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King Fahd University of Petroleum & Minerals

Physics Department
Phys630 – "Phase Transition 1" Course
2nd semester (122) / 2013

Exercises # 3

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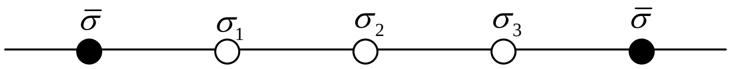
<u>Exercise 8.13</u>

Show that the Bethe- peierls approximation does not predict a phase transition for 1D Ising model with no external field. (As a matter of fact, the Bethe-Peierls approximation gives the exact result for the energy of the 1D Ising model with or with out an external field.)

<u>The solution:</u>

We always look for the solution to Ising model at (external magnetic field H =0) for spins 1 and 3 and a different external field H' for spin 2 .

So, the Hamiltonian is given by



$$E = -H(\sigma_1+\sigma_3) - H'\sigma_2 - V(\bar{\sigma}\sigma_1+\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\bar{\sigma})$$

$$\Rightarrow Z = \sum_{\sigma_1,\sigma_2,\sigma_3} e^{-\beta E} = \sum_{\sigma_1,\sigma_2,\sigma_3} e^{\beta H(\sigma_1+\sigma_3)+\beta H'\sigma_2+\beta V(\bar{\sigma}\sigma_1+\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\bar{\sigma})}$$

$$\Rightarrow Z = \sum_{\sigma_1,\sigma_2,\sigma_3} e^{h(\sigma_1+\sigma_3)+h'\sigma_2+\nu(\bar{\sigma}\sigma_1+\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\bar{\sigma})} \quad \textit{where } \beta H = h,\ \beta H' = h' \textit{ and } \beta V = \nu$$

$$\Rightarrow Z = \sum_{\sigma_2} e^{h'\sigma_2} \sum_{\sigma_1,\sigma_3} e^{h(\sigma_1+\sigma_3)+\nu(\bar{\sigma}\sigma_1+\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\bar{\sigma})} = \sum_{\sigma_2} e^{h'\sigma_2} \sum_{\sigma_1,\sigma_3} e^{\sigma_1(h+\nu\bar{\sigma}+\nu\sigma_2)+\sigma_3(h+\nu\bar{\sigma}+\nu\sigma_2)}$$

We put values of σ_1 *and* σ_3 *where* $\sigma_1 = \pm 1$ *and* $\sigma_3 = \pm 1$

$$\Rightarrow Z = \sum_{\sigma_2} e^{h'\sigma_2}\left(e^{1(h+\nu\bar{\sigma}+\nu\sigma_2)+1(h+\nu\bar{\sigma}+\nu\sigma_2)} + e^{-1(h+\nu\bar{\sigma}+\nu\sigma_2)+-1(h+\nu\bar{\sigma}+\nu\sigma_2)} + e^{1(h+\nu\bar{\sigma}+\nu\sigma_2)-1(h+\nu\bar{\sigma}+\nu\sigma_2)} + e^{-1(h+\nu\bar{\sigma}+\nu\sigma_2)+1(h+\nu\bar{\sigma}+\nu\sigma_2)}\right)$$

$$\Rightarrow Z = \sum_{\sigma_2} e^{h'\sigma_2}\left(e^{2(h+\nu\bar{\sigma}+\nu\sigma_2)} + e^{-2(h+\nu\bar{\sigma}+\nu\sigma_2)} + 2\right) \quad \text{then put } \sigma_2 = \pm 1$$

$$\Rightarrow Z = e^{h'}\left(e^{2(h+\nu\bar{\sigma}+\nu)} + e^{-2(h+\nu\bar{\sigma}+\nu)} + 2\right) + e^{-h'}\left(e^{2(h+\nu\bar{\sigma}-\nu)} + e^{-2(h+\nu\bar{\sigma}-\nu)} + 2\right)$$

$$\Rightarrow \boxed{Z = 4\left[e^{h'}\cosh^2(h+\nu\bar{\sigma}+\nu) + e^{-h'}\cosh^2(h+\nu\bar{\sigma}-\nu)\right]} --- (2)$$

This equation is the partition function Z of the system, to get $\bar{\sigma}$ we start the following steps:

$$\boxed{\left\langle \sigma_1+\sigma_3 \right\rangle = \frac{1}{Z}\frac{\partial Z}{\partial h} = \frac{{}^{2}\not{8}\left[e^{h'}\cosh(h+\nu\bar{\sigma}+\nu)\sinh(h+\nu\bar{\sigma}+\nu)+e^{-h'}\cosh(h+\nu\bar{\sigma}-\nu)\sinh(h+\nu\bar{\sigma}-\nu)\right]}{\not{4}\left[e^{h'}\cosh^2(h+\nu\bar{\sigma}+\nu)+e^{-h'}\cosh^2(h+\nu\bar{\sigma}-\nu)\right]}} ---(3)$$

