Clebsch-Gordan coefficients (3j symbols)

The relation between the coupling and uncouple representation is given by the Clebsch-Gordan coefficients (3j symbols)

$$\left| j_1 j_2 jm \right\rangle = (-1)^{j_2-j_1-m}\sqrt{2j+1}\sum_{m_1,m_2}\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix}\left| j_1 j_2 m_1 m_2 \right\rangle,$$

where,

$$m = m_1 + m_2,$$
$$j = j_1 + j_2,\ j_1 + j_2 - 1, \cdots, \left| j_1 - j_2 \right|,$$
$$m = j,\ j-1, \cdots, -j.$$

with

$$j_1, j_2, j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \cdots,$$

$$\Delta(j_1 j_2 j): \begin{cases} j_1 + j_2 + j = n & \text{(an integer)} \\ j_1 + j_2 - j & \geq 0 \\ j_1 - j_2 + j & \geq 0 \\ -j_1 + j_2 + j & \geq 0 \end{cases},$$

$$m_1 = j_1, j_1 - 1, \cdots, -j_1,$$
$$m_2 = j_2, j_2 - 1, \cdots, -j_2,$$
$$m = j, j-1, \cdots, -j.$$

Properties:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix} = \begin{pmatrix} j_3 & j_1 & j_2 \\ m_3 & m_1 & m_2 \end{pmatrix}$$
$$= \begin{pmatrix} j_2 & j_3 & j_1 \\ m_3 & m_1 & m_2 \end{pmatrix} = \begin{pmatrix} j_3 & j_1 & j_2 \\ m_2 & m_3 & m_1 \end{pmatrix}$$

$$(-1)^{j_1+j_2+j_3}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix}$$
$$= \begin{pmatrix} j_3 & j_2 & j_1 \\ m_3 & m_2 & m_1 \end{pmatrix} = \begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix} = \begin{pmatrix} j_1 & j_3 & j_2 \\ m_1 & m_3 & m_2 \end{pmatrix}$$

Special formula for 3j Symbols

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} = 0 \quad if \quad j_1 + j_2 + j_3 \quad \text{is odd}$$

$$\begin{pmatrix} j & j & 1 \\ m & -m & 0 \end{pmatrix} = (-1)^{j-m}\frac{m}{\sqrt{j(j+1)(2j+1)}}$$

$$\begin{pmatrix} j & j & 0 \\ m & -m & 0 \end{pmatrix} = (-1)^{j-m}\frac{1}{\sqrt{2j+1}}$$

Note that:

$$j_z \left| j_1 j_2 jm \right\rangle = (j_{1z} + j_{2z})\left| j_1 j_2 m_1 m_2 \right\rangle = (m_1 + m_2)\left| j_1 j_2 m_1 m_2 \right\rangle$$
$$= m\left| j_1 j_2 m_1 m_2 \right\rangle$$

Numerical Values of 3j Symbols.[a,b]

