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Electromagnetic wave propagation in an active medium and the equivalent Schrödinger equation with an energy-dependent complex potential

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We study the massless limit of the Klein-Gordon (K-G) equation in 1+1 dimensions with static complex potentials in order to provide an alternative, but equivalent, representation of plane electromagnetic (em) wave propagation in an active medium. In the case of a dispersionless em medium, the analogy dictates that the potential in the K-G equation is complex and energy dependent. We study also the nonrelativistic Schrödinger equation with a potential that has the same energy dependence as that of the K-G equation. The behavior of the solutions of this Schrödinger equation is compared with those found elsewhere in the literature for the propagation of electromagnetic plane waves in a uniform active medium with complex dielectric constant. In particular, both equations exhibit a discrepancy between the time-dependent and stationary results; our study attributes this discrepancy to the appearance of time-growing bound eigenstates corresponding to poles in the transmission and reflection amplitudes located in the upper half of the wave-number plane. The omission of these bound states in the expansion in stationary states leads to the observed discrepancy. Furthermore, it was demonstrated that there is a frequency- (energy) -and-size-dependent gain threshold above which this discrepancy appears. This threshold corresponds to the value of the gain at which the first pole crosses the real axis.

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I. INTRODUCTION

The interest in the amplification effects of classical and quantum waves in disordered media has been strongly motivated by experimental results on the amplification of light.[1] The amplification was shown to strongly enhance the coherent backscattering and, consequently, increase reflection. These results on the reflection naturally lead one to expect an enhancement of the transmission in such amplifying systems. However, for amplifying periodic systems, many workers[2] found that the transmission coefficient starts increasing exponentially with length up to a certain maximum where it oscillates and then decreases exponentially. Thus, for large systems, the wave propagation is suppressed as if the system became absorbing. Actually, it was generally shown by Beenakker *et al.*[2] that there is a dual symmetry between absorption and amplification for the propagation of radiation through a disordered medium with a complex dielectric constant. However, on physical grounds, one usually expects that with sufficient gain, the wave should be able to overcome the losses from backscattering and propagate through the system with increased intensity. The above mentioned work seems to confirm the paradoxial result that stimulated emission of radiation suppresses the transmission through the system. Setting aside the issue of the practical relevance of gain media in electronic systems, it was recently shown by Xiaozheng and Soukoulis[4] that the numerical solution of the Schrödinger equation with complex potential also exhibited a similar paradoxical behavior of the transmission coefficient. In this case, the positive imaginary part of the potential is interpreted as a gain since it leads to an increase in the probability density as time goes on. Thus, optical transmission through a segment of complex dielectric material[3] or the analogous electronic transmission through a complex scattering potential[4] exhibited a transition from amplification to absorption at a critical value of the gain (or length) when treated using the stationary wave equation or the stationary Schrödinger equation. However, when the time-dependent wave or Schrödinger equations were solved for an initial pulse by the finite difference time domain (FDTD) method no region of absorption was observed,[5] even for values of the gain above threshold. Thus, this numerical simulation shows that the Schrödinger equation in the presence of gain behaves similarly to the electromagnetic (em) wave equation, that is the paradoxical results seem to be generic. It seems that for systems with gain, the stationary solutions become irresponsive to time-dependent perturbation for gains above a certain threshold. Soukoulis *et al.*[4,5] correctly pointed out the nature of the discrepancy between the time-dependent wave equation and the stationary one. However, there is no satisfactory explanation of the origin of this apparent paradox.

It is the purpose of this work to try to clarify the origin of the discrepancy between the time-dependent and stationary wave equations (in all our work, this designates both em and Schrödinger wave equations) in the case of systems with gain. Since the above mentioned problem seems to be common to both wave and Schrödinger evolution equations with gain, we try to keep this parallelism in our analysis setting aside the issue of practical relevance for electronic systems. In optical systems, one can phenomenologically understand the increase of light intensity due to an increase in photons by means of coherent amplification, as by stimulated emission of radiation in an active lasing medium. However, in electronic systems, and due to particle number conservation, one cannot imagine such a violation of current conservation. To establish the connection between the electromagnetic and inertial systems, we start with the relativistic massless Klein-Gordon (K-G) equation, which should reproduce the wave equation. From there we deduce the corresponding equivalent potential components that reproduce the wave equation with a given complex dielectric function. The potential turns out to be energy dependent. For consistency, we assert that we should study the Schrödinger equation with this energy-dependent potential in order to coherently compare it with the corresponding wave equation. Actually, the effect of inertia in the relativistic K-G equation is by itself a serious issue that distinguishes the behavior of massive particles like electrons in the Schrödinger equation from the wave equation, which holds for the massless photons. Our approach follows first the usual path of calculating the transmission or reflection amplitude as a function of wave number k. Then comes the novelty of studying the pole structure of the transmission in the complex wave-number (or energy or frequency) plane and tuning the value of the gain (the complex potential) till we see one of the wave-number eigenvalues in the lower half of the complex wave-number plane approaching the real axis and crossing it to the upper half. We propose that this crossover at the critical value of the gain marks the appearance of the discrepancy between the stationary and time-dependent behavior. This is so because poles in the upper half of the complex k plane correspond to time-growing bound eigenstates (i.e., eigenstates decaying exponentially as $|x|\to\infty$), which have to be included in the expansion along with the eigenstates of the continuous spectrum. It is exactly the omission of this sum over the discrete eigenstates which caused the apparent paradox mentioned before.

