**KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS**
**PHYSICS DEPARTMENT**

**PHYS-430: Thermal Physics**
**Term 061**
**EXAM # 3**

**Name: solution ID#**

**Please Solve all the Problems.**

**Problem 1.**

Part A.

Real (non-ideal) gases with interatomic interactions are often described by the van der Waals model:

$$P = \frac{nRT}{V - na} - \frac{bn^2}{V^2}$$

$$E = nc_V T - \frac{bn^2}{V}$$

a. Find the entropy as a function of T and V
b. Find the equation of state for an adiabatic process.

Solution:

$$a)\ \ dS = \frac{1}{T}dE + \frac{P}{T}dV = \frac{1}{T}\left(nc_v dT + \frac{bn^2}{V^2}dV\right) + \frac{1}{T}\left(\frac{nRT}{V-na} - \frac{bn^2}{V^2}\right)dV = \frac{nc_v dT}{T} + \frac{nRdV}{V-na}$$

$$S = S_0 + \int_{T_0}^{T}\left(\frac{\partial S}{\partial T}\right)_V dT + \int_{V_0}^{V}\left(\frac{\partial S}{\partial V}\right)_T dV = S_0 + n\left[c_v \ln\left(\frac{T}{T_0}\right) + R\ln\left(\frac{V-na}{V_0-na}\right)\right]$$

$b)$ Equation of an adiabatic process (S=0)

$$S = cons = \left[\ln\left(\frac{T}{T_0}\right)^{c_v} + \ln\left(\frac{V-na}{V_0-na}\right)^{R}\right] = \ln\left[T\left(V-na\right)\right]$$

*or*

$$\left(P + \frac{bn^2}{V^2}\right)\left(V-na\right)^{\alpha} = const$$

$$\alpha = 1 + \frac{R}{c_V}$$

Part B

Helmholtz Free energy of a certain system has the following form

$$F = V\left[kTn(\ln n - 1) - \frac{3knT}{2}\log T + \frac{an^2}{2} + \frac{bn^3}{3}\right]$$

$$n = N/V$$

a. Derive equation of state for this system
b. Express the internal energy as a function of V, T and n.

Solution:

$$a)\ \frac{\partial}{\partial V} n =$$

$$P = -\left(\frac{\partial F}{\partial V}\right)_{T,N} = nkT + \frac{an^2}{2} + \frac{2bn^3}{3}$$

$$b)\ S = -\left(\frac{\partial F}{\partial T}\right)_{V,N} = V\left[-kn(\ln n - 1) + \frac{3ka}{2}(\log T - 1)\right]$$

$$E = F + TS = V\left(\frac{an^2}{2} + \frac{bn^3}{3} + \frac{3nkT}{2}\right)$$

**Problem 2.**

Part A.

Find the specific heat of a monatomic ideal gas (N particles) moving in a gravitational field.

Solution:

For the ideal gas with out the gravitational field $U = \frac{3}{2}NkT$

The effect of the gravitational filed

$$U_g = N < mgh >= N \frac{\int_0^\infty mgh \; e^{-mgh/kT}}{\int_0^\infty e^{-mgh/kT}} = NkT$$

the total energy = $\frac{5}{2}NkT$

$$C = \frac{5}{2}Nk$$

Part B.

A system of three interacting spins $(s_i = \pm 1)$ is described by the hamiltonian

$H = J(s_1 s_2 + s_2 s_3 + s_1 s_3)$

Find the heat capacity of the system.

Make your comments on the specific heat at the low and high temperature limits.

Solution:

If you count the total number of possible arrangements (state) for these three spins you end up with 8 states, 2 of them with energy 3J and 6 states with energy –J.

$$Z = 2e^{-3\beta J} + 6e^{\beta J}$$

$$U = 3J\left(\frac{4}{3+e^{-4\beta J}} - 1\right)$$

$$C = \frac{3e^{-4\beta J}}{\left(3+e^{-4\beta J}\right)^2}\left(4\beta J\right)^2$$

$$T \to 0: E \to J, C \to \frac{(4J\beta)^2}{3} e^{-4\beta J}$$

$$T \to \infty: U \to 3J^2 / kT, C \to 3\left(J / kT\right)^2$$

**Problem 3**

Suppose that the multiplicity function for a system of energy U, volume V and number of particles N is

$\Omega(U, N, V) = cU^{\alpha N} V^{\beta N}$

$C, \alpha$ and $\beta$ are constants

a. What are the temperature and pressure of the system
b. If the system was an ideal gas, what can you say about the meaning of the above constants?