Also

$$\boxed{\left\langle \sigma_2 \right\rangle = \frac{1}{Z}\frac{\partial Z}{\partial h'} = \frac{\left[e^{h'}\cosh^2(h+\nu\bar{\sigma}+\nu)-e^{-h'}\cosh^2(h+\nu\bar{\sigma}-\nu)\right]}{\left[e^{h'}\cosh^2(h+\nu\bar{\sigma}+\nu)+e^{-h'}\cosh^2(h+\nu\bar{\sigma}-\nu)\right]}} ----(4)$$

but $\left\langle \sigma_1+\sigma_3 \right\rangle = 2\left\langle \sigma_2 \right\rangle$ so from Eqs(3) and(4) we get

$$\frac{\not{2}\left[e^{h'}\cosh(h+\nu\bar{\sigma}+\nu)\sinh(h+\nu\bar{\sigma}+\nu)+e^{-h'}\cosh(h+\nu\bar{\sigma}-\nu)\sinh(h+\nu\bar{\sigma}-\nu)\right]}{\left[e^{h'}\cosh^2(h+\nu\bar{\sigma}+\nu)+e^{-h'}\cosh^2(h+\nu\bar{\sigma}-\nu)\right]}$$

$$= \not{2}\,\frac{\left[e^{h'}\cosh^2(h+\nu\bar{\sigma}+\nu)-e^{-h'}\cosh^2(h+\nu\bar{\sigma}-\nu)\right]}{\left[e^{h'}\cosh^2(h+\nu\bar{\sigma}+\nu)+e^{-h'}\cosh^2(h+\nu\bar{\sigma}-\nu)\right]}$$

when $h = h' = 0$ we obtain

$$\left[\cosh(\nu\bar{\sigma}+\nu)\sinh(\nu\bar{\sigma}+\nu)+\cosh(\nu\bar{\sigma}-\nu)\sinh(\nu\bar{\sigma}-\nu)\right] = \left[\cosh^2(\nu\bar{\sigma}+\nu)-\cosh^2(\nu\bar{\sigma}-\nu)\right]$$

To solve this equation as polynomial equation we use these following substitutions

$x = e^{\nu\bar{\sigma}}$ *and* $y = e^{\nu\bar{\sigma}}$ in order to obtain

$$\frac{1}{4}\left[\underbrace{\cosh(\nu\bar{\sigma}+\nu)}_{(xy+x^{-1}y^{-1})}\underbrace{\sinh(\nu\bar{\sigma}+\nu)}_{(xy-x^{-1}y^{-1})}+\underbrace{\cosh(\nu\bar{\sigma}-\nu)}_{(xy^{-1}+x^{-1}y)}\underbrace{\sinh(\nu\bar{\sigma}-\nu)}_{(xy^{-1}-x^{-1}y)}\right] = \frac{1}{4}\left[\underbrace{\cosh^2(\nu\bar{\sigma}+\nu)}_{(xy+x^{-1}y^{-1})^2}-\underbrace{\cosh^2(\nu\bar{\sigma}-\nu)}_{(xy^{-1}+x^{-1}y)^2}\right]$$

$$\Rightarrow \left(xy+x^{-1}y^{-1}\right)\left(xy-x^{-1}y^{-1}\right)+\left(xy^{-1}+x^{-1}y\right)\left(xy^{-1}-x^{-1}y\right) = \left(xy+x^{-1}y^{-1}\right)^2-\left(xy^{-1}+x^{-1}y\right)^2$$

$$\Rightarrow \left(x^2y^2-x^{-2}y^{-2}\right)+\left(x^2y^{-2}-x^{-2}y^2\right) = x^2y^2+x^{-2}y^{-2}+2-x^2y^{-2}-x^{-2}y^2-2$$

$$\Rightarrow \not{x^2y^2}-x^{-2}y^{-2}+x^2y^{-2}-\not{x^{-2}y^2} = \not{x^2y^2}+x^{-2}y^{-2}+\not{2}-x^2y^{-2}-\not{x^{-2}y^2}-\not{2}$$

$$\Rightarrow -x^{-2}y^{-2} + x^{2}y^{-2} = +x^{-2}y^{-2} - x^{2}y^{-2}$$

$$-2x^{-2}y^{-2} + 2x^{2}y^{-2} = 0$$

$$(x^{2} + x^{-2})y^{-2} = 0$$

The trivial solution is obtained by

$$(x^{2} - x^{-2})y^{-2} = 0 \quad \Rightarrow (x^{4} - 1)y^{-2} = 0$$

$x = 1$ only because $x = e^{\nu\bar{\sigma}} = 1$ this means that $\nu\bar{\sigma} = 0$ but $\nu \neq 0$ *so* $\bar{\sigma} = 0$

where $\nu \neq 0$ (ν is spin-spin interacting energy)

also $y \neq 0$ because $y = e^{\nu}$ and $\nu \neq 0$.

So the Bethe- peierls approximation does not predict a spontaneous magnetization

for any value of ν .