To sum up, we stress that the main point in our present paper is the explanation of the apparent discrepancy between time-domain and frequency-domain solutions in linear equations with gain, regardless of whether they are em, Klein-Gordon, or Schrödinger wave equations. The key to the removal of the apparent discrepancy is the existence of time-growing exact-bound eigenstates corresponding to eigenvalues in the upper half of the complex k plane in systems with gain; the latter destroys the Hermitian nature of the Hamiltonian and, consequently, the obligatory reality of its eigenvalues.

II. FORMULATION OF THE PROBLEM

The propagation of the electromagnetic waves in a medium free of charges and currents is described by the wave equation

$$\left(\nabla^2-\hat{\mu}\hat{\varepsilon}\frac{\partial^2}{\partial t^2}\right)F(t,r)=0, \tag{1}$$

where $\hat{\mu}$ is the permeability, $\hat{\varepsilon}$ the permittivity of the medium, and F stands for the electromagnetic fields, E or B. The relative permeability and permittivity μ and ε are defined by $\mu=\hat{\mu}/\mu_0$ and $\varepsilon=\hat{\varepsilon}/\varepsilon_0$, where $\mu_0\varepsilon_0=1/c^2$ and c is the speed of light in free space. These two parameters are generally complex, space-dependent, and frequency-dependent corresponding to an absorbing or active, nonuniform, and dispersive medium. The time-independent wave equation for oscillatory electromagnetic fields of the form $F(t,r)=F_0(r)e^{-i\omega t}$ becomes

$$\left[\nabla^2+\frac{\omega^2}{c^2}\mu(\omega,r)\varepsilon(\omega,r)\right]F_0(r)=0, \tag{2}$$

assuming that the permittivity and the permeability are piecewise constant. It has been found,[5] and our calculations have verified this finding, that there is a discrepancy between the time-dependent and the frequency-dependent solutions of the wave equation in an active medium. More explicitly, the time evolution of a wave packet passing through an active medium of length L shows that the transmitted packet is amplified by a factor e^{aL}, where a is a positive quantity; on the other hand, the frequency-dependent solution for *real* ω shows amplification only up to a critical value of L, while for larger values instead of amplification it exhibits attenuation. In this paper, we provide an explanation of this paradoxical behavior of the frequency-dependent solution, and we reconcile it with the time-dependent solution. Besides the electromagnetic (em) wave equation (2), we study also the relativistic Klein-Gordon (K-G) equation, which, under certain conditions, is equivalent to Eq. (2); its nonrelativistic limit reduces to the Schrödinger equation, thus, establishing the equivalence of all three equations (under certain correspondences). All three equations exhibit the above mentioned discrepancy between the time-dependent and the frequency-dependent solutions in the presence of gain. The resolution of this discrepancy is the same for the three equations.

The time-dependent and the frequency-dependent free K-G equations are given below,

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi(t,r)=\left(\frac{mc}{\hbar}\right)^2\psi(t,r), \tag{3}$$

$$\left(\nabla^2+\frac{\mathcal{E}^2}{\hbar^2c^2}\right)\psi_0(r)=\left(\frac{mc}{\hbar}\right)^2\psi_0(r), \tag{4}$$

where $\psi(t,r)=\psi_0(r)e^{-i\mathcal{E}t/\hbar}$, $\mathcal{E}$ is the relativistic energy, and m is the mass of the particle. In the massless limit, these equations look very similar to those above. We confine our discussion to the case where ($\mu=1$ and to problems in one dimension corresponding to the propagation of plane waves where Eq. (2) is written as

$$\left[\frac{d^2}{dx^2}+\frac{\omega^2}{c^2}\varepsilon(\omega,x)\right]F_0(x)=0. \tag{2$'$}$$

Now the effects of the property of the medium, which is contained in the complex function ε, could be incorporated in the one-dimensional version of Eq. (4) by an equivalent effect which is introduced in the form of a complex potential interaction as follows:[6,7]

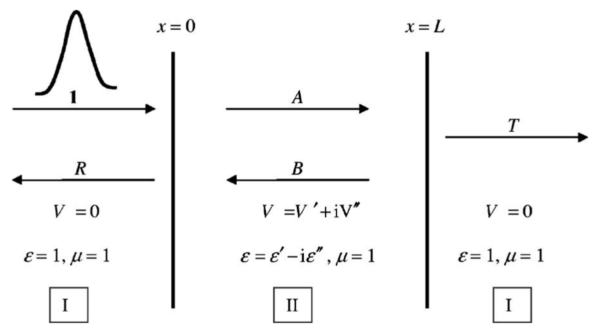

FIG. 1. Wave packet propagation through an amplifying system of length L and complex potential $V=V'+iV''$ (or dielectric constant $\varepsilon=\varepsilon'-i\varepsilon''$ for an em wave). T and R stand for the transmission and reflection amplitudes for a unit incident wave. A and B are the amplitudes of the forward and backward wave in the system and μ is the permeability of the medium.

$$\left\{\frac{d^2}{dx^2}+\frac{1}{\hbar^2c^2}[\mathcal{E}-V(x)]^2\right\}\psi_0(x)=\left(\frac{mc}{\hbar}\right)^2\psi_0(x), \quad (5)$$

where V is the time component of a vector potential whose space component is taken to vanish.