Solution:

$a)\ S = k \ln\Omega = \ln c + \alpha N \ln U + \beta N \ln V$

$$\frac{1}{T} = \frac{\alpha k N}{U} \Rightarrow kT = \frac{U}{\alpha N}$$

$$\frac{P}{T} = \frac{\beta k N}{V} \Rightarrow PV = \frac{\beta U}{\alpha}$$

$b)$ For the ideal gas $PV = NkT = \dfrac{\beta U}{\alpha} \Rightarrow \beta = 1$

Also for the ideal gas $U = \dfrac{3}{2} NkT \Rightarrow \alpha = \dfrac{3}{2}$

**Problem 4.**

The vapor pressure (in mm Hg) of solid ammonia is given by

$$\ln P = 23.03 - 3754/T$$

and that of liquid ammonia

$$\ln P = 19.49 - 3063/T$$

a. What is the temperature of the triplet point
b. What are the latest heats (per mole) of sublimation (solid to gas) and vaporization (liquid to gas) at the triple point?
 Hint: You may assume that the volume of liquid and that of the solid are negligible compared to that of the gas.

Solution:

a) At the triple point the pressure and the temperature for all phases are equal. Equating the pressure in the above equations gives $T_{tr}$ =195 K.
b) Using the Clausius-Calpeyron equation

$$\frac{dP}{dT} = \frac{L}{\Delta V} \approx \frac{L}{V_g}$$

For the ideal gas with n=1 $PV = RT \Rightarrow \frac{dP}{dT} = \frac{LP}{RT^2}$

For the soild-gas latent heat, use the firts equation

and differentiate to get

$$\frac{dP}{dT} = e^{23.03-3754/T} \frac{3754}{T^2} = P\frac{3754}{T^2} = \frac{LP}{RT^2} \Rightarrow L = 31.2kJ$$

similarly for the liquid-gas latent heat L=25.5kJ

**Problem 5**

You are given a system of two identical particles which may occupy any of three energy levels

$\varepsilon_n = n\varepsilon, \quad n = 0,1,2.$

The lowest energy state, $\varepsilon_0$ is doubly degenerate. The system is in thermal equilibrium at temperature T. For each of the following cases determine the partition function and the energy.

a. The particles obey Fermi-Dirac statistics
b. The particles obey Bose-Einstein statistics
c. The particles (now distinguishable) obey Boltzmann statistics

Solution

$a)\ Z = 1 + e^{-\beta\varepsilon} + e^{-3\beta\varepsilon}(1 + 2e^{\beta\varepsilon})$

$$E = \frac{\varepsilon e^{-\beta\varepsilon}(2 + 4e^{-\beta\varepsilon} + 3e^{-2\beta\varepsilon})}{\left(1 + e^{-\beta\varepsilon} + e^{-3\beta\varepsilon}(1 + 2e^{\beta\varepsilon})\right)}$$

b) $Z = 3 + 2e^{-\beta\varepsilon} + 3e^{-2\beta\varepsilon} + e^{-3\beta\varepsilon} + e^{-4\beta\varepsilon}$

$$E = \frac{\varepsilon}{Z} e^{-\beta\varepsilon}(2 + 6e^{-\beta\varepsilon} + 3e^{-2\beta\varepsilon} + 4e^{-3\beta\varepsilon})$$

c) $Z = 4 + 4e^{-\beta\varepsilon} + 5e^{-2\beta\varepsilon} + 2e^{-3\beta\varepsilon} + e^{-4\beta\varepsilon}$

$$E = \frac{2\varepsilon}{Z} e^{-\beta\varepsilon}(2 + 5e^{-\beta\varepsilon} + 3e^{-2\beta\varepsilon} + 2e^{-3\beta\varepsilon})$$

**Problem 6.**

A. In a Bose-Einstein condensation experiment, $10^7$ Rb-87 atoms were cooled down to T=200 nK. The atoms were confined to a volume of $10^{-15}$ $m^3$. Find the BEC temperature. Also Determine the number of atoms in the ground state at T=200 nK.

Solution

$$a)\ T_C = 0.527\left(\frac{h^2}{2\pi mk}\right)\left(\frac{N}{V}\right)^{2/3}$$

$$T_C = 0.527\left(\frac{\left(6.626\times10^{-34}\right)^2}{2\pi(87\times1.67\times10^{-27})(1.38\times10^{-23})}\right)\left(\frac{10^7}{10^{-15}}\right)^{2/3} = 8.5\times10^{-6}K$$

$$\text{b)}\ N_0 = \left[1-\left(\frac{T}{T_C}\right)^{3/2}\right]N = 9.96\times10^6$$

B. Consider the collapse of the Sun into a white dwarf. For the sun,

$M = 2\times10^{30} Kg$

$R = 7\times10^{8} m$

$V = 1.4\times10^{27} m^3$

a. Calculate the Fermi energy for the sun.
b. What is the Fermi temperature?

Solution

a) $\varepsilon_F = \frac{h^2}{8m}\left(\frac{3N}{\pi V}\right)^{2/3}$

To find $\frac{N}{V}$, as in problem 7.23, we assume that the star contains one proton and one neutron for each electron.

No of nucleaons = $\frac{\text{mass of sun}}{\text{mass of nucleon}} = \frac{2\times10^{30}}{1.67\times10^{-27}} = 1.2\times10^{57}$

No of electrons is half this number = $6\times10^{56}$

$\varepsilon_F = 20.8$ eV

b) $T_F = 2.4\times10^{5} K$