Chapter 9

Center of Mass and Linear Momentum

1. The Center of Mass

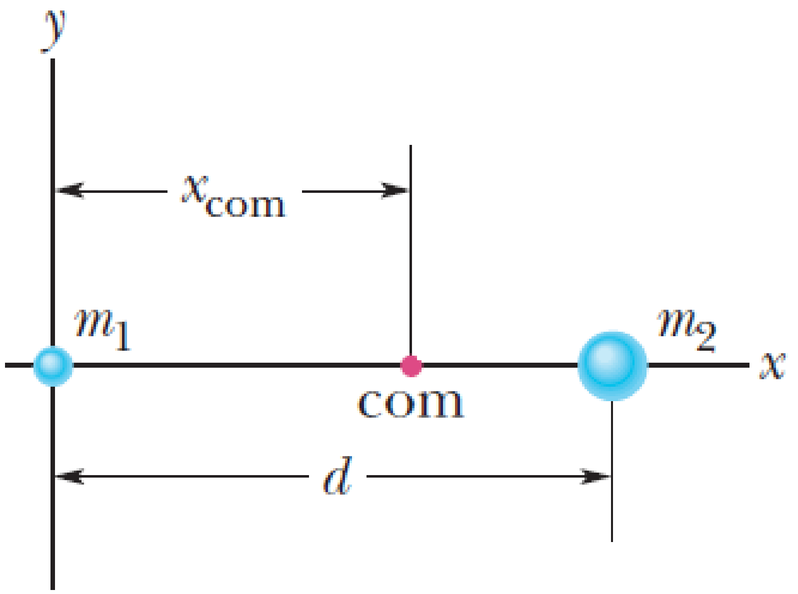
- The **center of mass** (com) of a system is the point that moves as though (1) all of the system's mass were concentrated there and (2) all external forces were applied there.
- We discuss here how to find the center of mass of a system of a few particle, and then we consider a system of many particles (a solid body). Later in the chapter, we discuss how the center of mass of a system moves when external forces act on the system.

1. The Center of Mass

- <u>System of Particles:</u>

Consider the configuration shown in the figure. We define the position of the center of mass (com) of this two particle system as

$$x_{\text{com}} = \frac{m_2}{m_1 + m_2} d.$$



When $m_2 = 0$, $x_{\text{com}} = 0$, when $m_1 = m_2$, $x_{\text{com}} = d/2$, and when $m_1 = 0$, $x_{\text{com}} = d$.

x_{com} lies between $x_{\text{com}} = 0$ and $x_{\text{com}} = d$.

1. The Center of Mass

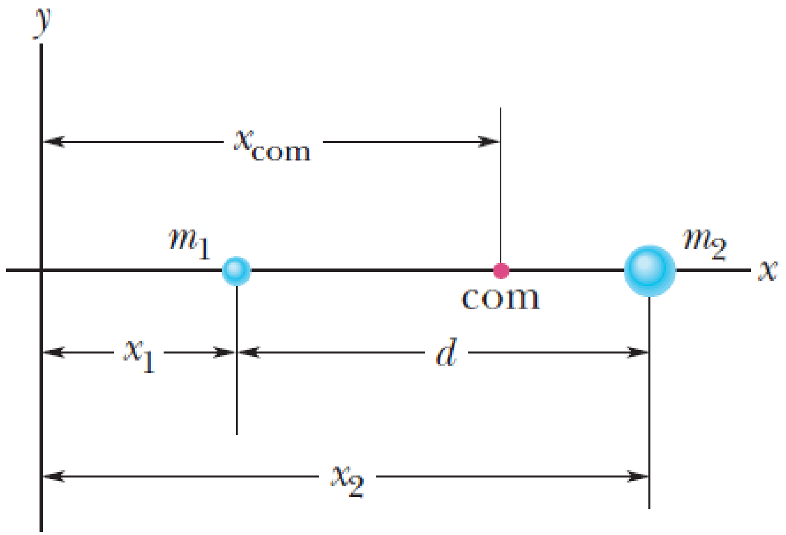
- <u>System of Particles:</u>

  Consider now the more situation shown in the figure. The position of the center of mass is now defined as

$$x_{\mathrm{com}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{m_1 x_1 + m_2 x_2}{M}.$$

  When $x_1 = 0$, then $x_2 = d$ and the previous situation is recovered.

  Despite the shift of the coordinate system, the center of mass is still the same distance form each particle.