j_1	j_2	j_3	m_1	m_2	m_3		j_1	j_2	j_3	m_1	m_2	m_3	
1	1	0	0	0	0	*$0\underline{1}$	3/2	3/2	1	1/2	-3/2	1	*$\underline{1}0\underline{1}$
2	1	1	0	0	0	$1\underline{11}$	3/2	3/2	1	1/2	-1/2	0	*$\underline{211}$
2	2	0	0	0	0	$00\underline{1}$	3/2	3/2	1	3/2	-3/2	0	$\underline{2}1\underline{1}$
2	2	2	0	0	0	*$10\underline{11}$,	3/2	3/2	1	3/2	-1/2	-1	*$\underline{1}0\underline{1}$
3	2	1	0	0	0	*$01\underline{11}$,	2	1	1	-1	0	1	*$\underline{1}0\underline{1}$
3	3	0	0	0	0	*$000\underline{1}$,	2	1	1	0	-1	1	$\underline{111}$
3	3	2	0	0	0	$2\underline{111}$,	2	1	1	0	0	0	$1\underline{11}$
4	2	2	0	0	0	$10\underline{11}$,	2	1	1	1	-1	0	*$\underline{1}0\underline{1}$
4	3	1	0	0	0	$2\underline{2}0\underline{1}$,	2	1	1	1	0	-1	*$\underline{1}0\underline{1}$
4	3	3	0	0	0	*$100\underline{1},\underline{1}$	2	1	1	2	-1	-1	$00\underline{1}$
4	4	0	0	0	0	$0\underline{2}$	2	3/2	1/2	0	-1/2	1/2	*$\underline{1}0\underline{1}$
4	4	2	0	0	0	*$2\underline{2}1\underline{1},\underline{1}$	2	3/2	1/2	1	-3/2	1/2	$\underline{2}0\underline{1}$
4	4	4	0	0	0	$120\underline{1},\underline{11}$	2	3/2	1/2	1	-1/2	-1/2	$\underline{2}1\underline{1}$
1/2	1/2	0	1/2	-1/2	0	$\underline{1}$	2	3/2	1/2	2	-3/2	-1/2	*$00\underline{1}$
1	1/2	1/2	0	-1/2	1/2	$\underline{11}$	2	3/2	3/2	-1	-1/2	3/2	$\underline{1}0\underline{1}$
1	1/2	1/2	1	-1/2	-1/2	*$0\underline{1}$	2	3/2	3/2	0	-3/2	3/2	*$\underline{2}0\underline{1}$
1	1	0	0	0	0	*$0\underline{1}$	2	3/2	3/2	0	-1/2	1/2	*$\underline{2}0\underline{1}$
1	1	0	1	-1	0	$0\underline{1}$	2	3/2	3/2	1	-3/2	1/2	$\underline{1}0\underline{1}$
1	1	1	-1	0	1	$\underline{11}$	2	3/2	3/2	1	-1/2	-1/2	0
1	1	1	0	-1	1	*$\underline{11}$	2	3/2	3/2	2	-3/2	-1/2	*$\underline{1}0\underline{1}$
1	1	1	0	0	0	0	2	3/2	3/2	2	-1/2	-3/2	$\underline{1}0\underline{1}$
1	1	1	1	-1	0	$\underline{11}$	2	2	0	0	0	0	$00\underline{1}$
1	1	1	1	0	-1	*$\underline{11}$	2	2	0	1	-1	0	*$00\underline{1}$
3/2	1	1/2	-1/2	0	1/2	*$\underline{11}$	2	2	0	2	-2	0	$00\underline{1}$
3/2	1	1/2	1/2	-1	1/2	$\underline{21}$	2	2	1	-1	0	1	*$\underline{1}0\underline{1}$
3/2	1	1/2	1/2	0	-1/2	$\underline{11}$	2	2	1	0	-1	1	$\underline{1}0\underline{1}$
3/2	1	1/2	3/2	-1	-1/2	*$\underline{2}$	2	2	1	0	0	0	0
3/2	3/2	0	1/2	-1/2	0	*$\underline{2}$	2	2	1	1	-2	1	*$0\underline{11}$
3/2	3/2	0	3/2	-3/2	0	$\underline{2}$	2	2	1	1	-1	0	*$\underline{111}$
3/2	3/2	1	-1/2	-1/2	1	$1\underline{11}$	2	2	1	1	0	-1	$\underline{1}0\underline{1}$

[a] The table gives values of $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}^2$ in prime notation which lists only the exponents of the prime numbers in the order 2, 3, 5, 7, 11, ... Negative exponents are underscored. For example,

$$210\underline{2} = \frac{2^2 \times 3^1 \times 5^0}{7^2} = \frac{12}{49}.$$

An asterisk indicates a negative radical. To look up the value of a 3j symbol use the symmetry properties

