KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS
PHYSICS DEPARTMENT

PHYS-430: Thermal Physics
Term 061
EXAM # 2

Name: ______________________________ **ID#** ______________
Please Solve all the Problems.

Problem 1.

A useful way to cool ^3He is to apply pressure P at a sufficiently low temperature T at a coexisting liquid-solid mixture. Describe qualitatively how this works on the basis of the following assumptions that the entropy of the solid comes entirely from the disorder associated with the nuclear spin (s=1/2) and the entropy of the liquid is $S_L = \gamma RT$, with $\gamma = 4.6$ K^{-1}. In addition the molar volume of the liquid is greater than that of the solid.
(Hint: solve for the minimum temperature)

Solution:
Starting from the Calusius-Clapeyron equation

$$\frac{dP}{dT} = \frac{S_L - S_S}{V_L - V_S}$$

And assuming that the entropy of the solid phase with spin ½ $S_S = kN_A \ln 2$

Then the minimum T is the point at which the slope of P vs. T is zero or $\frac{dP}{dT} = 0$

Then $T_{\min} = \frac{\ln 2}{\gamma} = \frac{\ln 2}{4.6} = 0.15$ K.

Problem 2.

Consider a system of N_0 non-interacting quantum mechanical oscillators in equilibrium at temperature T. The energy levels of a single oscillator are
$E_m = (m + 1/2)\gamma / V$ with $m = 0, 1, 2, 3, ..$

(a) Find the energy and the specific heat as functions of T.
(b) Sketch the energy and the specific heat as a function of temperature.
(c) Determine the equation of state for the system.
(d) What is the fraction of particles in the m-th level.

Solution:

(a) $Z = \sum_{m=0}^{\infty} \exp[-\beta(m + 1/2)\gamma / V] = \dfrac{\exp(-\beta\gamma / 2V)}{1 - \exp(-\beta\gamma / V)} = \frac{1}{2}\operatorname{csc}h(\beta\gamma / 2V)$

$$U = -N_0 \frac{\partial}{\partial\beta} \ln Z = \frac{N_0\gamma}{2V} \coth \frac{\gamma\beta}{2V}$$

$$C = \left(\frac{dU}{dT}\right)_V = N_0 k \left(\frac{\gamma}{2VkT}\right)^2 \operatorname{csc}h^2\left(\frac{\gamma}{2VkT}\right)$$

(b) In the limit of low T

$$U \to \frac{N_0\gamma}{2V}$$

$$C \to 0$$

In the high T limit

$$U \to N_0 kT$$

$$C \to N_0 k$$

You can now make a rough plot for these two functions

(c) $P = \dfrac{N_0}{\beta} - \dfrac{\partial}{\partial V} \ln Z = \dfrac{N_0\gamma}{2V^2} \coth\left(\dfrac{\gamma}{2VkT}\right)$

$$(d)\ f_m = \frac{N_0}{Z} \exp[-\beta(m + 1/2)\gamma / V]$$

$$= 2N_0 \exp[-\beta(m + 1/2)\gamma / V] \sinh\left(\frac{\gamma\beta}{2V}\right)$$

Problem 3.

Find the pressure, entropy and specific heat of an ideal Boltzmann gas of indistinguishable particles in the relativistic limit where the energy of a particle is related to its momentum by $\varepsilon = cp$.

Solution:

$$z_1 = \frac{4\pi V}{(hc)^3}\int_0^\infty \varepsilon^2 e^{-\varepsilon/kT}\, d\varepsilon = \frac{8\pi V}{(hc)^3}(kT)^3$$

$$Z = \frac{z_1^{\,N}}{N\,!} = \frac{1}{N\,!}\left[\frac{8\pi V}{(hc)^3}\right]^N (kT)^{3N}$$

$$P = \frac{1}{\beta}\frac{\partial}{\partial V}\ln Z = \frac{NkT}{V}$$

$$S = k\left(\ln Z - \beta\frac{\partial}{\partial \beta}\ln Z\right) = Nk\left(3\ln kT + \ln\frac{8\pi V}{N\,(hc)^3} + 4\right)$$

$$U = \frac{\partial}{\partial \beta}\ln Z = 3NkT$$

$$C = 3Nk$$

Problem 4.

