Power Series (Arfken 9.5)

It often happens that a differential equation cannot be solved in terms of **elementary** functions (that is, in closed form in terms of polynomials, rational functions, e^x , sin *x*, cos *x*, $\text{Ln}(x)$, etc.). A power series solution is all that is available. Such an expression is nevertheless an entirely valid solution, and in fact, many specific power series that arise from solving particular differential equations have been extensively studied and hold prominent places in mathematics and physics. The series solutions method is mainly used to find power series solutions of differential equations whose solutions can not be written in terms of familiar functions such as polynomials, exponential or trigonometric functions. This means that in general you will not be able to perform the last few steps of what we just did (less worries!), all we can try to do is to come up with a general expression for the coefficients of the power series solutions.

Definition I

A power series for the quantity $(x-a)$, about the point a, is an infinite series of the form:

$$\sum_{r=0}^{\infty} C_r (x-a)^r = C_o + C_1(x-a) + C_2(x-a)^2 + \cdots \qquad (1)$$

Such that $C_r, \quad r = 0, 1, 2, \cdots$ are the coefficients of the series and they are constants. The quantity "a" called the center of the series.

If $a=0$, we have a power series for x as follows:

$$\sum_{r=0}^{\infty} C_r x^r = C_o + C_1 x + C_2 x^2 + \cdots$$

Examples:

1− $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{r=0}^{\infty} x^r$

2− $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots = \sum_{r=0}^{\infty} \dfrac{x^r}{r!}$

3− $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \cdots = \sum_{r=0}^{\infty} (-1)^r \dfrac{x^{2r}}{(2r)!}$

4− $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \cdots = \sum_{r=0}^{\infty} (-1)^r \dfrac{x^{2r+1}}{(2r+1)!}$

A series is useful only if it **converges** (that is, if it approaches a finite limiting sum), so the natural question is, for what values of *x* will a given power series converge? Every power series in *x* falls into one of three categories:

- **Category 1**:
  The power series converges only for $x = 0$.
- **Category 2**:
  The power series converges for $|x| < R$ and **diverges** (that is, fails to converge) for $|x| > R$ (where *R* is some positive number).
- **Category 3**:
  The power series converges for all *x*.

To solve a Differential equation, for example $y''+P(x)y'+Q(x)y=0$, using power series method, we consider

$$y(x)=C_o+C_1x+C_2x^2+\cdots=\sum_{r=0}^{\infty}C_rx^r$$

By differentiation, we find

$$y'(x)=C_1+2C_2x^2+3C_3x^3+\cdots=\sum_{r=0}^{\infty}rC_rx^{r-1}$$

And

$$y''(x)=2C_2+3.2.C_3x+4.3.C_4x^2+\cdots=\sum_{r=0}^{\infty}r(r-1)C_rx^{r-2}$$

$$\vdots$$

Then substitute in the DIFFERENTIAL EQUATION we get equation in the form:

$$d_o+d_1x+d_2x^2+d_3x^3+\cdots=0$$

Last equation must verifies iff

$$d_o=d_1=d_2=d_3=\cdots=0$$

So we can easily evaluate the constants $C_1,C_2,C_3,\cdots$, and therefore finding the solution $y(x)$.

Definition II

A function $f(x)$ is said to be analytic function (remain finite) at a point $x=a$ if it can be expressed as a power series of $(x-a)$ i.e.:

$$f(x)=\sum_{r=0}^{\infty}C_r(x-a)^r=C_o+C_1(x-a)+C_2(x-a)^2+\cdots \tag{2}$$

This means that $\lim_{x\to a}f(x)=C_o$.

Definition III

A point $x=a$ is said to be **ordinary point** for the Differential equation

$$y''+P(x)y'+Q(x)y=0 \tag{3}$$

iff the functions $P(x)$ and $Q(x)$ are analytic at point $x=a$.

Definition IV

If one of the functions $P(x)$ or $Q(x)$ in (3) is not analytic (goes to infinity) at $x=a$, then $x=a$ is called a **singular point** for (3).

Definition V

A point $x=a$ is said to be **regular (removable) singular point** for the differential equation (3) iff

$$\lim_{x\to a}(x-a)P(x)=a_o, \quad \lim_{x\to a}(x-a)^2Q(x)=b_o$$

where a_o and b_o are finite values.

$$\text{In summary: } a\equiv\begin{cases}\text{ordinary point}\\ \text{singular point}\\ \text{regular (removable) singular point}\end{cases}$$

Example: Find and classify the singularities of the following differential equation for $y(x)$.

$$(x-2)y''-xy'+y=0$$

No need to study the point at infinite.