Now, for a given medium specified by $\varepsilon(\omega,x)$ and boundary conditions, we choose the complex potential $V(x)$ in Eq. (5) with $m=0$ that will give the same electromagnetic wave equation, Eq. (2′). The potential obtained by this equivalence requirement will be our guide in writing the nonrelativistic Schrödinger equation that gives the quantum mechanical analogue of the wave propagation equation.

Let us consider now the system shown in Fig. 1 where medium I is free space and the waves are incident from left. The constant parameters $\{\varepsilon',\varepsilon'',V',V''\}$ are real. Equation (2′) in medium II gives

$$\left[\frac{d^2}{dx^2}+\frac{\omega^2}{c^2}(\varepsilon'-i\varepsilon'')\right]F_{II}(x)=0. \quad (6)$$

Notice that as a result of the general relations of $\mathrm{Re}(\varepsilon)$ being an even function of $\mathrm{Re}(\omega)$, while $\mathrm{Im}(\varepsilon)$ being an odd function of $\mathrm{Re}(\omega)$, we have for a dispersionless medium with gain allowing propagation that $\mathrm{Re}(\varepsilon)$ is positive for all frequencies, $\mathrm{Im}(\varepsilon)$ is negative for positive $\mathrm{Re}(\omega)$, and positive for negative $\mathrm{Re}(\omega)$; for a lossy medium it is the other way around. Now, Eq. (5) with $m=0$ and $\mathcal{E}=\hbar\omega$, gives

$$\left[\frac{d^2}{dx^2}+\frac{\omega^2}{c^2}\left(1-2\frac{1}{\hbar\omega}V+\frac{1}{\hbar^2\omega^2}V^2\right)\right]\psi_{II}(x)=0. \quad (7)$$

Comparing these two equations, we obtain

$$\varepsilon'=1-2\frac{1}{\hbar\omega}V'+\frac{1}{\hbar^2\omega^2}(V'^2-V''^2), \quad (8a)$$

$$\varepsilon''=2\frac{1}{\hbar\omega}V''-2\frac{1}{\hbar^2\omega^2}V'V''. \quad (8b)$$

Now, if the medium is assumed to be nondispersive (i.e., the permittivity is independent of the frequency ω), then we conclude that the constant potential V should be proportional to ω. In other words, the vector potential in the K-G equation is *energy dependent*. It should be proportional to $\mathcal{E}$. Consequently, we write it as $V\equiv\mathcal{E}v$, where v is a dimensionless parameter. Thus, we can now write our previous Eqs. (8) as

$$\varepsilon'=1-2v'+v'^2-v''^2, \quad (9a)$$

$$\varepsilon''=2v''(1-v'). \quad (9b)$$

Therefore, v' and v'' are determined in terms of the parameters ε' and ε'' as follows:

$$(1-v')^2=\frac{1}{2}(\varepsilon'+\sqrt{\varepsilon'^2+\varepsilon''^2}), \quad (10a)$$

$$v''=\frac{\varepsilon''/2}{1-v'}, \quad (10b)$$

where for a propagating medium with gain $\mathrm{Re}(1-v)$ is positive and $\mathrm{Im}(1-v)$ has the same sign as $\mathrm{Re}(k)$. We use the ansatz that $V=\mathcal{E}[v'(\varepsilon',\varepsilon'')+iv''(\varepsilon',\varepsilon'')]$ as a guide for our nonrelativistic problem. That is, in our investigation of the wave-packet propagation through the quantum mechanically equivalent system, we take the potential in the Schrödinger equation as $V=E(v'+iv'')$, where E is the nonrelativistic energy and $\{v',v''\}$ are real potential parameters. In Sec. IV, we will also show that this ansatz is supported numerically.

III. POLES OF THE REFLECTION COEFFICIENT

In Sec. II, an equivalence was obtained between two representations of the wave equation. One comes from the electromagnetic wave equation and the other comes from the massless limit of the K-G equation with vector potential that embodies the dielectric property of the medium. A necessary condition for the equivalence is that the potential in the K-G equation should be energy dependent. For a dispersionless electromagnetic medium, it had to be proportional to the energy. On the other hand, the permittivity could assume complex values and so could the potential. The nonrelativistic mechanical representation of the system is described by the Schrödinger equation with the same energy-dependent potential. In this section, we study the solution of this Schrödinger equation and calculate the resonance energies and possibly the discrete eigenenergies by locating the poles of the reflected or transmitted amplitude.

The problem under study is depicted in Fig. 1 where a wave packet is incident from the left with partial amplitude normalized to unity. Medium I is free space and medium II is a uniform dielectric material whose properties are represented by the uniform potential V in the following one-dimensional time-independent Schrödinger equation:

$$\left[\frac{d^2}{dx^2}+\frac{2m}{\hbar^2}(E-V)\right]\psi(x)=0, \quad (11)$$

where m is the inertial mass associated with the wave packet and E is the nonrelativistic energy. The potential is energy dependent and is written as $V=Ev$, where v is a dimensionless parameter which is generally complex. Note that in the

K-G picture, we used $\mathcal{E}=\hbar\omega$ as equivalence between the relativistic energy in the massless K-G equation and the frequency in the electromagnetic wave equation. However, this correspondence no longer holds in the case of the Schrödinger equation; indeed, by comparing Eq. (11) (after setting $V=Ev$) with Eq. (6), we see that $(2m/\hbar^2)E$ correspond to ω^2/c^2 and $1-v$ to ε. Hence the wave number k in region I of Fig. 1 is given by $\sqrt{2mE/\hbar^2}$ for the case of Eq. (11), while $k=\omega/c$ for the case of Eq. (6).

