Real-space renormalization

Until the early 1970s all calculations of critical exponents were made either:

1- by exactly solving a model for its thermodynamic properties and then examining its behavior in the critical regime (as with the Ising model in one and two dimensions) or
2- by direct numerical simulations (as with the three-dimensional Ising model) or by extrapolation from approximate solutions which become invalid in the vicinity of a phase transition (high-temperature expansions, for example).

However, in a now-famous paper published in 1966, Kadanoff presented arguments which, he maintained, would allow one to simplify calculations in the critical regime to the point at which critical exponents could be extracted, without ever working out the partition function for the problem. However, though Kadanoff's ideas showed great physical intuition about the processes giving rise to critical phenomena, they lacked the mathematical precision which might have given one faith in his results. A more quantitative realization of the argument was given by Wilson and others, who introduced the so-called renormalization group techniques (Wilson and Kogut 1974). In their present state, these techniques divide roughly into two classes. There are those which were developed by pursuing the analogy between statistical mechanics and quantum field theory, which we will refer to as '**field-theoretical**' or '**k-space**' techniques, and then there are the real-space renormalization techniques, which are simpler and closer in spirit to the original ideas of Kadanoff. In the later chapters of this book we will discuss field-theoretical methods extensively; this chapter deals with the real-space methods. The term '**real-space**' in this context refers to the fact that these techniques involve quantities dependent on position coordinates in ordinary space. The field-theoretical techniques, conversely, are simplest when the equations are written in terms of spatially Fourier-transformed quantities, hence the name 'k-space' techniques.

Renormalizing the lattice

Real-space renormalization techniques are applicable only to:

☞ models based on a lattice. And more than this,
☞ the lattice must be regular in a very special kind of way; it must have a 'discrete scaling symmetry'.

To understand what this means consider taking a lattice and **blocking it**. This means dividing the sites of the lattice into groups or **blocks**, and then replacing each block by just one single site, which may be at the position occupied by one of the sites in that block, or at some other position within the area covered by the block. The lattice has a **discrete scaling symmetry** if we can block it in this way so as to produce a lattice exactly like the one we started with, except for an increase in the lattice parameter "a", $a \to a' \equiv ba$. The process of **renormalizing** the lattice is then completed by reducing all dimensions in the new lattice by a factor of b so that we end up with exactly the same lattice that we started with.

Actually, one thing does change when we renormalize our lattice. If we group sites into blocks containing p sites on average, then the renormalized lattice will contain fewer sites than the original by a factor of $p = \frac{N_{old}}{N_{new}}$. This will be important when we come to define site-averaged quantities, such as susceptibility, on our renormalized lattice. Since we have scaled our whole lattice down by a factor of b, its volume must have shrunk by a factor of b^d where d is the number of spatial dimensions. Clearly then, if the sites in the renormalized lattice are arranged in exactly the same way as those in the original lattice, their number must have been reduced by a factor of $p = b^d$.

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The most common lattice displaying a discrete scaling symmetry is the square lattice. Figure 1 illustrates the renormalization of the square lattice in two dimensions. In this case $p = 4$, $b = 2$ and the lattice is left with a quarter the number of sites it started with. Clearly this transformation may also be performed on a rectangular lattice. Actually this is not the only way to block a square lattice. Figure 2 illustrates an alternative method. In this case $b = \sqrt{2}$ and the number of sites is halved. Figure 3 shows how one would renormalize a triangular lattice. Here $b = \sqrt{3}$ and the renormalized lattice has a third as many sites as the original one.