Answer: Look at the following

1- $x=0$ is an ordinary point for this Differential equation Notice that:

$$P(x)=-\frac{x}{x-2},\qquad Q(x)=\frac{1}{x-2},$$

$$\lim_{x\to 0}P(x)=0,\qquad \lim_{x\to 0}Q(x)=-\frac{1}{2}$$

2- $x=2$ is a singular point for this Differential equation because

$$\lim_{x\to 2}P(x)=\infty,\qquad \lim_{x\to 2}Q(x)=\infty$$

3- Also, $x=2$ is a regular singular point for this Differential equation because

$$\lim_{x\to 2}(x-2)P(x)=-2,\qquad \lim_{x\to 2}(x-2)^2Q(x)=0$$

H. W. Find and classify the singularities of the following differential equations in Table 9.4.

Table 9.4

	Equation	Regular singularity $x=$	Irregular singularity $x=$
1.	Hypergeometric $x(x-1)y''+[(1+a+b)x-c]y'+aby=0.$	$0, 1, \infty$	–
2.	Legendre[a] $(1-x^2)y''-2xy'+l(l+1)y=0.$	$-1, 1, \infty$	–
3.	Chebyshev $(1-x^2)y''-xy'+n^2y=0.$	$-1, 1, \infty$	–
4.	Confluent hypergeometric $xy''+(c-x)y'-ay=0.$	0	∞
5.	Bessel $x^2y''+xy'+(x^2-n^2)y=0.$	0	∞
6.	Laguerre[a] $xy''+(1-x)y'+ay=0.$	0	∞
7.	Simple harmonic oscillator $y''+\omega^2y=0.$	–	∞
8.	Hermite $y''-2xy'+2\alpha y=0.$	–	∞

H. W. Find and classify the singularities of the following differential equations for $y(x)$.
No need to study the point at infinite.

$$x^3y''+x^4y'+2y=0 \tag{1}$$

$$(x^2-3x+2)y''-xy'+x^2y=0 \tag{2}$$

$$y''+2y'+2y=0 \tag{3}$$

$$(e^x-1)y''+xy=0 \tag{4}$$

Answer:

$$y''+xy'+\frac{2}{x^3}y=0 \tag{1}$$

Irregular (Essential) Singular point at x=0.

$$y''-\frac{x}{(x-1)(x-2)}y'+\frac{x^2}{(x-1)(x-2)}y=0 \tag{2}$$

Regular singular points at x=1 and x=2.

$$y'' + 2y' + 2y = 0 \tag{3}$$

No singular points, all ordinary points.

$$y'' + \frac{x}{e^x - 1}y = 0 \tag{4}$$

Regular Singular points occur for $x=2n\pi i$ except for x=0 which is an ordinary point since the singularity is removable

$$\lim_{x\to 0}\frac{x}{e^x - 1} = 1 \tag{5}$$

Fuchs's Theorem:

Can the power series method solve any differential equation? The answer is the Fuchs' theorem. It states that: **"We can always get at least one power-series solution, provided we are expanding about a point that is an ordinary point or at worst a regular singular point".**

Theorem: without proof

If the functions $P(x)$, $Q(x)$ and $h(x)$ in the differential equation

$$y'' + P(x)y' + Q(x)y = h(x)$$

Are analytic at $x = a$, then every solution $y(x)$ of the differential equation is analytic at $x = a$ and can be represented by a power series in powers of $(x - a)$.

Simple differential equation

Example: Solve the differential equation:

$$y'' + \omega^2 y = 0$$

Answer: for the equation: $y'' + \omega^2 y = 0$ the **auxiliary equation** $\Rightarrow r^2 + \omega^2 = 0 \quad \Rightarrow r = \pm i\,\omega$

Then the answer is: $y(x) = ae^{r_1 x} + be^{r_2 x} = ae^{i\omega x} + be^{-i\omega x} = A\sin(\omega x) + B\cos(\omega x)$

Power Series Solution:

Example: Using the power series method, solve the differential equation:

$$y''(t) + 4y(t) = 0 .$$

Answer: Using the above technique it is not hard to see that the solutions are of the form

$$y(t) = A\sin(2t) + B\cos(2t)$$

where A and B are constants.

We want to illustrate how to find power series solutions for a second-order linear differential equation. The generic form of a power series is

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

We have to determine the right choice for the coefficients (a_n).

