EXACT ONE-DIMENSIONAL ISING MODEL

The one-dimensional Ising model consists of a chain of N spins, each spin interacting only with its two nearest neighbors. The simple Ising problem in one dimension can be solved directly in several ways.

First *the chain is considered as open ended* and the Hamiltonian in the form:

$$H = -J\sum_{i=1}^{N-1}\sigma_i\sigma_{i+1} \quad J > 0$$

The partition function is given by

$$Z_N = \sum_{\{\sigma_i=\pm 1\}} e^{K\sum_{i=1}^{N-1}\sigma_i\sigma_{i+1}} \tag{7.5.1}$$

where $K = \beta J$. The exponential can be factored as a product of terms of the form $e^{K\sigma_i\sigma_{i+1}}$, each of which can be written as (Note that: $(\sigma_i\sigma_{i+1})$ can only be +1 or -1, $\cosh(\pm x) = x$ and $\sinh(\pm x) = \pm x$)

$$\begin{aligned} e^{K\sigma_i\sigma_{i+1}} &= \cosh(K\sigma_i\sigma_{i+1}) + \sinh(K\sigma_i\sigma_{i+1}) = \cosh(K) + \sigma_i\sigma_{i+1}\sinh(K) \; * \\ &= (1+\sigma_i\sigma_{i+1}y)\cosh(K) \end{aligned} \tag{7.5.2}$$

where $y = \tanh(K)$.

Here we have used:

$$* \, e^{c\sigma\sigma'} = \begin{Bmatrix} e^{c} & \sigma\sigma' = 1 \\ e^{-c} & \sigma\sigma' = -1 \end{Bmatrix} = \cosh(c) + \sigma\sigma'\sinh(c)$$

which holds because can only +1 or -1.

The partition function (7.5.1) then becomes

$$\begin{aligned} Z_N &= \cosh^{N-1} K \sum_{\{\sigma_i=\pm 1\}} \prod_{i=1}^{N-1} (1+\sigma_i\sigma_{i+1}y) \\ &= (\cosh K)^{N-1} \sum_{\{\sigma_i=\pm 1\}} (1+\sigma_1\sigma_2 y)(1+\sigma_2\sigma_3 y)\cdots(1+\sigma_{N-1}\sigma_N y). \end{aligned} \tag{7.5.3}$$

Summation over $\sigma_1 = \pm 1$ gives $(1+\sigma_2 y)(\cdots) + (1-\sigma_2 y)(\cdots)$, where $(\cdots)$ represents

$$\prod_{i=2}^{N-1}(1+\sigma_i\sigma_{i+1}y),$$

and the indicated summation yields 2(. . .). Next, summation over $\sigma_2 = \pm 1$ gives another factor 2 and a product of $N-3$ terms. Continued summations finally produce the result

$$Z_N = \{2\cosh(K)\}^{N-1} \tag{7.5.4}$$

Average energy and the specific heat:

In the thermodynamic limit, the free energy per spin is given by:

$$f = -k_B T \lim_{N\to\infty} \frac{1}{N} \ln Z_N = -k_B T \ln\{2\cosh(K)\},$$
$$U = \langle E\rangle = \lim_{N\to\infty}\left[-\frac{1}{N}\frac{\partial \ln Z_N}{\partial\beta}\right] = -J\tanh(K) \tag{7.5.5}$$

and the heat capacity C is

$$C = \frac{\partial\langle E\rangle}{\partial T} = k_B K^2 \cosh^2(K) \tag{7.5.6}$$

The energy and heat capacity are smoothly varying, always finite functions of temperature, exhibiting no phase transition. Thus the molecular-mean-field- approximation is incorrect, no matter how plausible, for a one-dimensional system, and its validity in n-dimensions then is immediately to be doubted.

Another direct technique for the open, one-dimensional chain makes use of a change in variables $\eta_i = \sigma_i \sigma_{i+1} = \begin{cases} 1 & \text{if } \sigma_i = \sigma_{i+1} \\ -1 & \text{if } \sigma_i = -\sigma_{i+1} \end{cases}, \quad 1 \le i \le N-1$, where it may be seem from Eq. (7.5.1) or (7.5.3) that Z_N is independent of η_i. For the open chain, the η's are independent, and Z_N can be written as

$$Z_N = \sum_{\{\eta_i=\pm1\}} e^{K\sum_{i=1}^{N-1}\sigma_i\sigma_{i+1}} = \left(\sum_{\eta_1=\pm1} e^{K\eta_1}\right)\left(\sum_{\eta_2=\pm1} e^{K\eta_2}\right)\cdots$$
$$= \{2\cosh(K)\}^{N-1}$$

where each sum on a η_i yields a factor $2\cosh(K)$, and the final result is the product of $N-1$ such independent factors, in agreement with Eq. (7.5.4). For closed chains in one dimension and for Ising lattices of higher dimension, no such simple technique will work because the η's are no longer independent.

