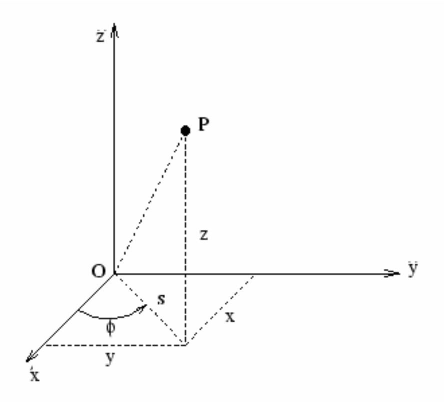
Laplace's equation in Cylindrical Coordinates

1- Circular cylindrical coordinates

The circular cylindrical coordinates (s,ϕ,z) are related to the rectangular Cartesian coordinates (x,y,z) by the formulas (see Fig.):

Circular cylindrical coordinates.



$$x = s\cos\phi, \qquad y = s\sin\phi, \qquad z = z. \qquad \left(0\le s<\infty, 0\le\phi\le 2\pi, -\infty<z<\infty\right)$$

The inverse relations are:

$$s=\sqrt{x^2+y^2}, \quad \tan\phi=\frac{y}{x}, \quad z=z.$$

An infinitesimal length $d\ell$ is

$$d\ell=\sqrt{(ds)^2+(sd\phi)^2+(dz)^2}$$

An infinitesimal volume element is:

$$d\Im = s\ ds\ d\phi\, dz .$$

The gradient, divergence, curl and Laplacian become, in cylindrical coordinates are:

Gradient

$$\nabla V = \frac{\partial V}{\partial s}\hat{s}+\frac{1}{s}\frac{\partial V}{\partial\phi}\hat{\phi}+\frac{\partial V}{\partial z}\hat{z}$$

Divergent

$$\nabla\bullet\vec{v}=\frac{1}{s}\frac{\partial}{\partial s}(sv_s)+\frac{1}{s}\frac{\partial}{\partial\phi}\left(v_\phi\right)+\frac{\partial}{\partial z}\left(v_z\right)$$

Curl

$$\nabla\times\vec{v}=\left(\frac{1}{s}\frac{\partial v_z}{\partial\phi}-\frac{\partial v_\phi}{\partial z}\right)\hat{s}+\left(\frac{1}{s}\frac{\partial v_s}{\partial z}-\frac{\partial v_z}{\partial s}\right)\hat{\phi}+\frac{1}{s}\left[\frac{\partial}{\partial s}\left(sv_\phi\right)-\frac{\partial v_s}{\partial\phi}\right]\hat{z}$$

Laplacian

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right)+\frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2}+\frac{\partial^2 V}{\partial z^2}=0$$

Laplace's equation in two dimensions (Consult Jackson (page 111))

Example: Solve Laplace's equation by separation of variables in *cylindrical* coordinates, assuming there is no dependence on *z* (cylindrical symmetry). Make sure that you find *all* solutions to the radial equation. Does your result accommodate the case of an infinite line charge?
Answer: For a system with cylindrical symmetry the electrostatic potential does not depend on *z*. This immediately implies that $\frac{\partial V}{\partial z}=0$. Under this assumption Laplace's equation reads:

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right)+\frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2}=0$$

Consider as a possible solution of *V*:

$$V(s,\phi)=\Re(s)\Phi(\phi)$$

Substituting this solution into Laplace's equation we obtain

$$\frac{\Phi(\phi)}{s}\frac{\partial}{\partial s}\left(s\frac{\partial \Re(s)}{\partial s}\right)+\frac{\Re(s)}{s^2}\frac{\partial^2 \Phi(\phi)}{\partial \phi^2}=0$$

Multiplying each term in this equation by s^2 and dividing by $\Re(s)\Phi(\phi)$ we obtain

$$\frac{s}{\Re(s)}\frac{\partial}{\partial s}\left(s\frac{\partial \Re(s)}{\partial s}\right)+\frac{1}{\Phi(\phi)}\frac{\partial^2 \Phi(\phi)}{\partial \phi^2}=0$$