As in other techniques for solving differential equations, once we have a "guess" for the solutions, we plug it into the differential equation. Recall from the previous section that

$$y''(t) = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2}.$$

Plugging this information into the differential equation we obtain:

$$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + 4\sum_{n=0}^{\infty} a_n t^n = 0.$$

Our next goal is to simplify this expression such that only one summation sign "$\sum$ " remains. The obstacle we encounter is that the powers of both sums are different, t^{n-2} for the first sum and t^n for the second sum. We make them the same by shifting the index of the first sum by 2 units to obtain

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n + \sum_{n=0}^{\infty} 4a_n t^n = 0.$$

Now we can combine the two sums as follows:

$$\sum_{n=0}^{\infty}\left((n+2)(n+1)a_{n+2}t^n + 4a_n t^n \right) = 0,$$

and factor out t^n:

$$\sum_{n=0}^{\infty}\left((n+2)(n+1)a_{n+2} + 4a_n \right) t^n = 0.$$

Next we need a result you probably already know in the case of polynomials: A polynomial is identically equal to zero if and only if **all** of its coefficients are equal to zero. This result also holds true for power series:

Theorem: A power series is identically equal to zero if and only if **all** of its coefficients are equal to zero.

This theorem applies directly to our example: The power series on the left is identically equal to zero, consequently all of its coefficients are equal to 0:

$$(n+2)(n+1)a_{n+2} + 4a_n = 0 \text{ for all } n = 0, 1, 2, 3, \ldots$$

Solving these equations for the "highest index" n+2, we can rewrite as

$$a_{n+2} = -\frac{4}{(n+1)(n+2)}a_n \text{ for all } n = 0, 1, 2, 3, \ldots$$

These equations are known as the **"recurrence relations"** of the differential equation. The recurrence relations contain all the information about the coefficients we need.

Recall that

$$y(t) = \sum_{n=0}^{\infty} a_n t^n = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots ,$$

in particular $y(0)=a_0$, and

$$y'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1} = a_1 + 2a_2 t + 3a_3 t^2 + \ldots ,$$

in particular $y'(0)=a_1$. This means, we can think of the first two coefficients a_0 and a_1 as the initial conditions of the differential equation.

How can we evaluate the next coefficient a_2? Let us read our recurrence relations for the case n=0:

$$a_2 = -\frac{4}{1 \cdot 2}a_0.$$

Reading off the recurrence relation for $n = 1$ yields

$$a_3 = -\frac{4}{2 \cdot 3} a_1.$$

Continue:

$$a_4 = -\frac{4}{3 \cdot 4} a_2 = -\frac{4}{3 \cdot 4} \cdot \left(-\frac{4}{1 \cdot 2} a_0\right) = \frac{2^4}{4!} a_0.$$

$$a_5 = -\frac{4}{4 \cdot 5} a_3 = -\frac{4}{4 \cdot 5} \cdot \left(-\frac{4}{2 \cdot 3} a_1\right) = \frac{2^5}{2 \cdot 5!} a_1.$$

What do we know about the solutions to our Differential equation at this point? They look like this:

$$\begin{aligned}
y(t) &= \sum_{n=0}^{\infty} a_n t^n = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 \ldots \\
&= a_0 + a_1 t - \frac{4}{1 \cdot 2} a_0 t^2 - \frac{4}{2 \cdot 3} a_1 t^3 + \ldots \\
&= a_0 \left(1 - \frac{4}{1 \cdot 2} t^2 + \frac{2^4}{4!} t^4 \ldots\right) + a_1 \left(t - \frac{4}{3!} t^3 + \frac{2^5}{2 \cdot 5!} t^5 \ldots\right) \\
&= a_0 \left(1 - \frac{2^2}{2!} t^2 + \frac{2^4}{4!} t^4 \ldots\right) + \frac{a_1}{2} \left(2t - \frac{2^3}{3!} t^3 + \frac{2^5}{5!} t^5 \ldots\right) \\
&= a_0 \left(1 - \frac{1}{2!} (2t)^2 + \frac{1}{4!} (2t)^4 \ldots\right) + \frac{a_1}{2} \left((2t) - \frac{1}{3!} (2t)^3 + \frac{1}{5!} (2t)^5 \ldots\right)
\end{aligned}$$

Of course the power series inside the parentheses are the familiar functions $\cos(2t)$ and $\sin(2t)$:

$$y(t) = a_0 \cos(2t) + \frac{a_1}{2} \sin(2t)$$

so we have found the general solution of the Differential equation (with a_0 instead of B, and $a_1/2$ instead of A).