TABLE 1.6 (continued)

j_1	j_2	j_3	m_1	m_2	m_3		j_1	j_2	j_3	m_1	m_2	m_3	
2	2	1	2	-2	0	111	5/2	2	3/2	1/2	-1	1/2	*2111,
2	2	1	2	-1	-1	*011	5/2	2	3/2	1/2	0	-1/2	1011,
2	2	2	-2	0	2	1011,	5/2	2	3/2	3/2	-2	1/2	3111,
2	2	2	-1	-1	2	*0111,	5/2	2	3/2	3/2	-1	-1/2	1111,
2	2	2	-1	0	1	1011,	5/2	2	3/2	3/2	0	-3/2	*0111,
2	2	2	0	-2	2	1011,	5/2	2	3/2	5/2	-2	-1/2	*1101,
2	2	2	0	-1	1	1011,	5/2	2	3/2	5/2	-1	-3/2	1001,
2	2	2	0	0	0	*1011,	5/2	5/2	0	1/2	-1/2	0	11
2	2	2	1	-2	1	*0111,	5/2	5/2	0	3/2	-3/2	0	*11
2	2	2	1	-1	0	1011,	5/2	5/2	0	5/2	-5/2	0	11
2	2	2	1	0	-1	1011,	5/2	5/2	1	-1/2	-1/2	1	*0111,
2	2	2	2	-2	0	1011	5/2	5/2	1	1/2	-3/2	1	3111,
2	2	2	2	-1	-1	*0111	5/2	5/2	1	1/2	-1/2	0	1111,
2	2	2	2	0	-2	1011	5/2	5/2	1	3/2	-5/2	1	*0101,
5/2	3/2	1	-1/2	-1/2	1	*201	5/2	5/2	1	3/2	-3/2	0	*1111,
5/2	3/2	1	1/2	-3/2	1	211	5/2	5/2	1	3/2	-1/2	-1	3111,
5/2	3/2	1	1/2	-1/2	0	101	5/2	5/2	1	5/2	-5/2	0	1111,
5/2	3/2	1	3/2	-3/2	0	*011	5/2	5/2	1	5/2	-3/2	-1	*0101,
5/2	3/2	1	3/2	-1/2	-1	*101	5/2	5/2	2	-3/2	-1/2	2	2211,
5/2	3/2	1	5/2	-3/2	-1	11	5/2	5/2	2	-1/2	-3/2	2	*2211,
5/2	2	1/2	-1/2	0	1/2	101	5/2	5/2	2	-1/2	-1/2	1	0
5/2	2	1/2	1/2	-1	1/2	*011	5/2	5/2	2	1/2	-5/2	2	2001,
5/2	2	1/2	1/2	0	-1/2	*101	5/2	5/2	2	1/2	-3/2	1	0011,
5/2	2	1/2	3/2	-2	1/2	111	5/2	5/2	2	1/2	-1/2	0	*2111,
5/2	2	1/2	3/2	-1	-1/2	111	5/2	5/2	2	3/2	-5/2	1	*1001,
5/2	2	1/2	5/2	-2	-1/2	*11	5/2	5/2	2	3/2	-3/2	0	2111,
5/2	2	3/2	-3/2	0	3/2	*0111,	5/2	5/2	2	3/2	-1/2	-1	0011,
5/2	2	3/2	-1/2	-1	3/2	2211,	5/2	5/2	2	5/2	-5/2	0	2111,
5/2	2	3/2	-1/2	0	1/2	1011,	5/2	5/2	2	5/2	-3/2	-1	*1001,
5/2	2	3/2	1/2	-2	3/2	*0011,	5/2	5/2	2	5/2	-1/2	-2	2001,

(1.5-48) and (1.5-49) to (a) interchange the columns so that $j_1 \geqslant j_2 \geqslant j_3$ and (b) change the signs (if necessary) of m_1, m_2, and m_3 so that $m_2 \leqslant 0$.

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Examples:

$$\begin{pmatrix} 1/2 & 1/2 & 1 \\ 1/2 & 1/2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1/2 & 1/2 \\ -1 & 1/2 & 1/2 \end{pmatrix} = -\begin{pmatrix} 1 & 1/2 & 1/2 \\ 1 & -1/2 & -1/2 \end{pmatrix}$$

$$= (*0\underline{1}) = (-\sqrt{\frac{2^0}{3^1}}) = -\sqrt{\frac{1}{3}}$$

$$\begin{pmatrix} 1/2 & 1/2 & 1 \\ 1/2 & -1/2 & 0 \end{pmatrix} = (\underline{11}) = (\frac{1}{2^1 3^1})$$

$$\begin{pmatrix} 1/2 & 1/2 & 1 \\ -1/2 & 1/2 & 0 \end{pmatrix} = (\underline{11}) = (\frac{1}{2^1 3^1})$$