The first term in this equation depends only on *s* while the second term in this equation depends only on ϕ. This equation can therefore be only valid for every *s* and every ϕ if each term is equal to a constant. Thus we require that:

$$\frac{s}{\Re(s)}\frac{\partial}{\partial s}\left(s\frac{\partial \Re(s)}{\partial s}\right)=\gamma \equiv \text{ constant} \qquad \text{(A)}$$

and

$$\frac{1}{\Phi(\phi)}\frac{\partial^2 \Phi(\phi)}{\partial \phi^2}=-\gamma \qquad \text{(B)}$$

1- consider the case in which $\gamma=-m^2>0$. The differential equation for $\Phi(\phi)$ can be rewritten as

$$\frac{\partial^2 \Phi(\phi)}{\partial \phi^2}-m^2\,\Phi(\phi)=0$$

The most general solution of this differential solution is

$$\Phi_m(\phi)=C_m e^{m\phi}+D_m e^{-m\phi}$$

However, in cylindrical coordinates the angle ϕ must be unique, namely, $\Phi(\phi+2\pi)=\Phi(\phi)$ and therefore the general solution of the equation $\frac{d^2\Phi}{d\phi^2}-m^2\Phi=0$ is not satisfied for this solution, and we conclude that $\gamma=m^2>0$. The differential equation for $\Phi(\phi)$ can be rewritten as

$$\frac{\partial^2\Phi(\phi)}{\partial\phi^2}+m^2\ \Phi(\phi)=0$$

The most general solution of this differential solution is

$$\Phi_m\left(\phi\right)=C_m\cos\left(m\phi\right)+D_m\sin\left(m\phi\right)$$

The condition that $\Phi(\phi)=\Phi(\phi+2\pi)$ requires that m is an integer. Now consider the radial function $\Re\left(s\right)$. We will first consider the case in which $\gamma=m^2>0$.

2- Consider the following solution for $\Re\left(s\right)$:

$$\Re\left(s\right)=As^k, \qquad A=\text{ constant}$$

Substituting this solution into equation (A) we obtain

$$\frac{s}{As^k}\frac{\partial}{\partial s}\left(s\,\frac{\partial}{\partial s}\left(As^k\right)\right)=k^2=m^2$$

Therefore, the constant k can take on the following two values:

$$k_+=m, \qquad k_-=-m$$

The most general solution for $\Re\left(s\right)$ under the assumption that $m^2>0$ is therefore

$$\Re\left(s\right)=A_m s^m+\frac{B_m}{s^m}$$

Now consider the solutions for $\Re\left(s\right)$ when $m^2=0$. In this case we require that

$$\frac{\partial}{\partial s}\left(s\,\frac{\partial\Re\left(s\right)}{\partial s}\right)=0 \qquad \Rightarrow \qquad s\,\frac{\partial\Re\left(s\right)}{\partial s}=a_0=\text{ constant}$$

This equation can be rewritten as

$$\frac{\partial\Re\left(s\right)}{\partial s}=\frac{a_0}{s}$$

If $a_0=0$ then the solution of this differential equation is

$$\Re\left(s\right)=b_0=\text{ constant}$$

If $a_0\neq 0$ then the solution of this differential equation is

$$\Re\left(s\right)=a_0\ln(s)+b_0$$

Combining the solutions obtained for $m^2=0$ with the solutions obtained for $m^2>0$ we conclude that the most general solution for $\Re\left(s\right)$ is given by

$$\Re\left(s\right)=a_0\ln(s)+b_0+\sum_{m=1}^{\infty}\left[A_m s^m+\frac{B_m}{s^m}\right]$$

Therefore, the most general solution of Laplace's equation for a system with cylindrical symmetry is

$$\boxed{V\ \left(s,\phi\right)=a_0\ln(s)+b_0+\sum_{m=1}^{\infty}\left[\left(A_m s^m+\frac{B_m}{s^m}\right)\left(C_m\cos\left(m\phi\right)+D_m\sin\left(m\phi\right)\right)\right]}$$

Laplace’s equation in three dimensions

Laplace's equation in cylindrical coordinates takes the form:

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right)+\frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2}+\frac{\partial^2 V}{\partial z^2}=0 \tag{1}$$

Consider as a possible solution of *V*:

$$V(s,\phi,z)=\Re(s)\Phi(\phi)Z(z) \tag{2}$$

Substituting (2) into (1) we obtain

$$\frac{\nabla^2 V}{V}=\frac{\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial \Re(s)}{\partial s}\right)}{\Re(s)}+\frac{\frac{1}{s^2}\frac{\partial^2\Phi(\phi)}{\partial\phi^2}}{\Phi(\phi)}+\frac{\frac{\partial^2 Z(z)}{\partial z^2}}{Z(z)}=0 \tag{3}$$