Second *When the chain is closed*, with $S_1 = S_{N+1}$,**:**

For the open ended chain, the Hamiltonian has the form:

$$H = -J\sum_{i=1}^{N} S_i S_{i+1} \quad J > 0$$

The partition function is given by

$$Z_N = \sum_{\{S_i=\pm1\}} e^{K\sum_{i=1}^{N} S_i S_{i+1}}$$

When the chain is closed, with $S_1 = S_{N+1}$, direct evaluation of Z_N becomes slightly more difficult, but other procedures are often simpler for the closed chain than for the open one. For the closed chain, Z_N becomes:

$$Z_N = \left\{\cosh\left(K\right)\right\}^N \sum_{\{S\}} \prod_{i=1}^{N} \left(1 + yS_i S_{i+1}\right)$$

where $K = \beta J$ and $y = \tanh(K)$ .We work out the product and sort terms in powers of y :

$$Z_N = \left\{\cosh\left(K\right)\right\}^N \sum_{\{S\}} \left\{1 + y\left(S_1S_2 + S_2S_3 + \cdots + S_NS_1\right) + \right.$$
$$\left. + y^2\left(S_1S_2S_2S_3 + \cdots\right) + \cdots + y^N\left(S_1S_2S_2 \cdots S_NS_NS_1\right)\right\}$$

The terms, linear in y contain products of two different (neighboring) spins, like S_iS_{i+1}. The sum over all spin configurations of this product vanish, $\sum_{S_i \text{ or } S_{i+1}} \left(S_iS_{i+1}\right) = 0$, because there are two configurations with parallel spins ($S_iS_{i+1} = 1$) and two with antiparallel spins ($S_iS_{i+1} = -1$). Thus, the term linear in y vanishes after summation over all spin configurations. For the same reason also the sum over all spin configurations, which appear at the term proportional to y^2, vanish. In order for a term to be different from zero, all the spins in the product must appear twice (then, $\sum_{S_i} S_i^2 = 2$). This condition is fulfilled only in the last term, which after summation over all spin configurations gives $2^N y^N$. Therefore the partition function of the Ising model of a linear chain of *N* spins is:

$$Z_N = \left\{2\cosh\left(K\right)\right\}^{N-1}\left[1 + y^N\right]$$

a result that differs from that of $Z_N = \left\{2\cosh\left(K\right)\right\}^{N-1}$ for open chain. In the limit of very large *N*, however, the y^N contribution becomes vanishingly small, since $y = \tanh K < 1$ for all finite *J* and β.

Note that:

i- $\sum_{\{S\}} 1 = 2,$

ii- $\sum_{s_i} s_is_{i+1} = (1 + -1)s_{i+1} = 0,$

iii- $\sum_{S_i \text{ or } S_{i+1}} \left(S_iS_{i+1}\right) = \underbrace{\uparrow\uparrow}_{+1} + \underbrace{\uparrow\downarrow}_{-1} + \underbrace{\downarrow\uparrow}_{-1} + \underbrace{\downarrow\downarrow}_{+1} = 1 - 1 - 1 + 1 = 0$

iv- $\sum_{\{S\}_i} \left(S_iS_{i+1}S_jS_{j+1}\right) = \uparrow\uparrow\uparrow\uparrow + \uparrow\uparrow\uparrow\downarrow + \uparrow\uparrow\downarrow\uparrow + \uparrow\downarrow\uparrow\uparrow + \downarrow\uparrow\uparrow\uparrow + \downarrow\downarrow\uparrow\uparrow + \downarrow\uparrow\uparrow\downarrow + \cdots + \downarrow\downarrow\downarrow\downarrow$

$$= 1 - 4 + 6 - 4 + 1 = 0$$

v- $\sum_{s_i=-1}^{+1} s_i^{\ell} = \begin{cases} 2 & \ell = \text{even} \\ 0 & \ell = \text{odd} \end{cases}, \qquad \sum_{s_i} 1 = 2$

