hat{\text{k}}$$

$$\vec{L}_f = \vec{L}_i : 0.75\,\hat{\text{k}} = I\omega\hat{\text{k}} - 0.375\hat{\text{k}}$$

$$I\omega = 0.75 + 0.375 = 1.125 \quad \Rightarrow \omega = \frac{1.125}{4} = 0.281 \quad rad/s$$

-----------------------------------------------------------------------

**Q**: A horizontal disk of rotational inertia 4.25 kg.m$^2$ with respect to its axis of symmetry is spinning counterclockwise about its axis of symmetry, as viewed from above, at 15.5 rev/s on a frictionless massless bearing. A second disk, of rotational inertia 1.80 kg.m$^2$ with respect to its axis of symmetry, spinning clockwise as viewed from above about the same axis (which is also its axis of symmetry) at 14.2 rev/s, is dropped on top of the first disk. The two disks stick together and rotate as one about their common axis of symmetry at what new angular velocity (in units of radians per second)?




**Answer**: Define the counterclockwise as + and the clockwise as -.

Some preliminary work (expressing the given angular velocities in units of rad/s):

$$\omega_1 = 15.5\frac{\text{rev}}{\text{s}}\left(\frac{2\pi\,\text{rad}}{\text{rev}}\right) = 97.39\frac{\text{rad}}{\text{s}} \qquad \omega_2 = 14.2\frac{\text{rev}}{\text{s}}\left(\frac{2\pi\,\text{rad}}{\text{rev}}\right) = 89.22\frac{\text{rad}}{\text{s}}$$

Now we apply the principle of conservation of angular momentum for the special case in which there is no transfer of angular momentum to or from the system from outside the system. Referring to the diagram:

$$L_{\circlearrowleft} = L'_{\circlearrowleft}$$

We define counterclockwise, as viewed from above, to be the "+" sense of rotation.

$$\text{I}_1\,\omega_1 - \text{I}_2\,\omega_2 = (\text{I}_1 + \text{I}_2)\omega'$$

$$\omega' = \frac{\text{I}_1\,\omega_1 - \text{I}_2\,\omega_2}{\text{I}_1 + \text{I}_2}$$

$$\omega' = \frac{(4.25\,\text{kg}\cdot\text{m}^2)\,97.39\,\text{rad/s} - (1.80\,\text{kg}\cdot\text{m}^2)\,89.22\,\text{rad/s}}{4.25\,\text{kg}\cdot\text{m}^2 + 1.80\,\text{kg}\cdot\text{m}^2} = 41.9\frac{\text{rad}}{\text{s}}$$

(It is counterclockwise as viewed from above.)

--------------------------------------------------------

**Example:** A merry-go-round of radius R = 2.0 m is rotating about a frictionless pivot. It makes one revolution every 5.0 sec. The moment of inertia of the merry-go-round (about an axis through its center) is 500 kg·m$^2$. A child of mass m = 25 kg, originally standing at the rim, walks radially in to the exact center. The child can be considered as a point mass. What is the new angular velocity, in rad/sec, of the merry-go-round?

**Answer:** Apply the conservation of angular momentum (there are no net external torques on the system of merry-go-round and child). Thus we have

$$L = \text{constant} = I_i\, \omega_i = I_f\, \omega_f$$

or

$$\omega_f = I_i\, \omega_i / I_f$$

The initial angular velocity and the initial and final moments of inertia. Since T = 5 s, so the initial angular velocity is

$$\omega_i = 2\pi/T = 1.257 \text{ rad/s}$$

The initial moment-of-inertia is that of the merry-go-round plus that of the child located at the rim

$$I_i = 500 \text{ kg·m}^2 + mR^2 = 500 \text{ kg·m}^2 + (25 \text{ kg})(2 \text{ m})^2 = 600 \text{ kg·m}^2$$

Since the child ends up at the center (r = 0), she/he contributes no rotational inertia in the final situation, so the $I_f$ is just that of the merry-go-round, *i.e.*

$$I_f = 500 \text{ kg·m}^2$$

Plugging these in gives

$$\omega_f = (600 \text{ kg·m}^2)(1.257 \text{ rad/s})/(500 \text{ kg·m}^2) = 1.51 \text{ rad/sec}$$

--------------------------------------------------------------------------------

**Q:** A playground merry-go-round of radius R = 1.60 m has a moment of inertia $I$ = 255 kg .m$^2$ and is rotating at 9.0 rev/min about a frictionless vertical axle. Facing the axle, a 22.0-Kg child hops onto the merry-go-round and manages to sit down on the edge. What is the new angular speed of the merry-go-round?