Taking $\left(\partial^2 Z/\partial z^2\right)/Z$ to the right-hand side of the equation we have an expression independent of *z* on the left, from which we conclude that either expression (on the right or on the left) must be a constant. Explicitly putting in the sign (which must still be determined from the boundary conditions) of the separation constant, we have:

1- Azimuthal direction:

$$\frac{1}{\Phi(\phi)}\frac{\partial^2\Phi(\phi)}{\partial\phi^2}=-\gamma \ . \tag{3a}$$

First consider the case in which $\gamma=-m^2>0$. The differential equation for $\Phi(\phi)$ can be rewritten as

$$\frac{\partial^2\Phi(\phi)}{\partial\phi^2}-m^2\,\Phi(\phi)=0$$

The most general solution of this differential solution is

$$\Phi_m(\phi)=C_m e^{m\phi}+D_m e^{-m\phi} \ . \tag{3b}$$

However, in cylindrical coordinates the angle ϕ must be unique, namely, $\Phi(\phi+2\pi)=\Phi(\phi)$ and therefore the general solution of the equation $\frac{d^2\Phi}{d\phi^2}-m^2\Phi=0$ is not satisfied for this solution, and we conclude that $\gamma=m^2>0$. The differential equation for $\Phi(\phi)$ can be rewritten as

$$\frac{\partial^2\Phi(\phi)}{\partial\phi^2}+m^2\,\Phi(\phi)=0 \tag{4}$$

The most general solution of (4) is

$$\Phi(\phi)=A\sin(m\phi)+B\cos(m\phi) \tag{5}$$

The condition that $\Phi(\phi)=\Phi(\phi+2\pi)$ requires that *m* is an integer.

Next, consider the second part, i.e.:

$$\frac{\frac{\partial^2 Z(z)}{\partial z^2}}{Z(z)}=\lambda^2 \quad\Rightarrow\quad Z(z)=A\sinh(\lambda z)+B\cosh(\lambda z) \tag{4}$$

Or, alternatively,

$$\frac{\dfrac{\partial^2 Z(z)}{\partial z^2}}{Z(z)} = -\lambda^2 \quad \Rightarrow \quad Z(z) = A\sin(\lambda z) + B\cos(\lambda z) \tag{5}$$

For the choice (4), Eq (3) reduces to:

$$\frac{s\dfrac{\partial}{\partial s}\left(s\dfrac{\partial \Re(s)}{\partial s}\right)}{\Re(s)} + \lambda^2 s^2 = -\frac{\dfrac{\partial^2 \Phi(\phi)}{\partial \phi^2}}{\Phi(\phi)} = m^2 \tag{6}$$

Finally, consider the radial function $\Re(s)$, in the form:

$$s\frac{\partial}{\partial s}\left(s\frac{\partial \Re(s)}{\partial s}\right) + \left(\lambda^2 s^2 - m^2\right)\Re(s) = 0 \tag{7}$$

is the Bessel's equation, having solutions

$$\Re(s) = EJ_m(\lambda s) + FN_m(\lambda s) \tag{8}$$

where J_m and N_m are Bessel and Neumann functions of order *m*. Had we picked the negative separation constant as in equation (7), we would have obtained for $\Re(s)$ the modified Bessel equation:

$$s\frac{\partial}{\partial s}\left(s\frac{\partial \Re(s)}{\partial s}\right) + \left(-\lambda^2 s^2 - m^2\right)\Re(s) = 0 \tag{9}$$

having as solutions the modified Bessel functions $I_m(\lambda s)$ and $K_m(\lambda s)$.