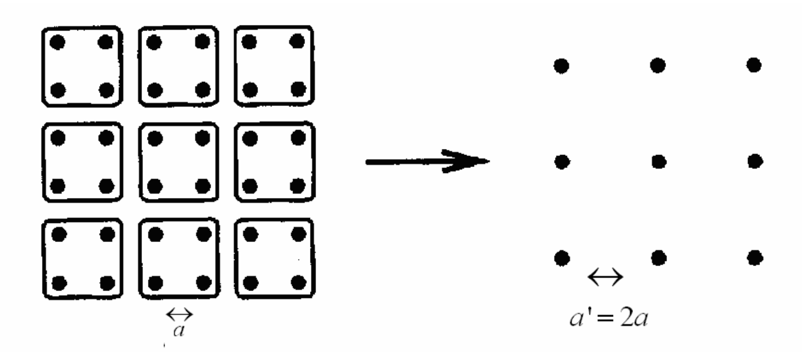


Figure 1 Renormalization of a square lattice. The linear dimensions of the lattice on the right must be shrunk by a factor of $b = 2$ to render it similar to the original one. The final lattice therefore has fewer sites than the original by a factor of $b^2 = 4$. $\left(p = b^d = b^2 = 4\right)$

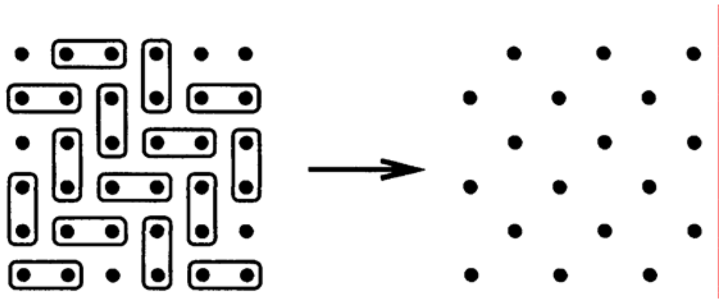


Figure 2 An alternative blocking scheme for renormalizing the square lattice. Here all length scales must be divided by a factor of $b = \sqrt{2}$. The final lattice will therefore have fewer sites than the original by a factor of $b^2 = 2$.

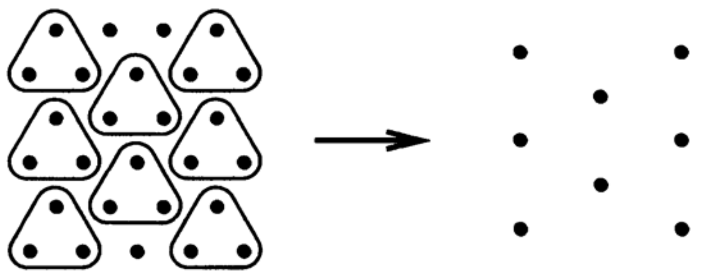
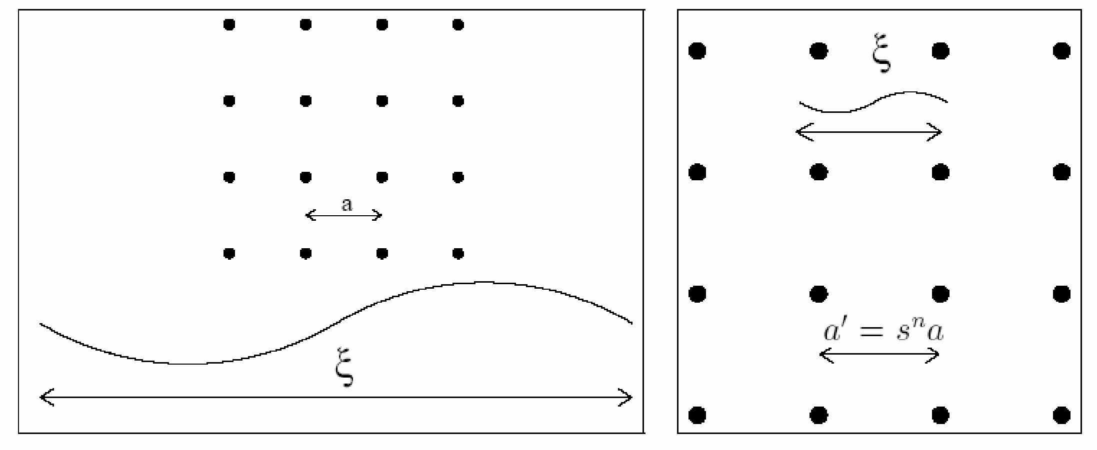


Figure 3 Renormalization of a triangular lattice. The linear dimensions of the lattice on the right must be shrunk by a factor of $b = \sqrt{3}$ to render it similar to the original one. The final lattice therefore has fewer sites than the original by a factor of $b^2 = 3$.