1. The Center of Mass

- <u>System of Particle:</u>

  For a system of n particles along the x axis,

$$x_{\text{com}} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots + m_nx_n}{M}$$

$$= \frac{1}{M}\sum_{i=1}^{n} m_i x_i \,.$$

  If the particles are distributed in three dimensions, the center of mass is identified by three coordinates:

$$x_{\text{com}} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i \,, \qquad y_{\text{com}} = \frac{1}{M}\sum_{i=1}^{n} m_i y_i \,, \qquad z_{\text{com}} = \frac{1}{M}\sum_{i=1}^{n} m_i z_i \,.$$

1. The Center of Mass

- System of Particle:

The position of the center of mass can be written as a position vector:

$$\vec{r}_{\mathrm{com}} = x_{\mathrm{com}}\,\hat{\mathrm{i}} + y_{\mathrm{com}}\,\hat{\mathrm{j}} + z_{\mathrm{com}}\,\hat{\mathrm{k}}.$$

The three scalar equations in the previous slide can be combined into a single equation:

$$\vec{r}_{\mathrm{com}} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{r}_i\,,$$

where $\vec{r}_{\mathrm{i}} = x_{\mathrm{i}}\,\hat{\mathrm{i}} + y_{\mathrm{i}}\,\hat{\mathrm{j}} + z_{\mathrm{i}}\,\hat{\mathrm{k}}.$

1. The Center of Mass

- Solid Bodies:

Solid objects contain so many particles, that we can best treat it as a continuous distribution of matter. The particles then become differential mass dm, and the sums become integrals:

$$x_{\mathrm{com}} = \frac{1}{M}\int x\,dm\,, \qquad y_{\mathrm{com}} = \frac{1}{M}\int y\,dm\,, \qquad z_{\mathrm{com}} = \frac{1}{M}\int z\,dm\,.$$

M here is the mass of the object.

These integrals are usually difficult to evaluate, unless an object has uniform density ρ.

1. The Center of Mass

- <u>Solid Bodies:</u>

  We then can write that

$$\rho = \frac{dm}{dV} = \frac{M}{V},$$

where dV is the volume occupied by a mass dm, and V is the total volume of the object.

The three integrals above can be rewritten for a uniform density object as

$$x_{\mathrm{com}} = \frac{1}{V}\int x\, dV\,, \qquad y_{\mathrm{com}} = \frac{1}{V}\int y\, dV\,, \qquad z_{\mathrm{com}} = \frac{1}{V}\int z\, dV\,.$$

1. The Center of Mass

- Solid Bodies:

  The determination of the center of mass becomes significantly easier when the object has a *point*, a *line*, or a *plane* of symmetry. The center of mass then lies at the *point*, on that *line*, or in that *plane*.

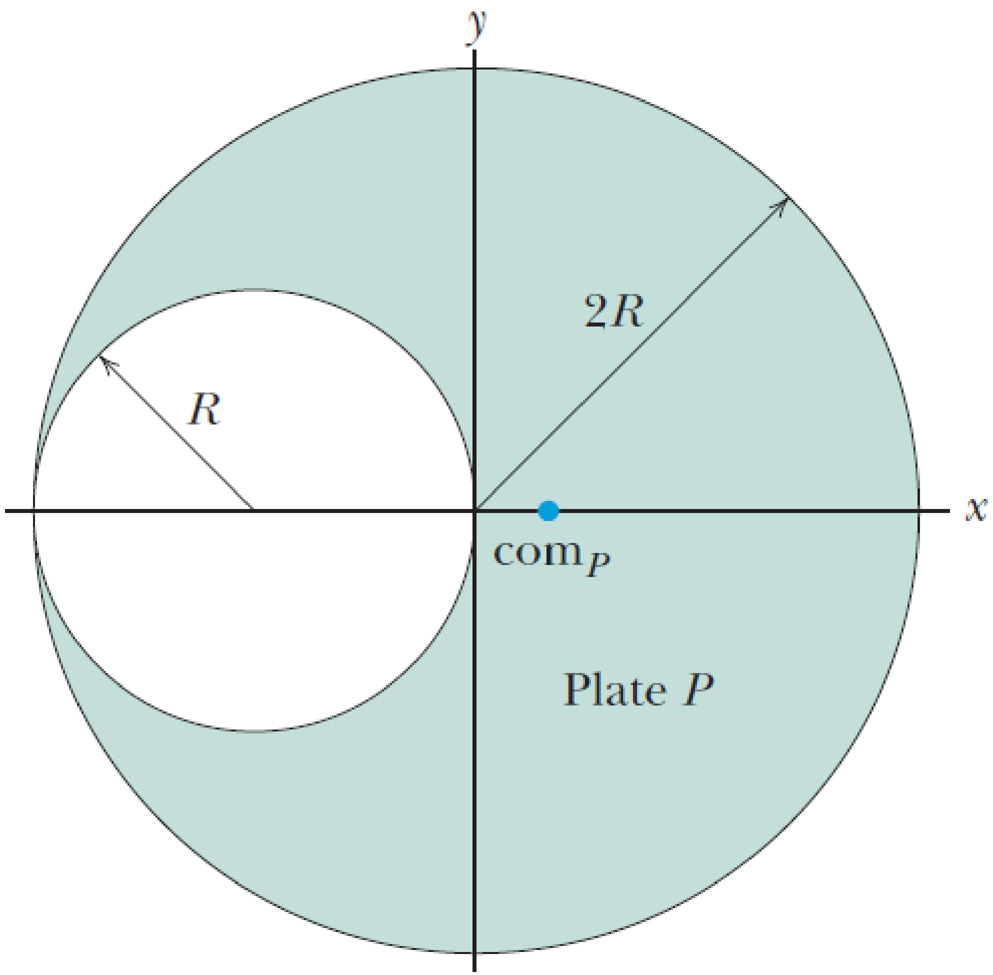
  For example, the center of mass of a uniform density sphere is the center of the sphere (the point of symmetry). The center of mass of a uniform density cone lies on the axis of the cone (the line of symmetry). The center of mass of a banana lies somewhere in the plane of symmetry (the plane which splits the banana into two identical parts).