1- $$\begin{pmatrix} \frac{1}{2} & 1 & \frac{3}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{pmatrix} = -\begin{pmatrix} \frac{3}{2} & 1 & \frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix} = \sqrt{(\underline{21})} = \sqrt{\frac{1}{2^2 3}} = \frac{1}{2\sqrt{3}}$$

2- $$\begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & -1 \end{pmatrix} = -\begin{pmatrix} 2 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 0 & -1 \end{pmatrix} = (\underline{1}0\underline{1}) = \left(\sqrt{\frac{3^0}{2\times 5}}\right) = \sqrt{\frac{1}{10}}$$

3- $$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} = (*\underline{1}0\underline{1}) = \left(-\sqrt{\frac{3^0}{2\times 5}}\right) = -\sqrt{\frac{1}{10}}$$

4- $$\begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = (-1)^{1+\frac{1}{2}+\frac{3}{2}} \begin{pmatrix} \frac{3}{2} & 1 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} = -(*\underline{11}) = -(-\sqrt{\frac{1}{2\times 3}}) = \sqrt{\frac{1}{6}}$$

5- $$\begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} = -\begin{pmatrix} \frac{3}{2} & 1 & \frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix} = \sqrt{(\underline{21})} = \sqrt{\frac{1}{2^2 3}} = \frac{1}{2\sqrt{3}}$$

(* Clebsh Gordan Coefficient *)

In[47]:= $j1 = \frac{1}{2};\ m1 = \frac{1}{2};\ j2 = \frac{1}{2};\ m2 = \frac{1}{2};\ j3 = 1;\ m3 = 1;$

In[48]:= $\frac{(-1)^{j1+j2+m3}}{\sqrt{2\,j3+1}}$ ClebschGordan[{j1, m1}, {j2, m2}, {j3, m3}]

Out[48]= $\frac{1}{\sqrt{3}}$

In[33]:= $j1 = 1;\ m1 = -1;\ j2 = \frac{1}{2};\ m2 = \frac{1}{2};\ j3 = \frac{1}{2};\ m3 = -\frac{1}{2};$

In[35]:= $\frac{(-1)^{j1+j2+m3}}{\sqrt{2\,j3+1}}$ ClebschGordan[{j1, m1}, {j2, m2}, {j3, m3}]

Out[35]= $\frac{1}{\sqrt{3}}$

In[25]:= $j1 = 1;\ m1 = 0;\ j2 = 1;\ m2 = 1;\ j3 = 2;\ m3 = 1;$

In[26]:= $\frac{(-1)^{j1+j2+m3}}{\sqrt{2\,j3+1}}$ ClebschGordan[{j1, m1}, {j2, m2}, {j3, m3}]

Out[26]= $-\frac{1}{\sqrt{10}}$

Example 1: Calculate CGC transformation matrix for coupling of two spins of non equivalent electrons in s-state.
Answer: use the expression

$$\left|JM\right\rangle_c = \sum_{m_1,m_2} (-1)^M \sqrt{2j+1}\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -M \end{pmatrix}\left|m_1 m_2\right\rangle$$

In this case we will have four degenerate states:

$$\chi_S = \left\{\begin{matrix} |11\rangle & = & |\alpha\rangle_1|\alpha\rangle_2 \\ |10\rangle & = & \frac{1}{\sqrt{2}}\left[|\beta\rangle_1|\alpha\rangle_2 + |\alpha\rangle_1|\beta\rangle_2\right] \\ |1-1\rangle & = & |\beta\rangle_1|\beta\rangle_2 \end{matrix}\right\} \text{ triplet states (Symmetric, Ortho or Even)}$$

$$\chi_A = \ |00\rangle = \frac{1}{\sqrt{2}}\left[|\beta\rangle_1|\alpha\rangle_2 - |\alpha\rangle_1|\beta\rangle_2\right] \quad \text{singlet states (Antisymmetric, Para or Odd)}$$

i- Start with the maximum M, i.e. $\left|\mathrm{S},\mathrm{M_S}\right\rangle = \left|1,1\right\rangle$. We have to remember that: $M = m_1 + m_2$, and the only possibility is: $m_1 = \frac{1}{2}, m_1 = \frac{1}{2}$.