(a) **(b)**

Figure 4: Cartoon of the renormalization idea: Close to the phase transition, the correlation length ξ diverges indication the onset of off-diagonal long range order. The Hamiltonian and all coupling constants, however, are given on a length scale a/ξ The Renormalization group transformation maps the Hamiltonian for the situation (a) onto a Hamiltonian for which $a' \geq \xi$. This is done by consecutive integrating out modes of short wave length and enlarging the effective lattice size to $a' = s^n a$. Then, all physical quantities are calculated perturbatively and there corresponding values with respect to the original model are obtained by using the scaling laws close to the fixed-point.

In summary: What is the renormalization group? The renormalization group consists of analytic and computational schemes to integrate systematically over degrees of freedom in a system near a critical point. After integration, the control parameters, for example temperature and magnetic field in a magnetic system, are rescaled to restore the system Hamiltonian to its original form. The behavior of the control parameters under this rescaling enables calculation of the critical behavior of the model.

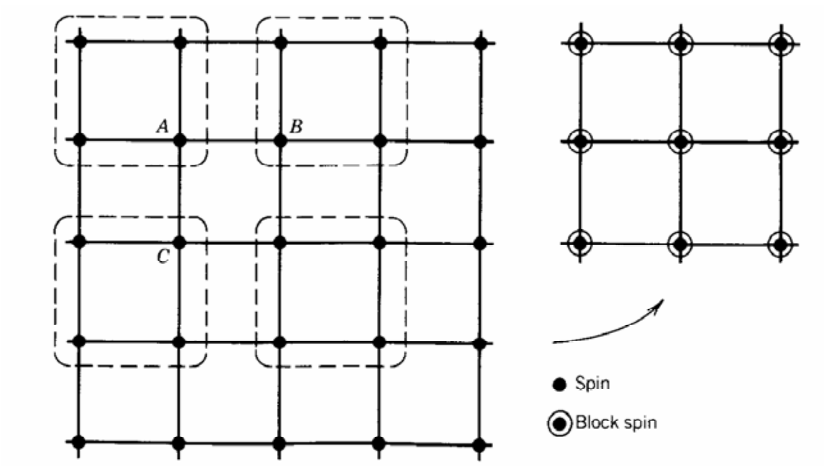


Figure: Block-spin transformation: averaging the spins in a block, and then rescaling the lattice to the original size. In more than one dimension, the indirect interaction between *B* and *C* gives rise to next-to-nearest-neighbor interactions of the block spins.

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Quite generally, one can expect the following advantages from such a renormalization group transformation:

i- The new coupling constants could be smaller. By repeated applications of the renormalization procedure, one could thus finally obtain a practically free theory, without interactions.

ii- The successively iterated coupling coefficients, also called "parameter flow", could have a *fixed point*, at which the system no longer changes under additional renormalization group transformations. Since the elimination of degrees of freedom is accompanied by a change of the underlying lattice spacing, or length scale, one can anticipate that the fixed points are under certain circumstances related to critical points. Furthermore, it can be hoped that the flow in the vicinity of these fixed points can yield information about the universal physical quantities in the neighborhood of the critical points.

The scenario described under (i) will in fact be found for the one-dimensional Ising model, and that described under (ii) for the two-dimensional Ising model.