By introducing k, both Eqs. (6) and (11) take the form

$$\left[\frac{d^2}{dx^2}+k^2(1-v)\right]\psi(x)=0. \tag{12}$$

Its solution for the configuration shown in Fig. 1 is

$$\psi(x)=\begin{cases} e^{ikx}+Re^{-ikx} & x<0 \\ Ae^{ik\sqrt{1-v}x}+Be^{-ik\sqrt{1-v}x} & 0<x<L \\ Te^{ikx} & x>L \end{cases}. \tag{13}$$

To find the full time-dependent solution, one must multiply (13) by $\exp(-iEt/\hbar)$ (for the Schrödinger case) or by $\exp(-i\omega t)$ (for the em case). Notice that for real k, $E\geqslant 0$, while ω can be either positive or negative depending on the sign of k.

If the incident wave packet $\psi(x\to-\infty)$ is constructed as usual from the partial amplitudes e^{ikx} using the Fourier expansion $1/2\pi\int_{-\infty}^{+\infty}f(k)e^{ikx}dk$ for a given choice of momentum distribution $f(k)$, the results are not the same as the solution of the time-dependent Schrödinger equation. Anyway, matching the wave function and its gradient at the left boundary $(x=0)$ and right boundary $(x=L)$ we find an expression for the reflection amplitude R. It could be written as

$$R=v\frac{e^{2i\eta\sqrt{1-v}}-1}{\alpha_-^2e^{2i\eta\sqrt{1-v}}-\alpha_+^2}, \tag{14a}$$

where $\alpha_\pm=\sqrt{1-v}\pm1$ and the dimensionless momentum parameter η is defined as $\eta=kL$. If one considers the wave equation Eq. (6) and solves along the same lines for the reflection coefficient, the result will be the expected one from the correspondence mentioned before

$$R=(\varepsilon-1)\frac{e^{2i\lambda\sqrt{\varepsilon}}-1}{\beta_+^2-\beta_-^2e^{2i\lambda\sqrt{\varepsilon}}}, \tag{14b}$$

where $\beta_\pm=\sqrt{\varepsilon}\pm1$ and the dimensionless parameter λ is defined by $\lambda=kL=L/c\omega$. The similarity between the Schrödinger equation and wave equation results (14) is very apparent under the substitution $1-v\to\varepsilon$. Thus, the parallelism between the wave equation and the corresponding Schrödinger equation is well established in our formalism. Consequently all results that will follow apply equally well to both systems under the prescribed transformation between potential and dielectric constants.

Our interest is focused on the conditions under which the reflection becomes infinite. This occurs when the denominator of Eqs. (14a) and (14b) become zero, i.e.,

$$(\alpha_+/\alpha_-)^2=e^{2i\eta\sqrt{1-v}}. \tag{15}$$

The zeros of R, other than $v=0$, are also interesting but will not be pursued here. When R becomes infinite so does T. Then, Eq. (12) admits a solution $\psi(x)$ of the form $\exp(-ikx)$ for $x<0$, $A\exp(ik\sqrt{1-v}x)+B\exp(-ik\sqrt{1-v}x)$ for $0<x<L$, and $T\exp(ikx)$ for $x>0$; if Eq. (15) is satisfied, this solution without the presence of an incident wave satisfies the continuity conditions at $x=0$ and $x=L$; furthermore, if $\operatorname{Im} k>0$, $\psi(x)\to0$ as $|x|\to\infty$, which means that it is a true bound eigenstate with eigenvalue k^2. We shall see that both Eq. (15) and $\operatorname{Im} k>0$ can be realized if there is gain, while it is impossible for a lossy system. Therefore, we take the momentum parameter η to be complex and write it as $\eta=\eta'+i\eta''$. Additionally, we write $\sqrt{1-v}=\gamma'+i\gamma''$ and, hence,

$$\frac{\alpha_+}{\alpha_-}=\frac{\gamma'+1+i\gamma''}{\gamma'-1+i\gamma''}\equiv\chi e^{i\phi}, \tag{16}$$

where for a propagating medium with gain, γ' is positive and γ'' has opposite sign than that of $\operatorname{Re}(k)$. We obtain the following solution for (15):

$$\eta''=-\frac{\gamma''}{\gamma'}\eta'-\frac{1}{\gamma'}\ln\chi, \tag{17a}$$

$$\eta'=\frac{\phi+n\pi-\dfrac{\gamma''}{\gamma'}\ln\chi}{\gamma'[1+(\gamma''/\gamma')^2]}, \tag{17b}$$