**Answer:** Without the child the merry-go-round has a moment of inertia I which will change to $I' = I + mr^2$ when the child hops onto the edge. However, the moment of inertia should be conserved.

$$L_i = L_f \Rightarrow I\omega = I'\omega'$$

$$\omega' = \frac{I}{I'}\omega = \frac{255}{255 + 22 \times 1.60^2} 9.0 \text{ rad/s} = 7.37 \text{ rev/s}$$

-----------------------------------------------------------------------

**Q:** A playground merry-go-round has a radius of 3.0 m and a rotational inertia of 600 kg. m$^2$. It is initially spinning at 0.80 rad/s when a 20 kg child crawls from the center to the rim. When the child reaches the rim the angular velocity of the merry-go-round is:

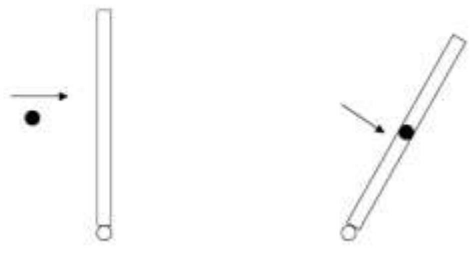
**Answer:** Conservation of angular momentum: $L_f = L_i$ (in rad/s) of the merry-go-round.

$$\Rightarrow\ (I_f + I)\omega_f = I_1\omega_f + \cancelto{0}{I_i}\omega_i \ \Rightarrow\ \omega_f = \frac{I_1\omega_f}{(I_i + I_f)}$$

$$\Rightarrow\ \omega_f = \frac{600 \times 0.8}{600 + 180} = 0.615 \text{ rad/s}$$

---

**Q:** A bullet, mass = 10 grams, is fired into the center of a door, of mass = 15 kg and width W = 1.0 meter, with a velocity of 400 m/s. The door is mounted on frictionless hinges. Find the angular speed of the door after the impact. [ $I_{door} = \frac{1}{3}M\,\mathrm{W}^2$ ]




**Answer:** Consider the type of collision involved here. There is no net external torque exerted on the bullet-door system so angular momentum is conserved. The bullet does exert a torque on the door but the door, in return exerts a torque on the bullet so the condition of zero external torques is met.

Computing angular momentum with respect to the door hinge:

Initial $\left.\begin{cases} \ell_{i,bullet} = mv(\mathrm{W}/2) = (0.01)(400)(0.5) = 2.0 \quad \mathrm{Kg.m^2/s} \\ \ell_{i,door} = 0 \end{cases}\right\} \Rightarrow L_{i,system} = 2.0 \quad \mathrm{Kg.m^2/s}$

Final $L_{f,system} = I_{f,system}\omega_f$

$$I_{door} = \frac{1}{3}M\,\mathrm{W}^2 = \frac{1}{3}\times 15\times 1^2 = 5.0 \text{ Kg.m}^2$$

$$I_{bullet} = mR^2 = (0.01)(0.5)^2 = 0.0025 \text{ Kg.m}^2;$$

$$I_{f,system} = I_{door} + I_{bullet}$$

Conservation of angular momentum requires:

$$L_{i,system} = I_{f,system}\omega_f \quad \Rightarrow \quad 2.0 = 5.0025\,\omega_f \quad \Rightarrow \omega_f = \underline{0.4 \text{ rad/s}}$$

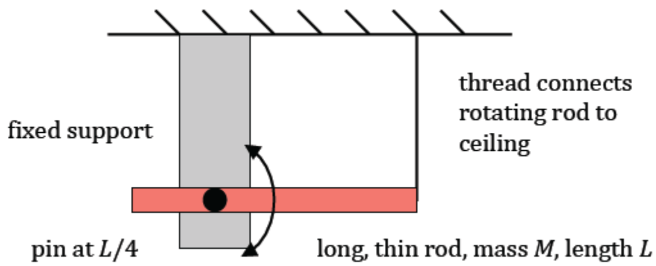
**Is energy conserved**?