Fixed Point (FP):- A point that remains unchanged under application of recursion relations is a fixed point. A fixed point is stable if nearby points flow towards it and unstable if nearby points flow away from it. Usually we use a set of recursion relations, such as those used in renormalization group calculations, lead to flows (changes in the variables upon successive iterations) in their parameter space. In other words, *it is a point at which the system becomes invariant under a change of length scale. That means the correlation length is either 0 or* ∞. *The latter corresponds to a critical point, which is physically interesting case. The case with zero correlation length (Trivial or Gaussian), as we have encountered in the one-dimensional Ising model, corresponding to infinite temperature, and usually can be recognized and rejected.*

Renormalization Group Theory

The goal of this section is to introduce several concepts of *Renormalization Group Theory* and to illustrate such concepts with the 1-dimensional Ising model.
Consider the task of computing the canonical partition function Z of the one-dimensional Ising model in the absence of an external magnetic field in the following:

$$Z(K,N) = \sum_{S_1}\sum_{S_2}\cdots\sum_{S_N} e^{K(S_1S_2+S_2S_3+\cdots+S_NS_1)} = \sum_{S_1}\sum_{S_2}\cdots\sum_{S_N} e^{KS_1S_2}e^{KS_2S_3}e^{\cdots}e^{KS_NS_1} \quad (317)$$

where *coupling parameter* $K = \beta J$ and N is the total number of spins. Note that according to Eq. (317),

$$\lim_{K\to 0} Z(K,N) = \prod_{j=1}^{N}\sum_{S_j=-1}^{1} 1 = 2^N. \quad (318)$$

The renormalization group strategy for the 1-dimensional Ising model can be described as follows:

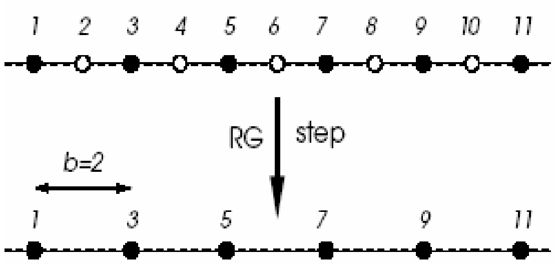

Figure: **Decimation scheme**: Every second spin **is integrated out** to reach a renormalized new system with only half the spins of the previous system.

Step (1): Sum over the even numbered spins in Eq. (317). Note that summing, e.g., over S_2 we obtain

$$Z(K,N) = \sum_{S_1,S_3,S_4,S_5,S_6,S_7\ldots} [e^{K(S_1+S_3)} + e^{-K(S_1+S_3)}]e^{KS_3S_4}e^{KS_4S_5}e^{KS_5S_6}e^{KS_6S_7}\ldots, \quad (319)$$

Summing over S_2 gives: $\sum_{S_2} e^{K(S_1S_2+S_2S_3)} = e^{K(S_1+S_3)} + e^{-K(S_1+S_3)}$

and for S_4 we obtain,

$$Z(K,N) = \sum_{S_1,S_3,S_5,S_6,S_7\ldots} [e^{K(S_1+S_3)} + e^{-K(S_1+S_3)}][e^{K(S_3+S_5)} + e^{-K(S_3+S_5)}]e^{KS_5S_6}e^{KS_6S_7}\ldots, \quad (320)$$

and summing over all even numbered spins we obtain

$$Z(K,N) = \sum_{S_1,S_3,S_5,S_7\ldots} [e^{K(S_1+S_3)} + e^{-K(S_1+S_3)}][e^{K(S_3+S_5)} + e^{-K(S_3+S_5)}][e^{K(S_5+S_7)} + e^{-K(S_7+S_9)}]\ldots. \quad (321)$$