where $n=0,\pm1,\pm2,\ldots$ which comes from $2n\pi$ difference in the phase of the two complex numbers on both sides of Eq. (15); however, n is restricted by the condition $\gamma''\eta'<0$. Therefore, the momentum poles of the reflection amplitude is indexed by the integer n as $k_n=k_n'+ik_n'$, where $k_n=\eta_n/L$. These poles correspond to true discrete eigenfrequencies or eigenenergies only if $\operatorname{Im} k_n\geqslant0$. When this condition is satisfied, the corresponding eigenstates are true bound states since then $|\psi_n(x)|\propto\exp(-k_n'|x|)$ as $|x|\to\infty$. For the em case, $\operatorname{Im} k_n\geqslant0$ implies $\operatorname{Im}\omega_n\geqslant0$, which in turn means that these eigenstates grow exponentially with time. It is this growing with time for these discrete eigenstates which provides the amplification of the wave packet beyond the critical value of L at which the transmission starts decreasing and thus resolves the discrepancy between the time and frequency approaches in systems with gain. Neither the growing with time of the eigenstates nor the nonreality of the eigenfrequencies are unexpected, since the Hamiltonian is not Hermitian (because $\operatorname{Im}\varepsilon\neq0$); we remind the reader that it is the Hermitian character which imposes the reality of the eigenfrequencies and the stationarity of the corresponding eigenstates. We also point out that besides the poles with $\operatorname{Im} k_n\geqslant0$, there are poles with $\operatorname{Im} k_n\leqslant0$; these poles do not correspond to eigenstates, since the associated solutions of Eq. (6) (with no incident wave) blow up as $|x|\to\infty$. However, these $\operatorname{Im} k_n<0$ poles may show up as sharp peaks in R and T (*resonances*) for $\omega=\operatorname{Re}\,\omega_n=c\operatorname{Re} k_n$ if $|\operatorname{Im} k_n|$ is very small.

The Schrödinger case requires some extra care. First, the discrete, complex eigenenergies E_n are not simply proportional to k_n as in the em case, but are given by

$$E_n = \frac{\hbar^2}{2m} k_n^2 = \frac{\hbar^2}{2mL^2} \eta_n^2 = \frac{\hbar^2}{2mL^2} (\eta_n'^2 - \eta_n''^2 + 2i \eta_n' \eta_n'').$$

Since the poles in the η plane are located symmetrically with respect to the Im η axis, E_n are located in pairs symmetrically to the Re E_n axis. Those with $\eta_n'' > 0$ correspond to true bound eigenstates [$\psi_n(x) \to 0$ for $|x| \to \infty$]; the subset with $\eta_n' > 0$ would give Im $E_n > 0$, and, hence, growing with time eigenstates, while the subset with $\eta_n' < 0$ would give Im $E_n < 0$ and, consequently, decaying with time eigenstates. Finally, the ones with $\eta_n'' < 0$ could produce, under some circumstances, sharp peaks in R and T vs E (resonances).

A second point, concerning the Schrödinger case is a requirement of self-consistency. For a pole E_n to be realized the choice of the potential $V \equiv Ev$ must be such as to produce one E_n satisfying the self-consistency requirement $E_n = E$; if E, v', v'', L are chosen randomly, the relation $E_n = E$ will not be satisfied no matter what n is. If we change only L, again this self-consistency relation is not expected to be satisfied, although for a certain length L, E_n may be very close to E [producing a resonance behavior in $T(E)$ or $R(E)$].

Before we close this section, it will be beneficial to digress on potential physical examples of such energy-dependent potentials for nonrelativistic particles. As pointed out by Formanek *et al.*,[8] that wave equations with energy-dependent potentials have been known for a long time.[9] They appeared along with the Klein-Gordon equation for a particle in an external electromagnetic field. In nonrelativistic quantum mechanics, they arise from momentum-dependent interactions, as shown by Green.[10] The Pauli-Schrödinger equation represents another example.[11,12] The one-dimensional Schrödinger equation with a potential $EV(x)$ proportional to energy was studied in Ref. 13. This equation is equivalent to the wave equation with variable speed. When $V(x) < 1$ is bounded from below, and satisfies two integrability conditions, the scattering matrix is obtained and its asymptotics for small and large energies are established. However, Hamiltonians with energy-dependent potentials sometimes contain unphysical bound states.[14] An important issue in nuclear physics is the removal of these unphysical states and this was considered first by Fiedeldey *et al.*[15]

Another example of an extra energy dependence (besides the k dependence) in the matrix elements of the Hamiltonian appears in the augmented plane wave method. There, this complicated energy dependence is due to the choice of the basis rather than the original Hamiltonian as in our case. However, from a mathematical point of view, the two cases share common features.