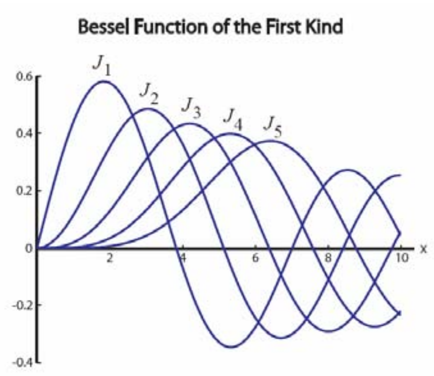
$$\Re(s) = EI_m(\lambda s) + FK_m(\lambda s) \tag{11}$$

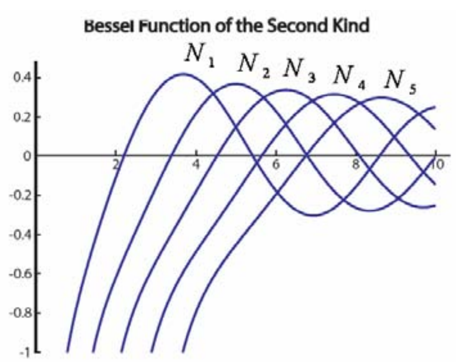
Note: $K_m(\lambda s)$ and $N_m(\lambda s)$ diverge at *r* = 0 and are therefore excluded from problems where the region of interest includes $r = 0$, while $J_m(\lambda s)$ and $I_m(\lambda s)$ diverges as $r \to \infty$ and will therefore be excluded from any exterior solution.

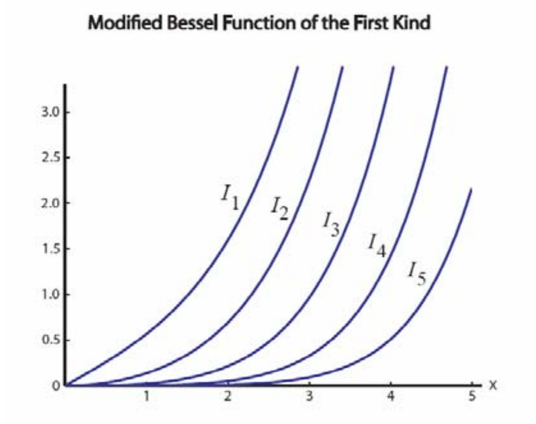
The complete solution is then of the form:

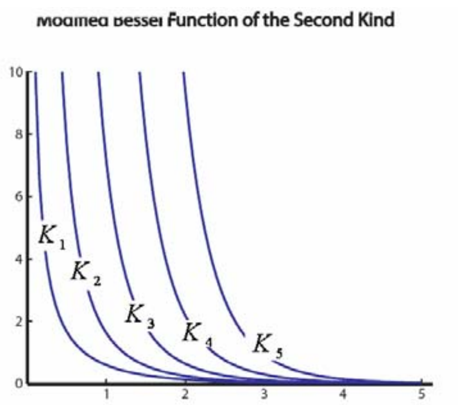
$$\begin{aligned} V(s,\phi,z) = &\sum_{\lambda,m}\begin{Bmatrix} J_m(\lambda s) \\ N_m(\lambda s)\end{Bmatrix}\cdot\begin{Bmatrix}\sin(m\phi) \\ \cos(m\phi)\end{Bmatrix}\cdot\begin{Bmatrix}\sinh(\lambda z) \\ \cosh(\lambda z)\end{Bmatrix} \\ &+\sum_{\lambda,m}\begin{Bmatrix} I_m(\lambda s) \\ K_m(\lambda s)\end{Bmatrix}\cdot\begin{Bmatrix}\sin(m\phi) \\ \cos(m\phi)\end{Bmatrix}\cdot\begin{Bmatrix}\sin(\lambda z) \\ \cos(\lambda z)\end{Bmatrix} \end{aligned} \tag{12}$$

where the braces $\{\ \}$ stand for the arbitrary linear combination of the two terms within.

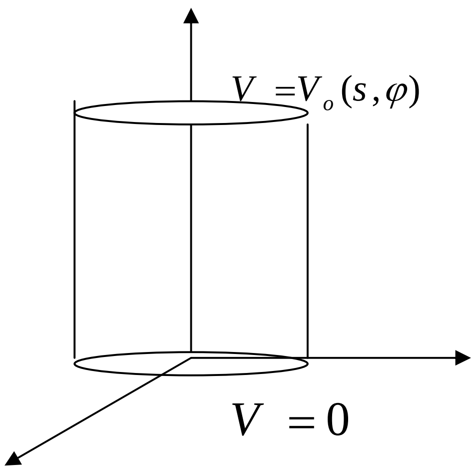








Example (6): A cylinder of radius “a” and height *L*, is placed parallel to the z axis. Its basis at $z = 0$ is grounded, and so is its face at $s = a$. The basis at $z = L$ is held at a given potential $V_o(s,\varphi)$ (a given function). Find the potential everywhere within the cylinder.