Step (2): Rewrite the remaining sum (i.e., the sum over odd numbered spins introduced by Eq. (321)) by implementing the *Kandanoff transformation*

$$e^{K(S+S')} + e^{-K(S+S')} = f(K)e^{K'SS'}, \quad (322)$$

where both $f(K)$ and K' are functions of K. Substituting Eq. (322) into Eq. (321) we obtain

$$Z(K,N) = f(K)^{N/2} \sum_{S_1,S_3,S_5,S_7...} e^{K'S_1S_3} e^{K'S_3S_5} e^{K'S_5S_7} .... = f(K)^{N/2} Z(K', N/2). \quad (323)$$

Note that such transformation allow us to rewrite the partition function $Z(K,N)$ in terms of a *renormalized* partition function $Z(K',N/2)$ (i.e., a partition function with new parameters that describes an Ising model with half the number of spins and a different coupling parameter K').

In order to determine the *renormalization group equations* (i.e., K' and $f(K)$ as a function of K) and show that $K' < K$, we note that when $S' = S = \pm 1$, Eq. (322) gives

$$e^{2K} + e^{-2K} = f(K) e^{K'}, \quad (324)$$

and when $S' = -S = \pm 1$, Eq. (322) gives

$$2 = f(K) e^{-K'}. \quad (325)$$

Therefore, solving for $f(K)$ in Eq. (325) and substituting into Eq. (324) we obtain

$$K' = \frac{1}{2} \ln[\cosh(2K)] \quad (326)$$

and substituting Eq. (326) into Eq. (325) we obtain

$$f(K) = 2\cosh^{1/2}(2K) \quad (327)$$

Eqs. (326) and (327) are called *renormalization group equations* since they provide the renormalization scheme.

Step (3): Go to (1), replacing $Z(K,N)$ by $Z(K',N/2)$.Step (3) is repeated each time on the subsequent (renormalized) partition function, (i.e., $Z(K'',N/4)$, $Z(K''',N/8)$, $Z(K^{IV},N/16)$, $Z(K^{V},N/32)$,...etc.) until the renormalized parameters become approximately constant (i.e., until the renormalized parameters reach a *fixed point* and become invariant under the Kadanoff transformation). Note that, according to Eq. (326), $K > K' > K'' > K'''$, etc., so after a few iterations the coupling parameter becomes negligibly small and the partition function can be approximated by using Eq. (318) as follows:

$$Z(K,N) = f(K) f(k') f(k'') \cdots Z(0,N)$$

and

$$\begin{aligned} \ln Z(K,N) \approx\ & \tfrac{N}{2}\ln[2\cosh^{1/2}(2K)] + \tfrac{N}{4}\ln[2\cosh^{1/2}(2K')] + \tfrac{N}{8}\ln[2\cosh^{1/2}(2K'')] + \\ & \tfrac{N}{16}\ln[2\cosh^{1/2}(2K''')] + \tfrac{N}{32}\ln[2\cosh^{1/2}(2K^{IV})] + \tfrac{N}{64}\ln[2\cosh^{1/2}(2K^{V})] + \tfrac{N}{2^6}\ln 2. \end{aligned} \quad (328)$$

The renormalization group strategy thus involves computing the total sum, introduced by Eq. (317), step by step. The success of the approach relies on the fact that after a few iterations the sum converges to an expression that can be easily computed.

Sometimes the partition function is known for a specific value of the coupling parameter (e.g., for $K' \approx 0$ in the 1-dimensional Ising model). The renormalization group theory can then be implemented to compute the partition function of the system for a different value K of the coupling constant. This is accomplished by inverting Eq. (326) as follows:

$$K = \frac{1}{2}\cosh^{-1}[\exp(2K')]. \quad (329)$$

and computing $Z(K,N)$ from $Z(K,N/2)$ according to Eq. (323).