A gas of N particles is absorbed on a surface of area A, forming a two dimensional gas at a temperature T on the surface. The energy of the absorbed particle is $\varepsilon = \vec{p}^{\,2} / 2m + \varepsilon_0$, where $\vec{p} = (p_x, p_y)$ and ε_0 is the surface binding energy per particle. Assuming that the particles are distinguishable find the classical partition function and derive the thermodynamics quantities U, C, S and μ.

Solution

$$Z = A^N \left(\frac{2\pi mkT}{h^2} \right)^N e^{-N\varepsilon_0/kT}$$

$$F = -kT \ln Z$$

$$P = -\left(\frac{\partial F}{\partial A} \right)_{T,N} = \frac{NkT}{A}$$

$$U = -\frac{\partial}{\partial \beta} \ln Z = NkT + N\varepsilon_0$$

$$C = Nk$$

$$S = -\left(\frac{\partial F}{\partial T} \right)_{A,N} = kN \left[1 + \ln\left(\frac{2\pi AmkT}{h^2} \right) \right]$$

$$\mu = -kT \left[\ln A + \ln\left(\frac{2\pi mkT}{h^2} \right) - \frac{\varepsilon_0}{kT} \right]$$

Problem 5.

A system is composed of N one-dimensional classical oscillators. Assume that the potential for the oscillators contains a small anharmonic term,

$$V(x) = \frac{k_0}{2} x^2 + \alpha x^4$$

Where $\alpha \ll 1$. Find U and C in the high and low T limits.

Solution

$$Z = \frac{1}{h} \int_{-\infty}^{\infty} e^{-p^2/2mkT} dp \int_{-\infty}^{\infty} e^{-V(x)/kT} dx$$

$$= \frac{\sqrt{2\pi mkT}}{h} \int_{-\infty}^{\infty} e^{-k_0 x^2/2kT} e^{-\alpha x^4/kT} dx$$

since $\alpha \ll 1$, $e^{-\alpha x^4/kT} \approx (1 - \alpha x^4/kT)$

$$Z = \frac{\sqrt{2\pi mkT}}{h} \int_{-\infty}^{\infty} e^{-k_0 x^2/2kT} (1 - \alpha x^4/kT) dx$$

use tables for the integrals to get

$$Z = \frac{2\pi kT}{h\sqrt{k_0/m}} \left(1 - 3\alpha kT/k_0^2\right)$$

$$U = -N \frac{\partial}{\partial \beta} \ln Z = NkT - 3\alpha N (kT)^2 \Big/ k_0^2 \left(1 - 3\alpha kT/k_0^2\right)$$

$$C = \frac{\partial U}{\partial T} = Nk - 6\alpha N k^2 T \Big/ k_0^2 \left(1 - 3\alpha kT/k_0^2\right) - 9N\alpha^2 k^3 T^2 / k_0^4 \left(1 - 3\alpha kT/k_0^2\right)^2$$

The limit $T \to \infty$ gives the above

When $T \to 0$, $U = NkT$ and $C = Nk$

Problem 6.

The specific Gibbs function of a gas is given by

$$g = RT\ \ln\left(\frac{P}{P_0}\right) - AP$$

where A is a function of the temperature T. Find expressions for

(a) the equation of state
(b) the specific entropy
(c) the specific Helmholtz function

Solution

$$(a)\ v = -\left(\frac{\partial g}{\partial P}\right)_T = \frac{RT}{P} - A$$

$$(b)\ s = -\left(\frac{\partial g}{\partial T}\right)_P = -R\ \ln\left(\frac{P}{P_0}\right) + P\left(\frac{\partial A}{\partial T}\right)$$

$$(c)\ f = g - Pv = RT\ \ln\left(\frac{P}{P_0}\right) - P(A + v)$$