Solution: Let us first consider the general solutions of the three equations, with the boundary conditions:

(a) B. Cs.

$$V(s,\phi,0) = 0 \tag{1a}$$

$$V(s,\phi,L) = V_0 \tag{1b}$$

$$V(a,\phi,z) = 0 \tag{2}$$

$$V(0,\phi,z) = \text{finite} \tag{3}$$

i- The angle ϕ must be unique, namely, $\Phi(\phi+2\pi) = \Phi(\phi)$ and therefore the general solution of the equation $\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0$ will be:

$$\Phi(\phi) = A_m \cos(m\phi) + B_m \sin(m\phi)$$

with m an integer.

ii- In our case, Z must vanish at $z = 0$, but not at $z = L$, which means we have the equation $\frac{d^2 Z}{dz^2} - k^2 Z = 0$ and the Z function is of the form:

$$Z(z) = C \sinh(kz) + D \cosh(kz),$$

iii- Due to the above items i and ii, $\Re$ must be the solution of the equation $\frac{d^2\Re}{ds^2} + \frac{1}{s}\frac{d\Re}{ds} + \left(k^2 - \frac{m^2}{s^2}\right)\Re = 0$ and taken to be of the form:

$$\Re(s) = EJ_m(k_m s) + FN_m(k_m s)$$

(b) The general solution of the Laplace's equation for the problem in cylindrical coordinates consists of a sum (superposition) of terms of the form:

$$V\ (s,\phi,z)=\Re(s)\Phi(\phi)Z(z)$$
$$=\sum_{m=0}^{\infty}\left[EJ_m(k_m s)+FN_m(k_m s)\right]\left[A_m\sin(m\phi)+B_m\cos(m\phi)\right]\left[C\sinh(kz)+D\cosh(kz)\right]$$

I- B.C. 1a and 1b implies $D=0$

II- B.C. 3 implies $F=0$

III- B.C. 2 implies $J_m(k_{mn}a)=0\Rightarrow k_{mn}=\dfrac{x_{mn}}{a}, \qquad n=1,2,3,\cdots$

x_{mn} is the n^{th} root of $J_m(k_{mn}a)$. Remember that, Bessel function has an infinite number of roots, and therefore κa takes an infinite number of discrete values, all of them are roots of the m^{th} Bessel function. Namely, $k_{mn}a$ is the n^{th} root of the m^{th} Bessel function.

It follows that the general solution of our problem is

$$V\ (s,\phi,z)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}J_m(k_{mn}s)\sinh(k_{mn}z)\left\{A_{mn}\sin(m\phi)+B_{mn}\cos(m\phi)\right\}$$

We now impose the boundary condition at z = L:

$$V\ (s,\phi)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}J_m(k_{mn}s)\sinh(k_{mn}L)\left\{A_{mn}\sin(m\phi)+B_{mn}\cos(m\phi)\right\}$$

This is a Fourier series in ϕ and a Fourier-Bessel series in *s*.

We now use this property with the boundary condition at $z=L$ to determine all the coefficients in terms of the given function $V_o(s,\varphi)$.

First, we use (Fourier trick) the delta functions of the trigonometric functions in the form:

$$\int_0^a\cos\left(\frac{n\pi y}{a}\right)\cos\left(\frac{m\pi y}{a}\right)dy=\frac{a}{2}\delta_{mn}=\begin{cases}0 & \text{if}\quad m\neq n\\ \dfrac{a}{2} & \text{if}\quad m=n\neq 0\\ a & \text{if}\quad m=n=0\end{cases}$$

$$\int_0^a\sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi y}{a}\right)dy=\frac{a}{2}\delta_{mn}=\begin{cases}0 & \text{if}\quad m\neq n\\ \dfrac{a}{2} & \text{if}\quad m=n\neq 0\end{cases}$$

$$\int_0^a\sin\left(\frac{n\pi y}{a}\right)\cos\left(\frac{m\pi y}{a}\right)dy=0$$

$$\int_0^a\sin\left(\frac{n\pi y}{a}\right)dy=\begin{cases}0 & \text{if } n \text{ is even}\\ \dfrac{2a}{n\pi} & \text{if } n \text{ is odd}\end{cases}$$

to obtain

$$\sum_m A_{mn}\sinh(k_{mn}L)J_m(k_{mn}s)=\frac{1}{\pi}\int_0^{2\pi}V(s,\varphi)\sin\left(m\varphi\right)d\varphi$$

$$\sum_m B_{mn}\sinh(k_{mn}L)J_m(k_{mn}s)=\frac{1}{\pi}\int_0^{2\pi}V(s,\varphi)\cos\left(m\varphi\right)d\varphi$$