IV. NUMERICAL RESULTS AND DISCUSSIONS

In all our numerical results with the Schrödinger equation, we use the units for which $\hbar = 2m = 1$ and corresponds to an energy unit $E = 0.658$ eV, unit of length $L_0 = 2.406$ Å, and a time unit corresponding to $T = 10^{-15}$ s. In Fig. 2, we show the numerical results for the poles of the transmission coefficient in the complex ω plane as obtained from the stationary electromagnetic wave equation for a system of length L and $\varepsilon = 4 - 0.2i$ which is valid for Re(k) positive. This figure compares quite well with Fig. 3, which corresponds to the stationary Schrödinger equation with a linearly energy-dependent potential. Even though we have presented, in Sec. III, a lucid argument that supports our assertion of linear energy dependence of the potential in the Schrödinger equation, we have also checked numerically for a few other energy dependences that only the linear energy dependence of the potential makes the pole structure of the electromagnetic wave equation similar to the Schrödinger equation. This numerical result strengthens our previous assertion that the Schrödinger equation problem will map onto the wave equation problem only if the potential is linearly dependent on the energy. In Fig. 4, we plot the transmission vs the imaginary part of the potential to detect the critical value of gain at which the transmission is maximum for a given system length of 200 in our units and incident energy of 1.209. From this figure, we see that the resonance occurs at $v'' \approx 0.023$. In Fig. 5, we show the pole that has a real part equal (or close) to the incident energy of the wave packet, in our case $E = 1.209$ (in our energy units), and study the behavior of this pole as we increase the imaginary part of the potential in the complex E plane as shown in Fig. 5. It is clear from Fig. 5 that, in accordance with our expectations, this important energy pole does cross the real axis at the critical value of the imaginary potential estimated to be $v'' \approx 0.023$ above which the energy pole moves into the upper half of the complex E plane and consequently causes amplification with time. To support this interpretation, we perform the numerical computation of the transmission coefficient from the time-dependent Schrödinger equation for values of the imaginary potential below, above, and at the critical value as shown in Fig. 6. The time-dependent Schrödinger equation is being derived from the stationary equation with a linearly dependent potential by substituting E by its operator form $i\hbar\, \partial / \partial t$ which gives

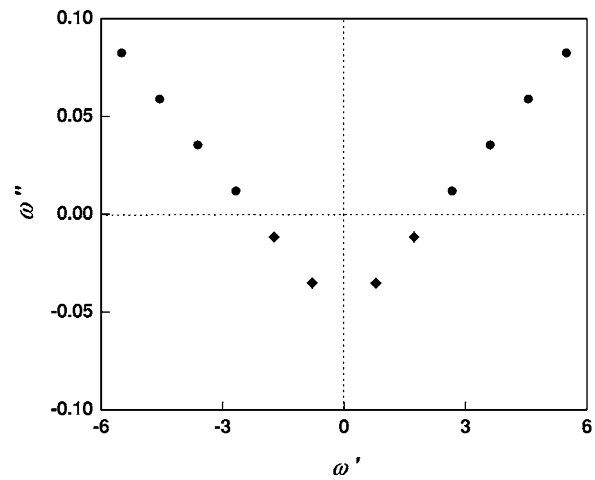

FIG. 2. The resonance poles (diamonds) and the discrete bound eigenstate poles (filled dots) of the transmittance obtained from the time-independent classical wave equation in the complex ω plane with $L = 10$ and $\varepsilon = 4 - 0.2i$.

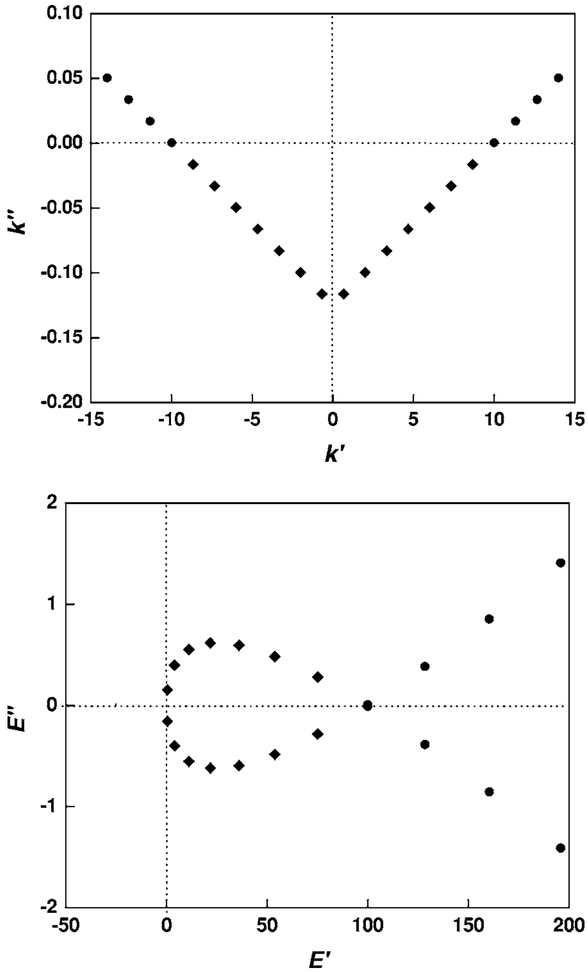

FIG. 3. The resonance poles (diamonds) and the discrete bound eigenstate poles (filled circles) of the transmittance obtained from the time-independent Schrödinger equation in the complex k (a) and E (b) planes for $L=10$ and for a potential of the form $V=(-1+0.05i)E$.

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+i\hbar v\frac{\partial}{\partial t}\right]\psi(x,t)=i\hbar\frac{\partial\psi(x,t)}{\partial t}. \qquad (18)$$

The initial wave form used in generating Fig. 6 was a Gaussian wave packet of the form

$$\psi(x,0)=\exp-\left[\frac{(x-x_0)^2}{4\sigma^2}\right]e^{ik_0x}, \qquad (19)$$

centered at x_0 with an average momentum of k_0 in our units, the normalization constant in (19) does not affect our numerical computations. In all cases, we used $\sigma=40.0$ units of length. The transmission coefficient was calculated by

$$|T(t)|^2=\int_L^\infty|\psi(x,t)|^2dx, \qquad (20)$$

where L is the system length. It is very clear from Fig. 6 that at values of the gain below the critical value, the transmission reaches a stationary state at large times, while at values of the gain above the critical value, the transmission increase steadily but step-like with time leading to an amplified output.