$K_i = \frac{1}{2}mv^2 = \frac{1}{2}(0.01)(400)^2(0.5) = 800 \text{ J}$,

$K_f = \frac{1}{2}I_{f,system}\omega_f^2 = 0.4 \text{ J}$ , it is 1/2000 of the initial value!

---

**Example:** A long, thin, rod of mass $M$ = 0.500 kg and length $L$ = 1.00 m is free to pivot about a fixed pin located at L/4. The rod is held in a horizontal position as shown above by a thread attached to the far right end.




a. Given that the moment of inertia about an axis of rotation oriented perpendicular to the rod and passing through its center of mass is $I_{CM} = \frac{1}{12} mR^2$ , determine the moment of inertia $I$ of the rod relative to the pivot at $L/4$.

**Answer:** We can determine the moment of inertia about this new axis of rotation by using the Parallel Axis Theorem:

$$I = I_{CM} + MD^2$$

$$I = \frac{1}{12} ML^2 + M\left(\frac{L}{4}\right)^2 = \frac{7}{48} ML^2 = \frac{7}{48}(0.5kg)(1m)^2 = 0.073 kg \bullet m^2$$

b. Calculate the tension $T$ in the thread that supports the rod.

**Answer:** There are a number of ways to solve this, but the easiest is to look at the sum of the Torques about an axis of rotation located at the pivot:

$$\sum \tau = I\alpha$$

$$\tau_{thread} - \tau_{gravity} = 0$$

$$\tau_{thread} = \tau_{gravity}$$

$$r \times F_{thread} = r \times F_{gravity}$$

$$\frac{3}{4} LT = \frac{1}{4} LMg \Rightarrow T = \frac{1}{3} Mg = \frac{1}{3} Mg(0.5)(9.8) = 1.63 \text{ N}$$

The thread is cut so that the rod is free to pivot about the fixed pin.

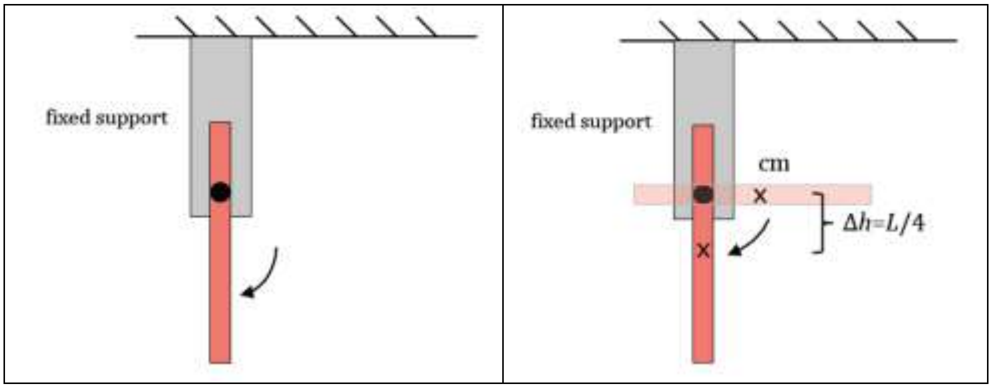
c. Determine the angular acceleration of the rod at the moment the thread is cut.

**Answer:**

$$\sum \tau = I\alpha$$

$$\alpha = \frac{\tau_{gravity}}{I} = \frac{r \times F_g}{I} = \frac{\frac{L}{4} Mg}{\frac{7}{48} ML^2} = \frac{12}{7}\frac{g}{L} = 16.8 rad/s^2$$

d. Determine the angular momentum of the rod relative to the pin at the moment the rod reaches a vertically-oriented position.