Then

$$\left|1,1\right\rangle_c = (-)^1\sqrt{2\times1+1}\begin{pmatrix} 1/2 & 1/2 & 1 \\ 1/2 & 1/2 & -1 \end{pmatrix}\left|\alpha\alpha\right\rangle = -\sqrt{3}\begin{pmatrix} 1/2 & 1/2 & 1 \\ 1/2 & 1/2 & -1 \end{pmatrix}\left|\alpha\alpha\right\rangle$$

$$= -\sqrt{3}\begin{pmatrix} 1 & 1/2 & 1/2 \\ 1 & -1/2 & -1/2 \end{pmatrix}\left|\alpha\alpha\right\rangle = -\sqrt{3}(*\sqrt{0\underline{1}})\left|\alpha\alpha\right\rangle = -\sqrt{3}(-\sqrt{\frac{2^0}{3^1}})\left|\alpha\alpha\right\rangle = \left|\alpha\alpha\right\rangle$$

ii- Now, we have to go to the next one, which is $\left|\mathrm{S},\mathrm{M_S}\right\rangle = \left|1,0\right\rangle$. Note that $\mathrm{M_S} = 0$ will be the resultant of two terms since: $\left|m_{s_1} m_{s_2}\right\rangle = \left|\alpha\beta\right\rangle$ **and** $\left|\beta\alpha\right\rangle$

$$\left|1,0\right\rangle_c = (-)^0\sqrt{2\times1+1}\begin{pmatrix} 1/2 & 1/2 & 1 \\ 1/2 & -1/2 & 0 \end{pmatrix}\left|\alpha\beta\right\rangle + (-)^0\sqrt{2\times1+1}\begin{pmatrix} 1/2 & 1/2 & 1 \\ -1/2 & 1/2 & 0 \end{pmatrix}\left|\beta\alpha\right\rangle$$

$$= \sqrt{3}\begin{pmatrix} 1/2 & 1/2 & 1 \\ 1/2 & -1/2 & 0 \end{pmatrix}\left|\alpha\beta\right\rangle + \sqrt{3}\begin{pmatrix} 1/2 & 1/2 & 1 \\ -1/2 & 1/2 & 0 \end{pmatrix}\left|\beta\alpha\right\rangle$$

$$= \sqrt{3}(\sqrt{\underline{11}})\left|\alpha\beta\right\rangle + \sqrt{3}(\sqrt{\underline{11}})\left|\beta\alpha\right\rangle = \sqrt{3}(\sqrt{\frac{1}{2^1 3^1}})\left|\alpha\beta\right\rangle + \sqrt{3}(\sqrt{\frac{1}{2^1 3^1}})\left|\beta\alpha\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\alpha\beta\right\rangle + \left|\beta\alpha\right\rangle\right)$$

iii- **H.W.** Do the rest and check the following table

Table CGC for $s_1 = s_2 = \frac{1}{2}$

$\left\|S,M_S\right\rangle$	$\left\|m_{s_1} m_{s_2}\right\rangle$			
	$\left\|\frac{1}{2},\frac{1}{2}\right\rangle$	$\left\|\frac{1}{2},-\frac{1}{2}\right\rangle$	$\left\|-\frac{1}{2},\frac{1}{2}\right\rangle$	$\left\|-\frac{1}{2},-\frac{1}{2}\right\rangle$
$\left\|1,1\right\rangle$	1	0	0	0
$\left\|1,0\right\rangle$	0	$\sqrt{1/2}$	$\sqrt{1/2}$	0
$\left\|0,0\right\rangle$	0	$-\sqrt{1/2}$	$\sqrt{1/2}$	0
$\left\|1,-1\right\rangle$	0	0	0	1

