

Introduction

It is difficult to trace the beginning of research in the field of phase transition physics. It might have been started by the experimental work by Andrew (1869) in which he studied the liquid-vapor critical point. That study motivated van der Waals theory (1873). At about the same time, a systematic study of magnetic ordering was undertaken. A cornerstone of that research was the molecular field theory (mean field theory) by Weiss (1907).

Mean field theory correctly describes the qualitative features of most phase transitions and, in some cases, the quantitative features. Since mean-field theory replaces the actual configurations of the local variables (e.g. spins) by their average value, it neglects the effects of fluctuations about this mean. These fluctuations may or may not be important. The more spins that interact with a particular test spin, the more the test spin sees an effective average or mean field. If the test spin interacts with two neighbors, the averaging is minimal and the fluctuations are large and important. The number of spins producing the effective field increases with the range of the interaction and with the dimension. Thus the mean field theory is a good approximation in high dimensions but fails to provide a quantitatively correct description of second-order critical points in low dimensions.

So, mean-field theory is an approximation for the thermodynamic properties of a system based on treating the order parameter as spatially constant. It is a useful description if spatial fluctuations are not important. It becomes an exact theory only when the range of interactions becomes infinite. It, nevertheless, makes quantitatively correct predictions about some aspects of phase transitions (e.g. critical exponents) in high spatial dimensions where each particle or spin has many nearest neighbors, and it makes qualitatively correct predictions in physical dimensions. Mean-field theory has the enormous advantage of being mathematically simple, and it is almost invariably the first approach taken to predict phase diagrams and properties of new experimental systems.

Landau was the first to notice (1937) that all second-order phase transition have an important constituent. At the transition point a new element of symmetry first appears. As the symmetry is a qualitative feature, the new element can appear only abruptly at some fixed values of thermodynamics variables. The latter means that the second-order phase transition breaks symmetry spontaneously and the symmetry of the system in the ordered state is lower than a symmetry of the Hamiltonian. To describe such a phenomenon Landau introduced a new entity- the order parameter. This parameter is equal to zero in the high-temperature (non ordered) phase (i.e. at temperatures above some critical value). Below the critical temperature in an ordered phase it acquires a finite value. For instance, in the case of a magnetic transition into a ferromagnetic state, the magnetization of this state can serve as an order parameter, the liquid-vapor transition order parameter can be characterized by the difference in the densities of the liquid and gas phases. From these examples it is seen that the order parameter is not always a scalar entity but can be a vector or even a tensor quantity. In the most general case, one may consider it as some macroscopic quantity having several components n . Sometimes, it is convenient to treat the order parameter as a vector in some isotropic space, not related to real configurational space. Landau theory is based on a power series expansion of the free energy in terms of the order parameter (or order parameters) for the transition of interest. It assumes the order parameter is small so that only the lowest order terms required by symmetry (and to keep the energy from diverging) are kept. The form of a Landau phenomenological free energy is determined entirely by the nature of broken symmetry of the ordered phase, i.e. by combinations of the order parameters that are left invariant under symmetry operations of

the interaction Hamiltonian. The undisputed usefulness of Landau theory rests in its simplicity. It is, of course, most useful in the vicinity of second-order phase transitions, where the order parameter is guaranteed to be small. It, however, can be used with care to treat first-order transitions, where there are discontinuous changes in order parameters, or to determine properties of ordered phases rather than of phase transitions themselves.

Landau Theory

Let us consider a physical system in d -dimensional space undergoing a phase transition in which there is only one monodimensional order parameter m and its conjugated field H . Landau theory assumes that its thermodynamic potential (generalized free energy density) F can be expanded as a power series in the order parameter m and its derivative at temperature near the critical temperature m small where only terms compatible with the symmetry of the system are included i.e.

$$F(m) = F_0 + a_2 m^2 + a_4 m^4 + a_6 m^6 + \dots - mH \quad \#$$

The even terms only are included because it is assumed they are invariant under a reversal in the sign of the magnetization. F_0 is weakly temperature dependent, and a_2, a_4, \dots are assumed to be constant or weakly temperature dependent coefficients. We can distinguish two cases, vanishing and nonvanishing field.

Case 1

In this case, $H = 0$, also we consider m is uniform $m = m_0$ which implies $\frac{dm}{dx} = 0$, then

$$F = F_0 + \frac{1}{2} a_2 m^2 + \frac{1}{4} a_4 m^4 + \frac{1}{6} a_6 m^6 + \dots \quad \#$$

In equilibrium: $\frac{dF}{dm} = 0$ and $\frac{d^2 F}{dm^2} > 0$, then

$$m_0 \left[a_2 + a_4 m_0^2 + \frac{1}{2} a_6 m_0^4 \right] = 0 \quad \#$$

Taking into account that the higher order coefficient a_6 must be positive, such that for $a_6 > 0$ to be stable at higher values of m . From Eq. (ref: a), we conclude that

at

$$T = T_c \quad m_0 = 0 \quad a_2 < 0 \quad \#$$

and at

$$T = T_c \quad m_0 \neq 0 \quad a_2 > 0 \quad \#$$

The question mark indicates that we do not know in advance the sign of a_2 at $T = T_c$, but we can investigate what constraints are put on it by the different possible signs in a_4 . Now, we distinguish three different possibilities for $T = T_c$.