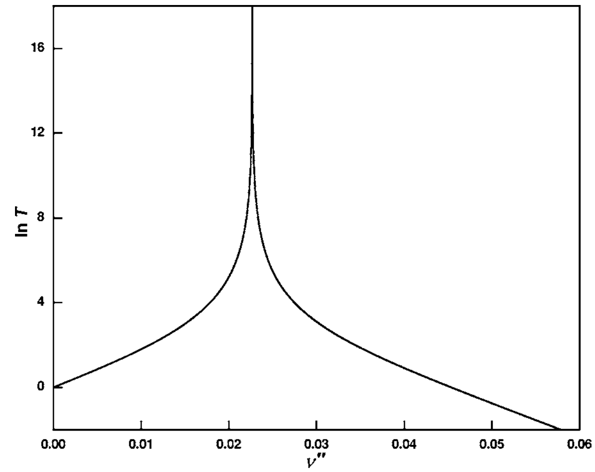

FIG. 4. The transmission coefficient obtained from the time-independent Schrödinger equation vs v'' for $E=1.209$, $L=200$, and $v=-1+iv''$. The figure shows an extremely sharp peak at $v''_c=0.022\,67$.

V. CONCLUDING REMARKS

We believe that our present work sheds light on the real origin of the discrepancy between time-domain and frequency-domain solutions of quantum and classical systems described by linear equations with gain, regardless of whether they are em, Klein-Gordon, or Schrödinger equations. The key idea to the removal of this apparent discrepancy is the existence of time-growing true-bound eigenstates in the upper half of the complex energy and frequency planes due to the non-Hermitian nature of the Hamiltonian with gain. While studying the origin of this problem, we succeeded in coming up with a mathematical mapping between the Schrödinger equation problem for a massive particle subject to a non-Hermitian potential and the electromagnetic

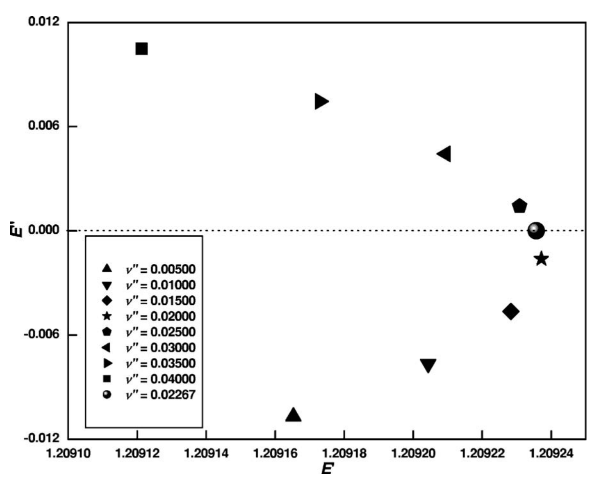

FIG. 5. The resonance pole location in the complex E plane, $L=200$, and $v=-1+iv''$ as v'' increases. The pole crosses the real axis when $v''_c=0.022\,67$.

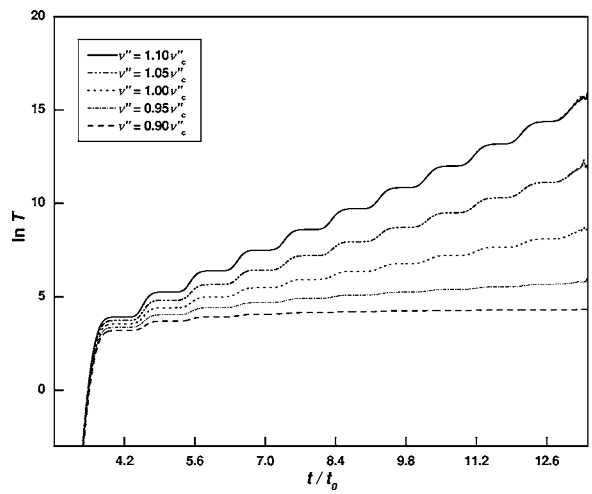

FIG. 6. The transmission coefficient obtained from the time-dependent Schrödinger equation for five different values of v'' vs time with $E=1.209$, $L=200$, and $V=(-1+iv'')E$, where $t_0=143$ time units.

wave propagation problem in a gain system. The equivalent Schrödinger equation problem resulted in a linearly energy-dependent complex potential. All numerical results of different physical quantities supported our ansatz about the similarity between the electronic and photonic systems. Setting this issue aside, we have computed the locations of the poles of the reflection and transmission amplitudes in the complex k, frequency, or energy plane. The poles cross the real axis when the length L (or, equivalently, the gain) reaches its critical value. This crossing signals the transfer of the amplification from $\mathrm{Re}(k)$ continuous spectrum to the time-growing bound eigenstates. Even though our present study is very informative and can be considered as being a satisfactory explanation of the paradoxial results between time-dependent and stationary evolution equation, a complete treatment of wave propagation in gain media can be achieved, in our view, by constructing the time-dependent solution from the whole spectrum of the time-independent solutions. Thus, both the discrete and continuous spectra should be included in the analysis similar to the approach of Hammer *et al.*[16] in dealing with the general solution of the Schrödinger equation. The general solution in this case reads