1. The Center of Mass

Example 1: The figure shows a uniform metal plate P of radius $2R$ from which a disk of radius R has been removed. Using the xy coordinate system shown, locate the center of mass $x_{com,P}$ of the remaining plate.

The center of mass of the removed disk S and the remaining plate P is the same as the center of mass of the whole disk C.

Center of mass of a disk is located at its center. Therefore, $x_{\text{com},C} = 0$ and $x_{\text{com},S} = -R$.



1. The Center of Mass

We then write

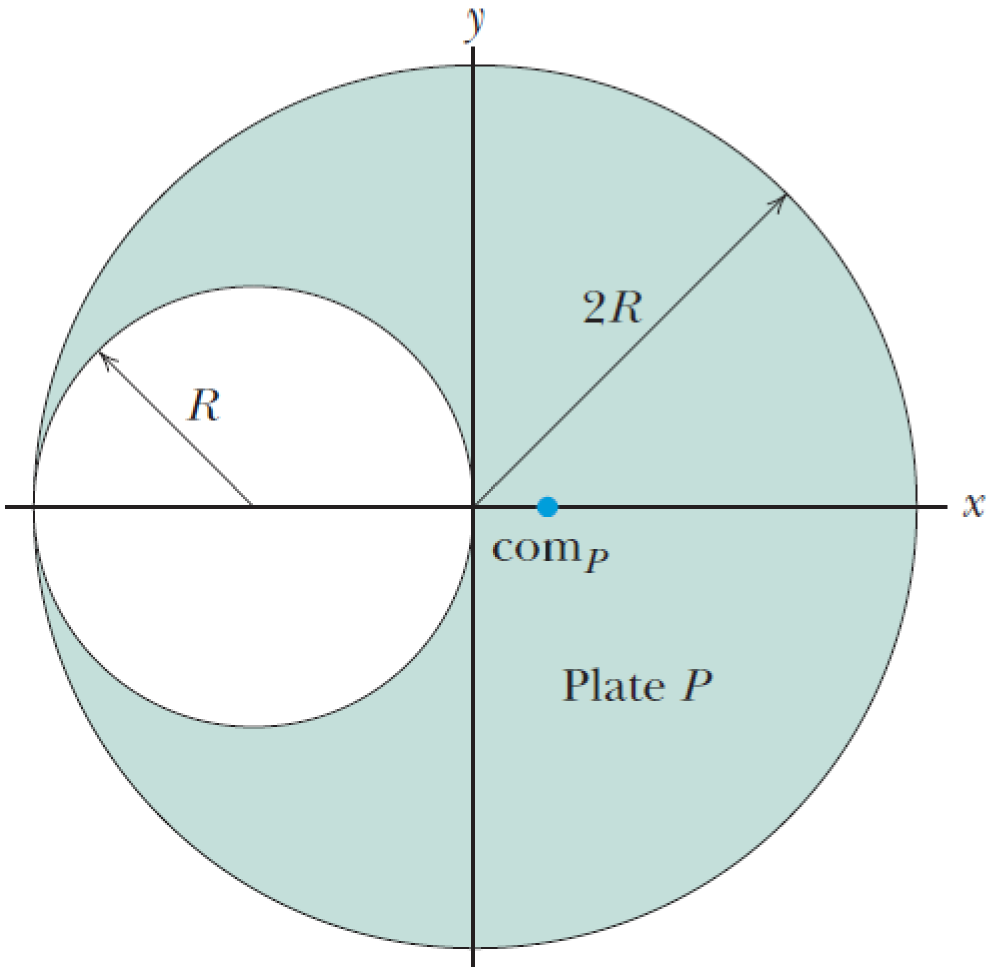
$$x_{\text{com},C} = \frac{m_S x_{\text{com},S} + m_P x_{\text{com},P}}{m_S + m_P},$$