Question: Can we have the inverse transformation? Answer yes:
Look at the above table, it is easy to proof that:

$$|11\rangle \quad = |\alpha\rangle_1 |\alpha\rangle_2 , \quad |1-1\rangle = |\beta\rangle_1 |\beta\rangle$$

And from:

$$|10\rangle = \frac{1}{\sqrt{2}}\left[|\beta\rangle_1 |\alpha\rangle_2 + |\alpha\rangle_1 |\beta\rangle_2\right], \quad |00\rangle = \frac{1}{\sqrt{2}}\left[|\beta\rangle_1 |\alpha\rangle_2 - |\alpha\rangle_1 |\beta\rangle_2\right]$$

We have

$$\Rightarrow |\beta\rangle_1 |\alpha\rangle_2 = \frac{1}{\sqrt{2}}\left[|10\rangle + |00\rangle\right], \quad |\alpha\rangle_1 |\beta\rangle = \frac{1}{\sqrt{2}}\left[|10\rangle - |00\rangle\right]$$

Example 2: Calculate CGC transformation matrix for coupling of two non equivalent electrons in p-state.

Answer:

$$|LM_L\rangle_c = \sum_{m_1 m_2} (-1)^{M_L} (2L+1)^{1/2} \begin{pmatrix} 1 & 1 & L \\ m_1 & m_2 & -M_L \end{pmatrix} |m_1 m_2\rangle .$$

Using values of the 3-j symbols from Appendix C we find

$$|2\ 2\rangle_c = (5)^{1/2} \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & -2 \end{pmatrix} |1\ 1\rangle = |1\ 1\rangle ,$$

$$|2\ 1\rangle_c = -(5)^{1/2} \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \end{pmatrix} |1\ 0\rangle - (5)^{1/2} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \end{pmatrix} |0\ 1\rangle$$

$$= (2)^{-1/2} |1\ 0\rangle + (2)^{-1/2} |0\ 1\rangle ,$$

$$|2\ 0\rangle_c = (5)^{1/2} \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix} |1\ -1\rangle + (5)^{1/2} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} |0\ 0\rangle$$

$$+ (5)^{1/2} \begin{pmatrix} 1 & 1 & 2 \\ -1 & 1 & 0 \end{pmatrix} |-1\ 1\rangle$$

$$= (6)^{-1/2} |1\ -1\rangle + (2/3)^{1/2} |0\ 0\rangle + (6)^{-1/2} |-1\ 1\rangle ,$$

etc. The complete C-G transformation matrix is given in Table 9-1. This matrix not only gives the transformation from the uncoupled to the coupled set of functions; since the matrix is orthogonal, its transpose is the transformation matrix from the coupled set back to the uncoupled set. We therefore have the equality

$$C(j_1 j_2 m_1 m_2; jm) = \langle j_1 j_2 m_1 m_2 | j_1 j_2 jm\rangle = \langle j_1 j_2 jm | j_1 j_2 m_1 m_2\rangle , \tag{9.12}$$

and the inverse transformation may be written

TABLE 9-1. THE CLEBSCH-GORDON TRANSFORMATION MATRIX FOR THE ORBITAL-ANGULAR-MOMENTUM COUPLING OF TWO NON-EQUIVALENT p ELECTRONS

LM_L \ $m_1 m_2$	1 1	1 0	0 1	1 −1	0 0	−1 1	−1 0	0 −1	−1 −1
2 2	1	0	0	0	0	0	0	0	0
2 1	0	$1/\sqrt{2}$	$1/\sqrt{2}$	0	0	0	0	0	0
1 1	0	$-1/\sqrt{2}$	$1/\sqrt{2}$	0	0	0	0	0	0
2 0	0	0	0	$1/\sqrt{6}$	$\sqrt{2/3}$	$1/\sqrt{6}$	0	0	0
1 0	0	0	0	$1/\sqrt{2}$	0	$-1/\sqrt{2}$	0	0	0
0 0	0	0	0	$1/\sqrt{3}$	$-1/\sqrt{3}$	$1/\sqrt{3}$	0	0	0
1 −1	0	0	0	0	0	0	$-1/\sqrt{2}$	$1/\sqrt{2}$	0
2 −1	0	0	0	0	0	0	$1/\sqrt{2}$	$1/\sqrt{2}$	0
2 −2	0	0	0	0	0	0	0	0	1