Secondly, we use the orthonormal property of the Bessel function, which can be written in the form

$$\boxed{\int_0^a J_\nu(x_{\nu n'}\frac{s}{a})J_\nu(x_{\nu n}\frac{s}{a})sds=\frac{a^2}{2}J_{\nu+1}^2(x_{\nu n})\delta_{nn'}}.$$

Then

$$A_{mn}=\frac{2}{a^2\pi J_{m+1}^2(x_{mn})\sinh(k_{mn}L)}\int_0^{2\pi}d\varphi\int_0^a sJ_m(k_{mn}s)V(s,\phi)\sin\left(m\phi\right)ds,$$

$$B_{mn}=\frac{2}{a^2\pi J_{m+1}^2(x_{mn})\sinh(k_{mn}L)}\int_0^{2\pi}d\varphi\int_0^a sJ_m(k_{mn}s)V(s,\phi)\cos\left(m\phi\right)ds$$

which completes the solution.

For the special but important case of azimuthal symmetry, for which V is independent of ϕ, i.e. $m=0$, we obtain:

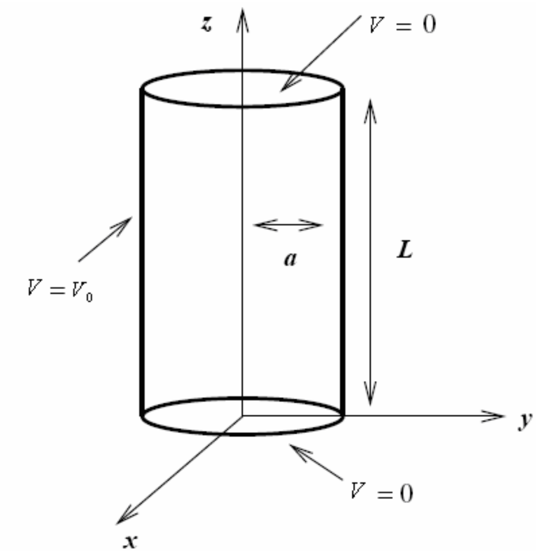
$$A_{mn}=0$$

$$B_{mn}=\frac{4\delta_{m,0}}{a^2J_1^2(x_{0n})\sinh(k_{0n}L)}\int_0^a sJ_0(k_{0n}s)V(s)ds$$

The reason we obtained discrete values for k was the demand that ϕ vanish at $s=a$. If we let $a\to\infty$, then k will be a continuous variable, and instead of a sum over k, we will obtain an integral. This is completely analogous to the transition **from a Fourier series to a Fourier transform, but we will not pursue it further.**

Example: A hollow right circular cylinder of radius "a" has its axis coincident with the z axis and its ends at z = 0 and z = L. The potential on the end faces is zero, while the potential on the cylindrical surface is given as a constant V_0 .Using the appropriate separation of variables in polar coordinates;

(a) Write down the boundary condition (conditions).
(b) Use the physical principal to write down the general solution.
(c) Use the boundary conditions in (a) to simplify the general solution in the separate coordinates. Write your reasons for dropping any term or terms.
(d) Find a series solution for the potential anywhere inside the cylinder.