One could also define the function $g(K)$ as follows

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$$Ng(K) \equiv \ln Z(K,N), \tag{330}$$

and substituting Eq. (329) into Eq. (323) we obtain

$$Ng(K) = \frac{N}{2}\ln 2 + \frac{N}{2}\ln(\cosh^{\frac{1}{2}}(2K)) + \frac{N}{2}g(K'). \tag{331}$$

Therefore, given the partition function $Z(K',N)$ for a system with coupling constant K', one can compute $g(K')$ and K according to Eqs. (330) and (329), respectively. The partition function $Z(K,N) = e^{Ng(K)}$ is then obtained by substituting the values of $g(K')$ and K in Eq. (330). Note that according to this procedure, $K > K'$ and the subsequent iterations give larger and larger values of K. This indicates that the flow of K has only two fixed points at $K = 0$ (e.g., at infinite temperature) and $K = \infty$ (e.g., at 0 K).

Systems with phase transitions, however, have nontrivial fixed points at intermediate values of K. For instance, following a similar procedure, as the one described in this section, it is possible to show that the 2-dimensional Ising model has an additional fixed point K_c and that the heat capacity $C = d^2 g(k)/dk^2$ diverges at K_c. Thus, K_c determines the critical temperature where the system undergoes a phase transition and spontaneously magnetizes.

Note that:

A- For the coupling constant $K' = \frac{1}{2}\ln[\cosh(2K)]$, we have two fixed points,

1- One stable (disordered) at $K = 0, \quad \Rightarrow T = \infty$ and
2- The other unstable (ordered) at $K = \infty, \quad \Rightarrow T = 0$.

We do not find a fixed point at a finite value of K which states that no phase transition occurs. The unstable fixed point corresponds to zero-temperature limit where the spins order in the ground state. The stable fixed point $K = 0$ is the limit of non-interacting spins. The renormalization group treatment which had been here performed exactly, shows that there is no phase transition at a finite temperature.

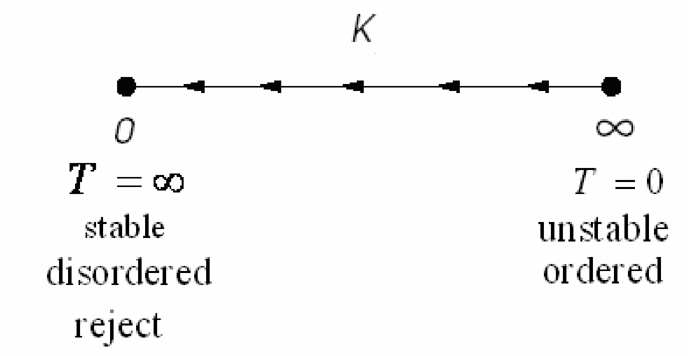

Figure: Flow of the coupling constant $K = \beta J$ of the one-dimensional Ising model under the renormalization group scheme. For any finite coupling and temperature the flow goes towards $K = 0$ the limit of completely decoupled spins.

Therefore we find that starting at any finite value of K leads us through successive application of the decimation procedure towards weaker and weaker coupling K. The fixed point of $K = 0$ eventually corresponding to non-interacting spins is the ultimate limit of disordered spins. Through the renormalization group procedure it is possible to generate an effective model which can be solved perturbatively and obtain the essential physics of the system.

B- For the coupling constant

$$K' = \frac{1}{2}\ln[\cosh(2K)] = \frac{1}{2}\ln[\frac{e^{2K}+e^{-2K}}{2}] = \frac{1}{2}\ln\frac{e^{2K}}{2}[1+e^{-4K}]$$
$$< \frac{1}{2}\ln\frac{e^{2K}}{2} = K - \frac{1}{2}\ln 2$$

i.e., K' is smaller than K, and so by repeatedly doubling the size of the blocks one can proceed to a very small value of interactions between spins far away from each other. Therefore, on going in the reverse direction from small to large interactions, one can start with a very small value of the interaction, say $K' = 0.01$, so that $Z(k') \approx 2^N$ and $f \approx \ln 2$. Using the above equations, one can find the renormalized interaction and the appropriate free energy.