i Ordinary Critical Point :

$$a_4 < 0 \quad m_0^2 = -\frac{a_2}{a_4} \quad a_2 > 0. \quad \#$$

Take $a_2 = a_0 (T - T_c)$, then

$$m_0 = T - T_c \quad \frac{a_0}{a_4} \left(T - T_c \right)^{\frac{1}{2}} \quad \boxed{\frac{1}{2}} \quad \#$$

ii Tri-Critical Point:

$$a_4 > 0 \quad m_0^4 = -\frac{a_2}{a_6} \quad a_2 < 0. \quad \#$$

Take $a_2 = a_0 (T - T_c)$, then

$$m_o(T) = \frac{a_0}{a_6} \left(\frac{1}{4} (T_c - T) \right)^{\frac{1}{4}} \quad \boxed{\frac{1}{4}} \quad \#$$

This case is a borderline between a continuous transition case *i* and a discontinuous transition case *iii*.

iii Discontinuous point:

$$a_4 = 0 \quad m_o^2 = \frac{1}{2a_2} \left[-a_4 + \sqrt{a_4^2 - 4a_2a_6} \right]^{\frac{1}{2}} \quad a_2 = \frac{a_4^2}{4a_6}. \quad \#$$

Eq. (ref: b) has a real solution only for $\frac{a_4^2}{4a_6} = a_2$. But $\frac{a_4^2}{4a_6} = a_2$, is possible only for $T = T_o$, such that

$$a_4^2 - 4a_0(T_o - T_c)a_6 = 0 \quad \#$$

Therefore m_o goes over from

$$T = T_o \quad m_o = 0, \\ T = T_o \quad m_o = \left[-\frac{a_4}{2a_2} \right]^{\frac{1}{2}}. \quad \#$$

in a discontinuous manner. That is the discontinuity or first order phase transition occurs at a given temperature T_o . We conclude that the sign of a_4 is very important in determining the character (continuous or discontinuous) of the phase transition.

Case 2

This is the nonvanishing case where $H = 0$, also m is uniform ($m = m_o$ and $m = 0$). The field H conjugated to the order parameter must satisfy $\frac{F}{m} = -H$, therefore

$$F = F_0 + \frac{1}{2}a_2m^2 + \frac{1}{4}a_4m^4 + \frac{1}{6}a_6m^6 = -mH \quad \#$$

In equilibrium: $\frac{F}{m} = 0$, then

$$H = a_0(T - T_c) + m + a_4m^3 + a_6m^5 \quad \#$$

We may analyse the critical exponents to be expected for two cases of continuous phase transitions corresponding to ordinary critical point and tricritical point.

i Ordinary Critical Point:

$$a_4 = 0, T = T_c \quad \#$$

The critical exponent describing the temperature dependence of the order parameter is obtained from Eq.(ref: o1) for

$$H = 0, T = T_c \quad m = \frac{a_0}{a_4} \left(\frac{1}{2} (T_c - T) \right)^{\frac{1}{2}} \quad \boxed{\frac{1}{2}} \quad \#$$

The critical isotherm exponent at $T = T_c$ is given by

$$m = a_4^{-\frac{1}{3}} H^{\frac{1}{3}} \quad \boxed{3} \quad \#$$

The susceptibility exponent at $T = T_c$, $T = T_c$ is given by

$$\chi^{-1} = T - T_c \quad \frac{H}{m} = \boxed{1}. \quad \#$$

From the condition $S = -\frac{E}{T}$, the specific heat exponent at $T = T_c$, $T > T_c$ is given by

$$c_p(T) \sim (T - T_c)^{-\alpha} \quad \boxed{0} \quad \#$$

ii Tri-Critical Point:

$$a_4 = 0, T = T_c \quad \#$$

The spontaneous order parameter dependence on temperature at $H = 0$ is given by

$$m_o(T) \sim (T_c - T)^{\frac{1}{4}} \quad \boxed{\frac{1}{4}} \quad \#$$

The critical isotherm exponent at $T = T_c$ is given by

$$m \sim H^{\frac{1}{5}} \quad \boxed{5} \quad \#$$

The susceptibility exponent at $T = T_c$, $T > T_c$ is given by

$$\chi^{-1} \sim \frac{H}{m} \sim a_0 (T - T_c) + 5a_6 m^4 \quad \boxed{1} \quad \#$$

The specific heat exponent at $T = T_c$, $T > T_c$ is given by

$$c_p(T) \sim (T - T_c)^{-\alpha} \quad \boxed{\frac{1}{2}} \quad \#$$

The mean field critical exponents are summarized in the below table

Exponents	Ordinary Critical Point	Tri-Critical Point
	0	1/2
	1/2	1/4
	1	1
	3	5
	1/2	1/2
	0	0

The feature(characteristic) of mean field theory which makes it universally applicable is that it identifies the order parameter of the system and tries to describe it as simply as possible. However, it assumes that the order parameter is uniform, regardless of what its neighbours are doing. In other words, it neglects fluctuations in the order parameter in which nearby parts of the system do something different from the average. This is the reason why the mean field theory fails to describe the critical exponents properly in systems with local interactions in low dimensionalities.