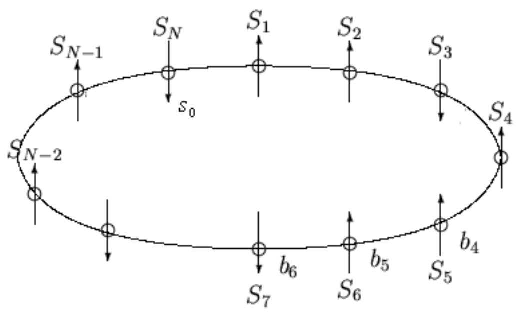
Ising Model and Transfer Matrix

Exact solutions of the Ising model are possible in 1 and 2 dimensions and can be used to calculate the exact critical exponents for the two corresponding universality classes.
In one dimension, the Ising Hamiltonian becomes:

$$\begin{array}{ccccccc} \uparrow & \uparrow & \downarrow & \cdots & \downarrow & \downarrow & \uparrow \\ s_o & s_1 & s_2 & \ldots & & & s_{N-1} \end{array}$$

$$H = -J\sum_{i=0}^{N} s_i s_{i+1} - h\sum_{i=0}^{N} s_i , \quad J, h > 0$$

which corresponds to N spins on a line. We will impose periodic boundary conditions on the spins so that $s_N = s_0$, $s_i = +1$ or -1. Thus, the topology of the spin space is that of a circle, see the figure.



With the definitions $K = \beta J$ and $H = \beta h$, the partition function is then:

$$Z = \sum_{\{s\}} e^{K(s_o s_1 + s_1 s_2 + \cdots + s_{N-1} s_o) + H(s_o + s_1 + s_2 + \cdots + s_{N-1})}$$

$$= \sum_{\{s\}} e^{H(\frac{s_0}{2}) + K(s_0 s_1) + \beta h(\frac{s_1}{2})} e^{H(\frac{s_1}{2}) + K(s_1 s_2) + \beta h(\frac{s_2}{2})} \cdots e^{H(\frac{s_{N-1}}{2}) + K(s_{N-1} s_0) + H(\frac{s_0}{2})}$$

$$\equiv \sum_{\{s\}} P_{0,1} P_{1,2} \cdots P_{N-1,0}$$

Where

$$P_{i,i+1} = e^{H(\frac{s_i}{2}) + K(s_i s_{i+1}) + H(\frac{s_{i+1}}{2})}$$

In order to carry out the spin sum, let us define a matrix P with matrix elements:

$$\langle s | P | s' \rangle = e^{H(\frac{s}{2}) + K(s s') + H(\frac{s'}{2})},$$

$$\langle 1 | P | 1 \rangle = e^{K+H} , \qquad \langle -1 | P | -1 \rangle = e^{K-H} ,$$

$$\langle -1 | P | 1 \rangle = \langle 1 | P | -1 \rangle = e^{-K}$$

The matrix **P** is called the *transfer matrix*. Thus, the matrix P is a 2 x2 matrix given by

$$P = \begin{matrix} & s_{i+1}=1 & s_{i+1}=-1 \\ s_i = 1 & e^{K+H} & e^{-K} \\ s_i=-1 & e^{-K} & e^{K-H} \end{matrix}$$

From the matrix rules, the larger λ_+ and smaller λ_- eigenvalues are calculated as:

$$\lambda_+ + \lambda_- = \mathrm{Tr}(P) = e^{K+H} + e^{K-H} = 2e^K \cosh(H)$$
$$\lambda_+ \lambda_- = \det(P) = e^{K+H} e^{K-H} - e^K e^K = e^{2K} - e^{-2K} = 2\sinh(K)$$
:

Solving for $\lambda_\pm$, one finds:

$$\lambda_\pm = e^K \left[\cosh(H) \pm \sqrt{\sinh^2(H) + e^{-4K}} \right]$$

The trace of P^N is given by:

$$Z_N = \mathrm{Tr}(P^N) = \lambda_+^N + \lambda_-^N = \lambda_+^N \left[1 + \left(\lambda_- / \lambda_+ \right)^N \right] \underset{N\to\infty}{\sim} \lambda_+^N$$

Free energy:

In the thermodynamic limit, the free energy per spin is given by:

$$f = -k_B T \lim_{N\to\infty} \frac{1}{N} \ln Z_N = -k_B T \ln \lambda_+$$

As $\beta \to \infty$

$$f = -k_B T \lim_{\beta\to\infty} \ln \lambda_+ = -k_B T \ln \left\{ e^K \left[\cosh(H) + \sinh(H) \right] \right\} = -J - h$$

which is the energy per spin as expected.