Solution:

(a) B. Cs.

$$V(s,\phi,0) = 0 \qquad (1a)$$
$$V(s,\phi,L) = 0 \qquad (1b)$$
$$V(a,\phi,z) = V_0 \qquad (2)$$
$$V(0,\phi,z) = \text{finite} \qquad (3)$$

i- The angle ϕ must be unique, namely, $\Phi(\phi+2\pi) = \Phi(\phi)$ and therefore the general solution of the equation $\frac{d^2\Phi}{d\phi^2} + \nu^2\Phi = 0$ will be:

$$\Phi(\phi) = A\cos(\nu\phi) + B\sin(\nu\phi)$$

with ν an integer.

ii- In our case, Z must vanish at $z=0$ and $z=L$, which means we have the equation $\frac{d^2Z}{dz^2} + k^2 Z = 0$ and the Z function is of the form:

$$Z(z) = C\sin(kz) + D\cos(kz),$$

iii- Due to the above items i and ii, $\Re$ must be the solution of the equation $\frac{d^2\Re}{ds^2} + \frac{1}{s}\frac{d\Re}{ds} + \left(k^2 - \frac{m^2}{s^2}\right)\Re = 0$ and taken to be of the form:

$$\Re(s) = EI_n(k_n s) + FK_n(k_n s)$$

(b) The general solution of the Laplace's equation for the problem in cylindrical coordinates consists of a sum (superposition) of terms of the form:

$V(s,\phi,z) = \Re(s)\Phi(\phi)Z(z)$

$$= \sum_{\nu=0}^{\infty}\left[E_\nu I_\nu(k\,s) + F_{n\nu}K_\nu(k\,s)\right]\left[A_\nu\sin(m\phi) + B_\nu\cos(m\phi)\right]\left[C_\nu\sin(k\,s) + D_\nu\cos(k\,s)\right]$$

For Z-direction:

B1- Boundary condition (1a) implies $D=0$.

B2- Boundary condition (1b) implies

$$Z(L) = C\sin(kL) = 0$$
$$\Rightarrow kL = n\pi$$
$$\Rightarrow k_n = \frac{n\pi}{L}, \qquad n = 1, 2, 3, \cdots$$

$n = 0$ gives trivial solution.
For Z-direction:
Since we're looking for the potential inside the cylinder and there is no charge at the origin, the solution must be finite (B. C. 3) as $s \to 0$, which requires F = 0. Then the potential expansion becomes

$$V(s,\phi,z) = \sum_{n=1}^{\infty}\sum_{\nu=0}^{\infty}\left[A_{n\nu}\sin(\nu\phi) + B_{n\nu}\cos(\nu\phi)\right]\sin(k_n z)\, I_\nu(k_n s) \tag{4}$$

at $s = a$

$$V(a,\phi,z) = V_0 = \sum_{n=1}^{\infty}\sum_{\nu=0}^{\infty}\left[A_{n\nu}\sin(\nu\phi) + B_{n\nu}\cos(\nu\phi)\right]\sin(k_n z)\, I_\nu(k_n a) \tag{5}$$

Multiplying (5) by $\sin(k_{n'} z)$ and integrate, we find:

$$\underbrace{\int_0^L V_0 \sin(k_{n'} z)\,dz}_{-V_0\frac{\cos(k_{n'} z)}{k_{n'}}} = \sum_{n=1}^{\infty}\sum_{\nu=0}^{\infty} I_\nu(k_n a)\left[A_{n\nu}\sin(\nu\phi) + B_{n\nu}\cos(\nu\phi)\right]\underbrace{\int_0^L \sin(k_n z)\sin(k_{n'} z)\,dz}_{\frac{L}{2}\delta_{nn'}}$$

So:

$$-\frac{LV_0}{n'\pi}\left[\underbrace{\cos(n'\pi)}_{(-1)^{n'}} - 1\right] = \frac{L}{2}\sum_{\nu=0}^{\infty} I_\nu(k_{n'} a)\left[A_{n'\nu}\sin(\nu\phi) + B_{n'\nu}\cos(\nu\phi)\right] \tag{6}$$

Relabeling $n = n'$, then for n odd,

$$\frac{2LV_0}{n\pi} = \frac{L}{2}\sum_{\nu=0}^{\infty} I_\nu(k_n a)\left[A_{n\nu}\sin(\nu\phi) + B_{n\nu}\cos(\nu\phi)\right] \tag{7}$$

Equation (7) implies $\nu = 0$ (No terms contain $\sin(\nu\phi)$ or $\cos(\nu\phi)$ in the left hand side), then $A_{n\nu} = 0$,

$$B_{n0} = \frac{4V_0}{n\pi}\frac{1}{I_0(k_n a)} \tag{8}$$

And finally,

$$\boxed{V(s,\phi,z) = \frac{4V_0}{\pi}\sum_{n=1,3,5,\cdots}^{\infty}\frac{1}{n}\frac{\sin(k_n z)\, I_0(k_n s)}{I_0(k_n a)}}$$