The one-dimensional Ising model does not show a phase transition at a finite temperature, and our analysis was only designed to show by means of a simple example how the RG method works.

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H.W. Discuss the possibility of the existence of the fixed point for 1D-Ising model. **.** Do it from David chandler, ***"Introduction to modern statistical mechanics"*** **(**Oxford University Press 1987).

Note that: the exact answer for the partition function is given by:

$$Z_N = Z_1^N, \quad Z_1 = 2\cosh(K)$$

where $K = \beta J$ and N is the total number of spins. Using the partition function in the form

$$Z(K,N) = \sum_{S_1}\sum_{S_2}\cdots\sum_{S_N} e^{K(S_1S_2+S_2S_3+\cdots+S_NS_1)} = \sum_{S_1}\sum_{S_2}\cdots\sum_{S_N} e^{KS_1S_2}e^{KS_2S_3}e^{\cdots}e^{KS_NS_1} \qquad (30)$$

(I) Remove a finite fraction of the degrees of freedom by averaging (summing) over them, say even numbered spins $s_2, s_4, \ldots$ then

$$Z(K,N) = \sum_{s_1,s_3,\ldots s_N=\pm1} \{e^{K(s_1+s_3)} + e^{-K(s_1+s_3)}\}\{e^{K(s_3+s_5)} + e^{-K(s_3+s_5)}\}\ldots \qquad (31)$$

(II) Cast Eq. (31) into a form that makes it looks the same as Eq. (30) with N/2 and (perhaps) a different coupling constant $\tilde{K}$. Suppose that, for all S, and $\tilde{S} = \pm 1$, we can write

$$e^{K(s+\tilde{s})} + e^{-K(s+\tilde{s})} = f(K)e^{\tilde{K}\, s\tilde{s}}$$

Then

$$Z(K,N) = [f(K)]^{N/2} Q\left(\tilde{K}, \frac{N}{2}\right) \qquad (32)$$

(III) Solve for $f(K)$ an $\tilde{K}$, one finds

$$\begin{aligned} \tilde{K} &= \frac{1}{2}\ln[\cosh(2K)] \\ f(K) &= 2\cosh^{\frac{1}{2}}(2K), \end{aligned} \qquad (33)$$

(IV) Define $\ln Z = N\, g(k)$ as a free energy, and since free energies are extensive, we expect $g(k)$ to be intensive-that is, independent of system size.

(V) From (32), we have $\ln\{Z(K,N)\} = \frac{N}{2}\ln[f(K)] + \ln Q\left(\tilde{K}, \frac{N}{2}\right)$, we have $g(k) = \frac{1}{2}\ln[f(K)] + \frac{1}{2}g\left(\tilde{K}\right)$, or since $f(k) = 2\cosh^{1/2}(2K)$, then

$$g(\tilde{K}) = 2g(K) - \ln[2\sqrt{\cosh(2K)}] \qquad (34)$$

Equations (33) and (34) are called renormalization group (RG) equation. An alternative set of RG equations would be

$$\begin{aligned} K &= \frac{1}{2}\cosh^{-1}(e^{2\tilde{K}}), \\ g(K) &= \frac{1}{2}g(\tilde{K}) + \frac{1}{2}\ln(2) + \frac{\tilde{K}}{2} \end{aligned} \qquad (35)$$

Equation (35) could be solved by iteration to find two fixed points at $K = 0$ and ∞. There is no phase transition in one-dimensional Ising model except at T= 0.
From (30) one expects $\ln Z = Ng(K)$.

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H.W. For the partition function:

$$Z\left(K,N\right)=\sum_{S_1}\sum_{S_2}\cdots\sum_{S_N}e^{NK_0+K\left(S_1S_2+S_2S_3+\cdots+S_NS_1\right)}$$

Prove that:

$$K'=\frac{1}{2}\ln[\cosh(2K)]$$

$$K_0'=2K_0+\frac{1}{2}\ln[4\cosh(2K)]$$