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In[3]:= sol2 = NDSolve[{∂{x,2}y[x] + 4 y[x] == 0, y[0] == 0, y'[0] == 1}, y, {x, 0, 2π}]

Out[3]= {{y → InterpolatingFunction[{{0., 6.28319}}, <>]}}

In[4]:= Plot[ y[x] /. sol2[[1]], {x, 0, 2π} ];
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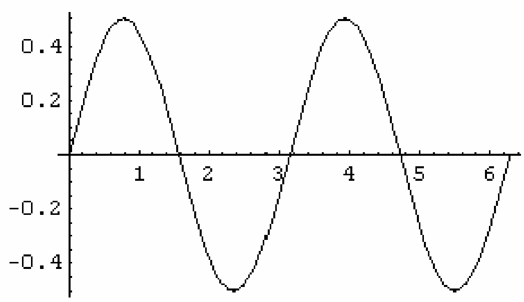

```
In[1]:= (* Example : solve the equation
                      y'' - 2 x y' +8 y = 0
          subject to the initial conditions y (0)=12 and y' (0) = 0.  *)

In[5]:= terms = 6

Out[5]= 6

In[6]:= lhs = y''[x] - 2 x y'[x] + 8 y[x];

In[7]:= serleft = Series[lhs, {x, 0, terms}]
```

Out[7]= $(8\,y[0]+y''[0])+(6\,y'[0]+y^{(3)}[0])\,x+\left(2\,y''[0]+\frac{1}{2}\,y^{(4)}[0]\right)x^2+\left(\frac{1}{3}\,y^{(3)}[0]+\frac{1}{6}\,y^{(5)}[0]\right)x^3+\frac{1}{24}\,y^{(6)}[0]\,x^4+\left(-\frac{1}{60}\,y^{(5)}[0]+\frac{1}{120}\,y^{(7)}[0]\right)x^5+\left(-\frac{1}{180}\,y^{(6)}[0]+\frac{1}{720}\,y^{(8)}[0]\right)x^6+O[x]^7$

```
In[8]:= eqs = LogicalExpand[serleft == 0]  (* equate the coefficient of x to zero  *)
```

Out[8]= $8\,y[0]+y''[0]==0\ \&\&\ 6\,y'[0]+y^{(3)}[0]==0\ \&\&\ 2\,y''[0]+\frac{1}{2}\,y^{(4)}[0]==0\ \&\&\ \frac{1}{3}\,y^{(3)}[0]+\frac{1}{6}\,y^{(5)}[0]==0\ \&\&\ \frac{1}{24}\,y^{(6)}[0]==0\ \&\&\ -\frac{1}{60}\,y^{(5)}[0]+\frac{1}{120}\,y^{(7)}[0]==0\ \&\&\ -\frac{1}{180}\,y^{(6)}[0]+\frac{1}{720}\,y^{(8)}[0]==0$

```
In[9]:= roots = Solve[eqs, Evaluate[Table[D[y[x], {x, i}], {i, 2, terms + 2}] /. x → 0]]
```

Out[9]= $\{\{y''[0]\to -8\,y[0],\ y^{(3)}[0]\to -6\,y'[0],\ y^{(4)}[0]\to 32\,y[0],\ y^{(5)}[0]\to 12\,y'[0],\ y^{(6)}[0]\to 0,\ y^{(7)}[0]\to 24\,y'[0],\ y^{(8)}[0]\to 0\}\}$

```
In[11]:= sol = Series[y[x], {x, 0, terms - 1}] /. roots[[1]]
```

Out[11]= $y[0]+y'[0]\,x-4\,y[0]\,x^2-y'[0]\,x^3+\frac{4}{3}\,y[0]\,x^4+\frac{1}{10}\,y'[0]\,x^5+O[x]^6$

```
In[12]:= y[0] = 12; y'[0] = 0;  (* use the B.C.  *)

In[13]:= nsol = Normal[sol]

General::spell1 : Possible spelling error: new symbol name "nsol" is similar to existing symbol "sol". More…
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Out[13]= $12-48\,x^2+16\,x^4$

```
In[14]:= ps = Plot[nsol, {x, -2, 2}]
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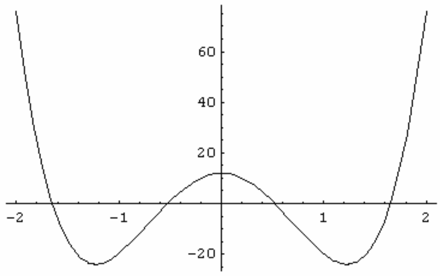


```
Out[14]= - Graphics -

In[23]:= sol2 = NDSolve[{ z''[x] - 2 x z'[x] + 8 z[x] == 0, z[0] == 12, z'[0] == 0}, z, {x, -2, 2}]

Out[23]= {{z → InterpolatingFunction[{{-2., 2.}}, <>]}}

In[24]:= pn = Plot[z[x] /. sol2[[1]], {x, -2, 2} ];
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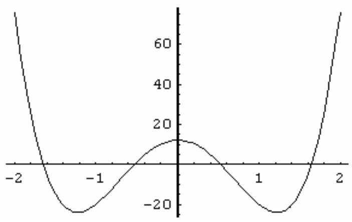

http://www.sosmath.com/diffeq/series/series04/series04.html
http://www.sosmath.com/diffeq/diffeq.html