**Answer:**
Just as the moving rod reaches the vertically-oriented position, it is struck in a head-on elastic collision at the lower end by a ball of mass $m$ = 0.500 kg traveling in a horizontal direction at velocity $v_0$ = 2.00 m/s as shown.

$$U_i + K_i = U_f + K_f$$

$$mgh + 0 = 0 + \frac{1}{2}I\omega^2$$

$$mg\left(\frac{L}{4}\right) = \frac{1}{2}\left(\frac{7}{48}ML^2\right)\omega^2$$

$$\omega = \sqrt{\frac{24g}{7L}} = 5.80 rad/s$$

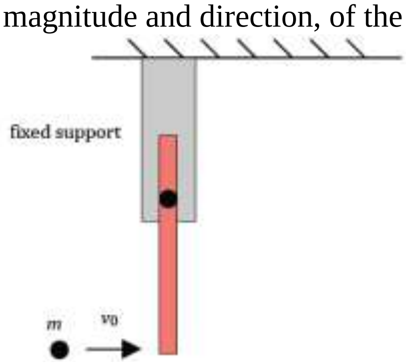
Now we can go on to determine the angular momentum of the rod:

$$L = I\omega$$

$$L = (0.073 kg \bullet m^2)(5.80 rad/s)$$

$$L = 0.42 kg \bullet m^2/s$$

e. Determine the velocity, both magnitude and direction, of the ball just after the collision.




**Answer:** This is an elastic collision, so kinetic energy is conserved in the collision, as well as linear momentum and angular momentum. We can solve for the final velocity of the ball (and the rod) after the collision using Conservation of K and Conservation of Angular Momentum. Let's start with the energy analysis:

$$K_{ball} + K_{rod} = K'_{ball} + K'_{rod}$$

$$\frac{1}{2}mv_i^2 + \frac{1}{2}I\omega_i^2 = \frac{1}{2}mv_f^2 + \frac{1}{2}I\omega_f^2$$

$$(0.5kg)(2.0m/s)^2 + (0.073kg \bullet m^2)(5.8rad/s)^2 = (0.5kg)v_f^2 + (0.073kg \bullet m^2)\omega_f^2$$

$$61.0 = 6.85v_f^2 + \omega_f^2$$

At this point we have two unknowns, so let's turn to Conservation of Angular Momentum to get another equation with those two unknowns. We'll describe the angular momentum *L* of both the rod and the ball relative to the rod's axis of rotation.

$$L_{ball} + -L_{rod} = L'_{ball} + L'_{rod}$$

$$r \times mv_i + I\omega_i = r \times mv_f + I\omega_f$$

$$(0.75m)(0.5kg)(2.0m/s) + -(0.073kg \bullet m^2)(5.80rad/s) =$$

$$(0.75m)(0.5kg)v_f + (0.073kg \bullet m^2)\omega_f$$

$$4.47 - 5.14v_f = \omega_f$$

At this point we have two expressions, both with the same unknown variables. Substitute in to get a quadratic equation that can be solved to get *vf* :

$$61.0 = 6.85v_f^2 + \omega_f^2 \qquad and \qquad 4.47 - 5.14v_f = \omega_f$$

$$61.0 = 6.85v_f^2 + (4.47 - 5.14v_f)^2$$

$$v_f = \{-0.62m/s, 2.0m/s\}$$

We have two possible solutions—which one is correct? The ball was traveling at 2.0 m/s before it struck the bar, so it can't possibly continue to have that velocity. Therefore, we choose the -0.62 m/s as the correct velocity of the ball after the elastic collision with the rod.

Solution:

$L_f = L_i$

$L_{wheel} + L_{hamster} = 0$

The wheel is a rotating object so its angular momentum is given by $L_{wheel} = -I\omega$ , where the minus sign indicates that it is into the paper. For a point particle, the angular momentum is $L_{hamster} = Rmv$ out of the paper. Thus we have

$-I\omega + Rmv = 0$ .

So the angular velocity of the wheel is

$\omega = Rmv / I$ = (0.3 kg)(0.12 m)(3.2 m/s) / (0.25 kg-m$^2$/s) = 0.461 rad/s .