$$|j_1 j_2 m_1 m_2\rangle = \sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m=-j}^{j} \langle j_1 j_2 jm | j_1 j_2 m_1 m_2\rangle |j_1 j_2 jm\rangle$$

$$= \sum_j \sum_m C(j_1 j_2 m_1 m_2; jm) |j_1 j_2 jm\rangle . \tag{9.13}$$

Example: for p-state $L=1, \quad S=\frac{1}{2},$

$$\left|\frac{3}{2},\frac{1}{2}\right\rangle \Rightarrow j=\frac{3}{2}, \quad m_j=\frac{1}{2}, \quad L=1, \quad S=\frac{1}{2}, \quad m_L=\begin{cases}0 & m_S=\frac{1}{2}\equiv\alpha \\ 1 & m_S=-\frac{1}{2}\equiv\beta\end{cases}$$

$$\left|\frac{3}{2},\frac{1}{2}\right\rangle=(-1)^{1-\frac{1}{2}-\frac{1}{2}}\sqrt{2\left(\frac{3}{2}\right)+1}\left\{\begin{pmatrix}1 & \frac{1}{2} & \frac{3}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2}\end{pmatrix}Y_{1,0}\alpha+\begin{pmatrix}1 & \frac{1}{2} & \frac{3}{2} \\ 1 & -\frac{1}{2} & -\frac{1}{2}\end{pmatrix}Y_{1,1}\beta\right\}$$

$$=\sqrt{4}\left\{\frac{1}{\sqrt{2}\sqrt{3}}Y_{1,0}\alpha+\frac{1}{\sqrt{4}\sqrt{3}}Y_{1,1}\beta\right\}=\sqrt{\frac{2}{3}}Y_{1,0}\alpha+\sqrt{\frac{1}{3}}Y_{1,1}\beta$$

$|J, M_J\rangle$

m_{j_1}	m_{j_2}	$\left\vert\frac{3}{2},\frac{3}{2}\right\rangle$	$\left\vert\frac{3}{2},\frac{1}{2}\right\rangle$	$\left\vert\frac{1}{2},\frac{1}{2}\right\rangle$	$\left\vert\frac{3}{2},-\frac{1}{2}\right\rangle$	$\left\vert\frac{1}{2},-\frac{1}{2}\right\rangle$	$\left\vert\frac{3}{2},-\frac{3}{2}\right\rangle$
1	$\frac{1}{2}$	1	0	0	0	0	0
1	$-\frac{1}{2}$	0	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{2}{3}}$	0	0	0
0	$\frac{1}{2}$	0	$\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{1}{3}}$	0	0	0
0	$-\frac{1}{2}$	0	0	0	$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{1}{3}}$	0
-1	$\frac{1}{2}$	0	0	0	$\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{2}{3}}$	0
-1	$-\frac{1}{2}$	0	0	0	0	0	1

Table CGC for $j_1=1, j_2=\frac{1}{2}$

	$\vert m_{j_1} m_{j_2}\rangle$					
$\vert J, M_J\rangle$	$\left\vert 1,\frac{1}{2}\right\rangle$	$\left\vert 1,-\frac{1}{2}\right\rangle$	$\left\vert 0,\frac{1}{2}\right\rangle$	$\left\vert 0,-\frac{1}{2}\right\rangle$	$\left\vert -1,\frac{1}{2}\right\rangle$	$\left\vert -1,-\frac{1}{2}\right\rangle$
$\left\vert\frac{3}{2},\frac{3}{2}\right\rangle$	1	0	0	0	0	0
$\left\vert\frac{3}{2},\frac{1}{2}\right\rangle$	0	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{2}{3}}$	0	0	0
$\left\vert\frac{1}{2},\frac{1}{2}\right\rangle$	0	$\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{1}{3}}$	0	0	0
$\left\vert\frac{1}{2},-\frac{1}{2}\right\rangle$	0	0	0	$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{1}{3}}$	0
$\left\vert\frac{3}{2},-\frac{1}{2}\right\rangle$	0	0	0	$\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{2}{3}}$	0
$\left\vert\frac{3}{2},-\frac{3}{2}\right\rangle$	0	0	0	0	0	1