The magnetization: becomes

$$m = \left(\frac{\partial f}{\partial h} \right) = -\left(\frac{\partial \ln \lambda_+}{\partial (H)} \right) = \frac{\sinh(H) + \dfrac{\sinh(H)\cosh(H)}{\sqrt{\sinh^2(H) + e^{-4K}}}}{\cosh(H) + \sqrt{\sinh^2(H) + e^{-4K}}}$$

Which is regular as $H \to 0$, since $\cosh(H) \to 1$ and $\sinh(H) \to 0$, itself vanishes. Thus, there is no magnetization at any finite temperature in one dimension, hence no nontrivial critical point.

Example: It was that the exact eigenvalues of the periodic Ising model is given by:

$$\lambda_\pm = e^K \left[\cosh(H) \pm \sqrt{\sinh^2(H) + e^{-4K}} \right], \quad \beta = 1/kT \text{ , } K = \beta J \text{ and } H = \beta h$$

For $H = 0$,simplify the expression:

$$\lambda_\pm = e^K \left[\cosh(H) \pm \sqrt{\sinh^2(H) + e^{-4K}} \right] = e^K \left[1 \pm \sqrt{e^{-4K}} \right] = e^K \pm e^{-K}, \quad \beta = 1/kT ,$$

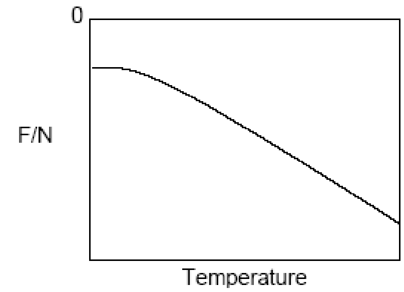
$$\Rightarrow \quad \lambda_+ = 2\cosh(K), \quad \lambda_- = 2\sinh(K),$$

Consequently

$$\lambda = \lambda_+^N + \lambda_-^N = \left[2\cosh(K)\right]^N + \left[2\sinh(K)\right]^N = \lambda_+^N \left[1 + \left(\lambda_- / \lambda_+\right)^N\right];$$

$$= \left\{2\cosh\left(K\right)\right\}^N \left(1 + y^N\right) \approx \left\{2\cosh\left(K\right)\right\}^N$$

Where $y = \tanh(K)$



Then find the following:

i- $F = -\frac{1}{\beta}\ln(\lambda)$ in the extreme limits, i.e. $T \to 0$ and $T \to \infty$.

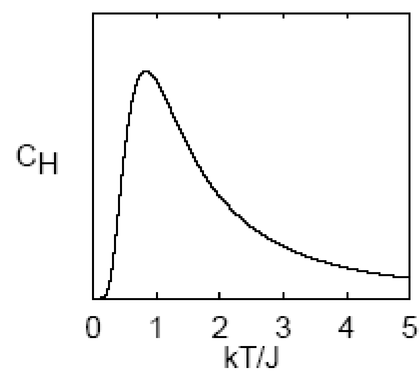
Ans: $F = -\frac{1}{\beta}\ln(\lambda) = -NkT\ \ln\left\{2\cosh\left(K\right)\right\}$

ii- $E = -\frac{\partial}{\partial \beta}\ln(\lambda)$

Ans: $E = -\frac{\partial}{\partial \beta}\ln(\lambda) = -NJ\ \tanh\left(K\right)$

ii- $C = -\frac{\partial E}{\partial T}$ in the extreme limits, i.e. $T \to 0$ and $T \to \infty$, and find T at maximum C.

Ans: $C = -\frac{\partial E}{\partial T} = NK\left[\frac{K}{\cosh(K)}\right]^2$



Note:

$$Z_N = \left\{2\cosh\left(K\right)\right\}^N ;$$

$$F = -\frac{1}{N}\ln Z_N = -\ln\left\{2\cosh\left(K\right)\right\} = -\ln\left\{e^{J/T} + e^{-J/T}\right\}$$

$$= -\ln\left(e^{J/T}\right) - \ln\left\{1 + e^{-2J/T}\right\}$$

As $T \to 0$

$$F \approx -\frac{J}{T} - e^{-2J/T}$$

The first term, even though it looks like it blows up at $T \to 0$, is actually regular. It simply says that the ground state energy is $-J$ per spin. It could be removed by a constant shift of energy, for example. The second term is singular. So the singular part of the free energy behaves as:

$$F_{\text{singular}} \approx -e^{-2J/T}$$

The correlation length is

$$\xi = \frac{1}{\ln\left(\frac{\lambda_+}{\lambda_-}\right)} = \frac{1}{\ln\left(\frac{1}{\tanh K}\right)} \approx \frac{1}{2} e^{2J/T}$$