$$\psi(x,t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} f(\omega) g(x,\omega) e^{-i\omega t} + \sum_n c_n g_n(x,\omega_n) e^{-i\omega_n t}$$

$$= \oint \frac{dz}{2\pi} f(z) g(x,z) e^{-izt}, \qquad (21)$$

where $f(\omega)$ and c_n coefficients are determined from the initial value of $\psi(x,0)=h(x)$. In the last integral, one has to choose a complex contour that will reflect the physical situation at hand. Keeping in mind that $g(x,\omega)$ and $g_n(x,\omega_n)$, i.e., the solutions of the stationary equation, are not orthonormal to each other because the operator is not Hermitian (due to the complex nature of ε), extra care and effort are needed in order to determine analytically or numerically $f(\omega)$ and c_n given $h(z)$.

The numerical results presented in this work constitute an important intermediate step toward a complete resolution of this interesting problem. Notice that the analytical determination of the poles in the transmission (or reflection) amplitudes, given by Eq. (17), was greatly facilitated as a result of assuming a dispersionless permittivity. Actually, the dielectric constant becomes frequency dependent, and this is the real physical situation, since one is dealing with lasing materials and their gain response or lasing action is certainly limited to a finite frequency range; hence, $\varepsilon(\omega)=\varepsilon'(\omega)+i\varepsilon''(\omega)$. One possible causal form of this dielectric constant is given by Ref. 17,

$$\varepsilon(\omega) = \varepsilon_0 + \frac{f_0}{\omega^2{}_0 - \omega^2 + i\Gamma\omega}, \qquad (22)$$

where f_0 is usually small compared to ε_0, ω_0 is the laser resonant frequency, and Γ represents the width of the response and is related to the lifetime of the lasing mode. All these parameters can be fixed experimentally for a given lasing system. We are planning, in the near future, to implement numerically this dispersive case and trace the path of the main lasing pole in the complex frequency plane.

We conclude by pointing out that, while the time-independent em and Schrödinger equations are equivalent under proper correspondences, the same is not true for their time-dependent counterparts. The Schrödinger equation is first order in time, while the em equation is second order in time. As a result of this, the Schrödinger wave packet increases its width linearly with time as it propagates in free space, while the em wave packet retains its shape. This important difference implies that the Schrödinger wave packet, in contrast to the em case, never leaves the gain area completely and, hence, is amplified without limit as shown in Fig. 6.

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[1] A. Z. Genack and D. M. Drack, Nature (London) **368**, 400 (1994); D. S. Wiersma, M. P. van Albada, and A. Lagendijk, Phys. Rev. Lett. **75**, 1739 (1995).
[2] V. Freilikher, M. Pustilnik, and I. Yurkevich, Phys. Rev. Lett. **73**, 810 (1994); C. W. J. Beenakker, J. C. J. Paasschens, and P. W. Brouwer, Phys. Rev. Lett. **76**, 1368 (1996); N. Zekri, H. Bahlouli, and A. Sen, J. Phys.: Condens. Matter **10**, 2405 (1998); P. Pradhan and N. Kumar, Phys. Rev. B **50**, R9644 (1994).
[3] Z. Q. Zhang, Phys. Rev. B **52**, 7960 (1995).
[4] Ma Xiaozheng and C. M. Soukoulis, Physica B **296**, 107 (2001).
[5] X. Jiang, Q. Li, and C. M. Soukoulis, Phys. Rev. B **59**, R9007 (1999); Qiming Li, K. H. Ho, and C. M. Soukoulis, Physica B **296**, 78 (2001).
[6] See, for example, W. Greiner, *Relativistic Quantum Mechanics: Wave Equation*, 3rd ed. (Springer, Berlin, 2000); J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).
[7] C. Y. Chen, D. S. Sun, and F. L. Lu, Phys. Lett. A **330**, 424 (2004); C. Y. Chen, C. L. Liu, F. L. Lu, and D. S. Sun, Acta Phys. Sin. **52**, 1579 (2003).
[8] J. Formanek, R. J. Lombard, and J. Mares, Czech. J. Phys. **54**, 289 (2004).
[9] H. Snyder and J. Weinberg, Phys. Rev. **57**, 307 (1940); L. I. Schiff, H. Snyder, and J. Weinberg, Phys. Rev. **57**, 307 (1940).
[10] A. M. Green, Nucl. Phys. **33**, 218 (1962).
[11] W. Pauli, Z. Physik **43**, 601 (1927).
[12] H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One- and Two-Electron Systems* (Springer-Verlag, Berlin, 1957).
[13] T. Aktosun and C. van der Mee, J. Math. Phys. **32**, 2786 (1991).
[14] J. M. Sparenberg, D. Baye, and H. Leeb, Phys. Rev. C **61**, 024605 (2000).
[15] H. Fiedeldy, S. A. Sofianos, A. Papastylianos, K. Amos, and L. J. Allen, Phys. Rev. C **42**, 411 (1990).
[16] C. L. Hammer, T. A. Weber, and V. S. Zidell, Am. J. Phys. **45**, 933 (1977).
[17] J. D. Jackson, *Classical Electrodynamics*, 3rd ed. (John Wiley & Sons, Inc., New York, 1999).