Let $\vec{J} = \vec{L} + \vec{S}$. Using the method described in the lecture, identify and calculate all non-zero coefficients for the $\ell = 2$, $s = 1/2$ case.
For starters, we need $j_{max} = \ell + s = 2 + \frac{1}{2} = \frac{5}{2}$ and $j_{min} = |2 - \frac{1}{2}| = \frac{3}{2}$
Starting from $|5/2, 5/2\rangle = |2, 1/2\rangle$, we apply $J_- = L_- + S_-$ to get

$$\begin{aligned}
J_-|5/2,5/2\rangle &= L_-|2,1/2\rangle + S_-|2,1/2\rangle \\
\sqrt{\frac{5}{2}\cdot\frac{7}{2} - \frac{5}{2}\cdot\frac{3}{2}}|5/2,3/2\rangle &= \sqrt{2\cdot 3 - 2\cdot 1}|1,1/2\rangle + |2,-1/2\rangle \\
\sqrt{5}|5/2,3/2\rangle &= 2|1,1/2\rangle + |2,-1/2\rangle \\
|5/2,3/2\rangle &= \frac{2}{\sqrt{5}}|1,1/2\rangle + \frac{1}{\sqrt{5}}|2,-1/2\rangle
\end{aligned}$$

Applying J_- again gives

$$\begin{aligned}
\sqrt{\frac{5}{2}\cdot\frac{7}{2} - \frac{3}{2}\cdot\frac{1}{2}}|5/2,1/2\rangle &= \frac{2}{\sqrt{5}}\left(\sqrt{2\cdot 3 - 1\cdot 0}|0,1/2\rangle + |1,-1/2\rangle\right) + \frac{1}{\sqrt{5}}\sqrt{2\cdot 3 - 2\cdot 1}|1,-1/2\rangle \\
\sqrt{8}|5/2,1/2\rangle &= \frac{2\sqrt{6}}{\sqrt{5}}|0,1/2\rangle + \frac{4}{\sqrt{5}}|1,-1/2\rangle \\
|5/2,1/2\rangle &= \frac{\sqrt{3}}{\sqrt{5}}|0,1/2\rangle + \frac{\sqrt{2}}{\sqrt{5}}|1,-1/2\rangle
\end{aligned}$$

After this point, the remaining terms can be found by symmetry, giving:

Table 1: coefficients: $\langle 2, 1/2, m_\ell, m_s | j, m_j\rangle$

m_ℓ, m_s

j, m_j		2,1/2	2,-1/2	1,1/2	1,-1/2	0,1/2	0,-1/2	-1,1/2	-1,-1/2	-2,1/2	-2,-1/2
	5/2,5/2	1	0	0	0	0	0	0	0	0	0
	5/2,3/2	0	$1/\sqrt{5}$	$2/\sqrt{5}$	0	0	0	0	0	0	0
	5/2,1/2	0	0	0	$\sqrt{2}/\sqrt{5}$	$\sqrt{3}/\sqrt{5}$	0	0	0	0	0
	5/2,-1/2	0	0	0	0	0	$\sqrt{3}/\sqrt{5}$	$\sqrt{2}/\sqrt{5}$	0	0	0
	5/2,-3/2	0	0	0	0	0	0	0	$2/\sqrt{5}$	$1/\sqrt{5}$	0
	5/2,-5/2	0	0	0	0	0	0	0	0	0	1
	3/2,3/2	0	$2/\sqrt{5}$	$-1\sqrt{5}$	0	0	0	0	0	0	0
	3/2,1/2	0	0	0	$\sqrt{3}\sqrt{5}$	$-\sqrt{2}/\sqrt{5}$	0	0	0	0	0
	3/2,-1/2	0	0	0	0	0	$\sqrt{2}/\sqrt{5}$	$-\sqrt{3}/\sqrt{5}$	0	0	0
	3/2,-3/2	0	0	0	0	0	0	0	$1/\sqrt{5}$	$-2/\sqrt{5}$	0