Appendix (Repeated)
Helmholtz's equation in cylindrical coordinates

In this appendix we will disuses the general solution of Helmholtz's equation $\nabla^2\psi + K^2\psi = 0$, in cylindrical coordinates. Starting with the

$$\frac{1}{s}\left[\frac{\partial}{\partial s}\left(s\,\frac{\partial\psi}{\partial s}\right)+\frac{\partial}{\partial\phi}\left(\frac{1}{s}\frac{\partial\psi}{\partial\phi}\right)+\frac{\partial}{\partial z}\left(s\,\frac{\partial\psi}{\partial z}\right)\right]+k^2 s\psi = 0 \qquad (1)$$

Expand equation (1) and multiply by *s*, we have:

$$s\,\frac{\partial^2\psi}{\partial s^2}+\frac{\partial\psi}{\partial s}+\frac{1}{s}\frac{\partial^2\psi}{\partial\phi^2}+s\,\frac{\partial^2\psi}{\partial z^2}+K^2 s^2\psi = 0 \qquad (2)$$

Using the method of separation of variables,

$$\psi = \Re(s)\Phi(\phi)Z(z) \qquad (3)$$

eq. (2) will be reduced to, after dividing by $s\Re\Phi Z$,

$$\frac{1}{\Re}\frac{\partial^2\Re}{\partial s^2}+\frac{1}{s\Re}\frac{\partial\Re}{\partial s}+\frac{1}{s^2\Phi}\frac{\partial^2\Phi}{\partial\phi^2}+sK^2 = -\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} \qquad (4)$$

Since the left hand side of (4) is a function of s and ϕ and the right hand side is a function of Z only, we may write

$$-\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = \bar{\lambda}^2 \qquad (6)$$

with solution

$$Z(z) = A\cos\left(\bar{\lambda}z\right)+B\sin\left(\bar{\lambda}z\right) \qquad (6)$$

where $\bar{\lambda}$ is a constant and

$$\frac{s^2}{\Re}\frac{\partial^2\Re}{\partial s^2}+\frac{s}{\Re}\frac{\partial\Re}{\partial s}+s^2\left[K^2-\bar{\lambda}^2\right] = -\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} \qquad (7)$$

On separating the variables in (7), we obtain

$$-\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} = m^2 \qquad (8)$$

with solution

$$\Phi(\phi) = C\cos\left(m\phi\right)+D\sin\left(m\phi\right) \qquad (9)$$

At this stage, $\bar{\lambda}$ and m are unknown constants. However, we will be interested in problems in which the dependence on the angle is uniquely defined, $\Phi(\phi+2\pi)=\Phi(\phi)$ and therefore $m = n$, where *n* is an integer. Clearly we know the general solution for the differential equations of $Z(z)$ and $\Phi(\phi)$. What about the function $\Re$? The third equation then reads

$$s^2\frac{\partial^2\Re}{\partial s^2}+s\,\frac{\partial\Re}{\partial s}+\left\{s^2\left[K^2-\bar{\lambda}^2\right]-m^2\right\}\Re = 0 \qquad (10)$$

For $K^2-\bar{\lambda}^2=\alpha^2$, where α^2 = constant , equation (10) becomes

$$s^2 \frac{\partial^2 \Re}{\partial s^2} + s \frac{\partial \Re}{\partial s} + \left\{ s^2 \alpha^2 - m^2 \right\} P = 0 \tag{10}$$

We now make the following change of the dimensionless variable ξ:

$$\xi = \alpha s$$

So that

$$\frac{d}{ds} = \frac{d}{d\xi} \frac{d\xi}{ds} = \alpha \frac{d}{d\xi},$$

$$\frac{d^2}{ds^2} = \alpha^2 \frac{d^2}{d\xi^2}$$

By use of this change of variable, eq. (10) reduces to:

$$\frac{\partial^2 \Re}{\partial \xi^2} + \frac{1}{\xi} \frac{\partial \Re}{\partial \xi} + \left\{ 1 - \frac{m^2}{\xi^2} \right\} \Re = 0 \tag{11}$$

Which is the Bessel's equation, and its solutions are called **Bessel** (1784-1846) **functions** (or **cylindrical functions**)