The last approximate equality works at $T \to 0$. Their product, in the limit $T \to 0$ is thus

$$\xi F_{\text{singular}} = -\frac{1}{2}$$

Which is a universal number (does not depend on parameters)

H.W. With the eigenvector of the Ising matrix in 2-dimensions, calculate the magnetization per spin, the correlation function , and the correlation length, and check if they behave in a sensible way.(Go to the discussion in sections 3.3 and 3.4 of Goldenfeld)

H.W. Write down the transfer matrix for the one-dimensional spin-1 Ising model in zero field which is described by the Hamiltonian

$$H = -J\sum_{i=0}^{N} s_i s_{i+1}, \quad J > 0, \quad s_i = \pm 1, 0$$

Hence calculate the free energy per spin of this model and show that it has the expected behavior in the limits $T \to 0$ and $T \to \infty$.

[Answer: $f = -kT \ln\{(1+2\cosh\beta J + [(2\cosh\beta J - 1)^2 + 8]^{1/2})/2\}$.]

While the one-dimensional Ising model is a relatively simple problem to solve, the two-dimensional Ising model is *highly* nontrivial. It was only the pure mathematical genius of Lars Onsager that was able to find an analytical solution to the two-dimensional Ising model. This, then, gives an exact set of critical exponents for the $d = 2$ and $n = 1$ universality class. To date, the three-dimensional Ising model remains unsolved.

Here, the Onsager results will be stated as:

In the thermodynamic limit, the final result at zero field is:

$$f(T) = -kT \ln\left[2\cosh(2\beta J)\right] - \frac{kT}{2\pi}\int_0^{\pi} d\phi \ln\frac{1}{2}\left(1+\sqrt{1-K^2\sin^2\phi}\right)$$

where

$$K = \frac{2}{\cosh(2\beta J)\coth(2\beta J)}$$

The energy per spin is

$$\epsilon(T) = -2J\tanh(2\beta J) + \frac{K}{2\pi}\frac{dK}{d\beta}\int_0^{\pi} d\phi \frac{\sin^2\phi}{\Delta(1+\Delta)}$$

where

$$\Delta = \sqrt{1-K^2\sin^2\phi}$$

The magnetization, then, becomes

$$m = \left\{1 - [\sinh(2\beta J)]^{-4}\right\}^{1/8}$$

for $T < T_c$ and 0 for $T > T_c$, indicating the presence of an order-disorder phase transition at zero field. The condition for determining the critical temperature at which this phase transition occurs turns out to be

$$2\tanh^2(2\beta J) = 1$$

$$kT_c \approx 2.269185 J$$

Near $T = T_c$, the heat capacity per spin is given by

$$\frac{C(t)}{k} = \frac{2}{\pi}\left(\frac{2J}{kT_c}\right)^2\left[-\ln\left(1-\frac{T}{T_c}\right)+\ln\left(\frac{kT_c}{2J}\right)-\left(1+\frac{\pi}{4}\right)\right]$$

Thus, the heat capacity can be seen to diverge logarithmically as $T \to T_c$.

The critical exponents computed from the Onsager solution are

$$\alpha = 0 \quad \text{(log divergence)}$$

$$\beta = \frac{1}{8}$$

$$\gamma = \frac{7}{4}$$

$$\delta = 15$$

which are a set of exact exponents for the $d = 2$ and $n = 1$ universality class.

$$\ln Z = n\ln(\lambda),\ \ \lambda = e^{\beta J}\cosh(\beta\mu B) + \sqrt{e^{2\beta J}\sinh^2(\beta\mu B) + e^{-2\beta J}}.$$

EXAMPLE:

In terms of $J, \mu B$ and T, find the average magnetization (per spin) for the 1-d Ising model.

Since the magnetic field enters the Hamiltonian as $-\mu\vec{B}\cdot\vec{\sigma}$, the average spin is

$$\langle\sigma\rangle = \frac{1}{n}\frac{d\ln Z}{d(\beta\mu B)}.$$

This gives,

$$\begin{aligned}\langle\sigma\rangle &= \frac{\sinh(\beta\mu B) + \frac{\cosh(\beta\mu B)\sinh(\beta\mu B)}{\sqrt{\sinh^2(\beta\mu B)+e^{-4\beta J}}}}{\cosh(\beta\mu B) + \sqrt{\sinh^2(\beta\mu B) + e^{-4\beta J}}} \\ &= \frac{\sinh(\beta\mu B)}{\sqrt{\sinh^2(\beta\mu B) + e^{-4\beta J}}}.\end{aligned}$$