or

$$0 = \frac{m_P x_{\text{com},P} - R\, m_S}{m_S + m_P}.$$

Solving for $x_{\text{com},P}$ we get

$$x_{\text{com},P} = R\,\frac{m_S}{m_P}$$



1. The Center of Mass

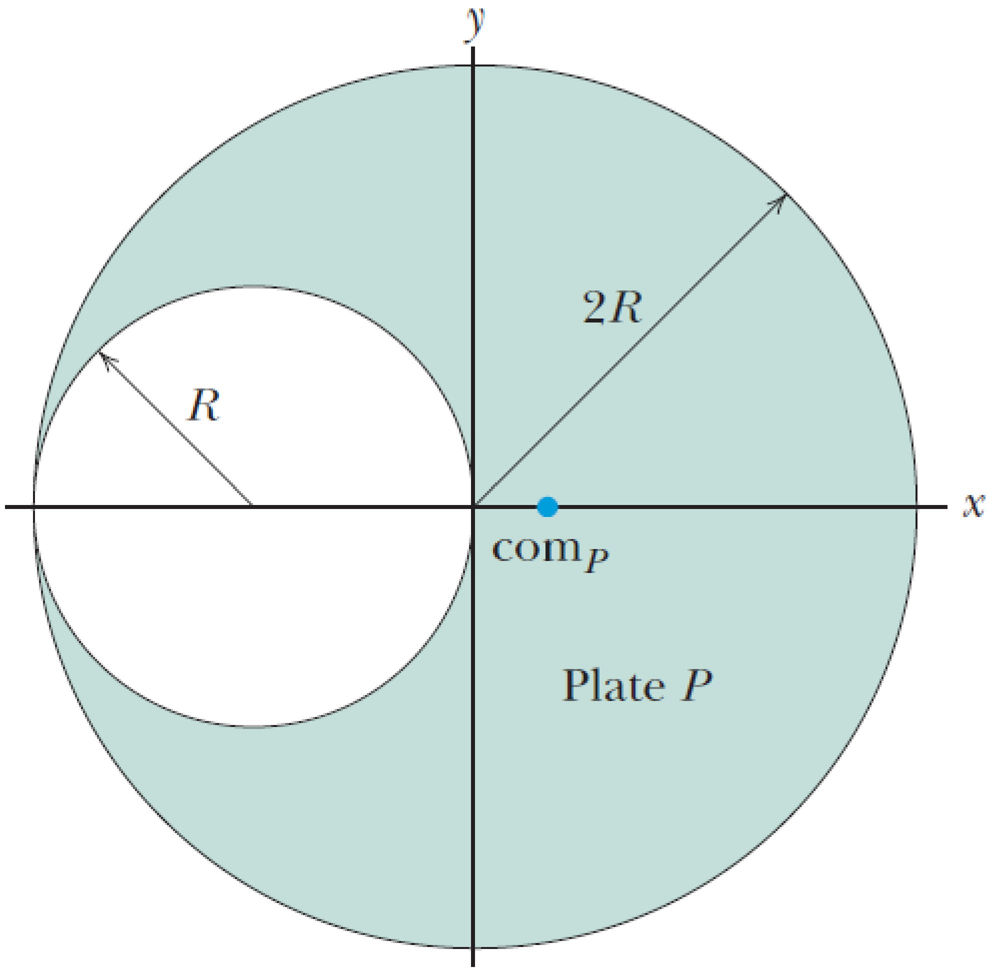
The masses of the removed disk and remaining plate are related to m_C by

$$m_S = \frac{R^2}{(2R)^2}\, m_C = \frac{m_C}{4},$$

$$m_P = m_C - m_S = \frac{3}{4}\, m_C\,.$$

We then find

$$x_{\text{com},P} = R\,\frac{m_S}{m_P} = R\,\frac{\frac{m_C}{4}}{\frac{3}{4}\, m_C} = \frac{R}{3}.$$



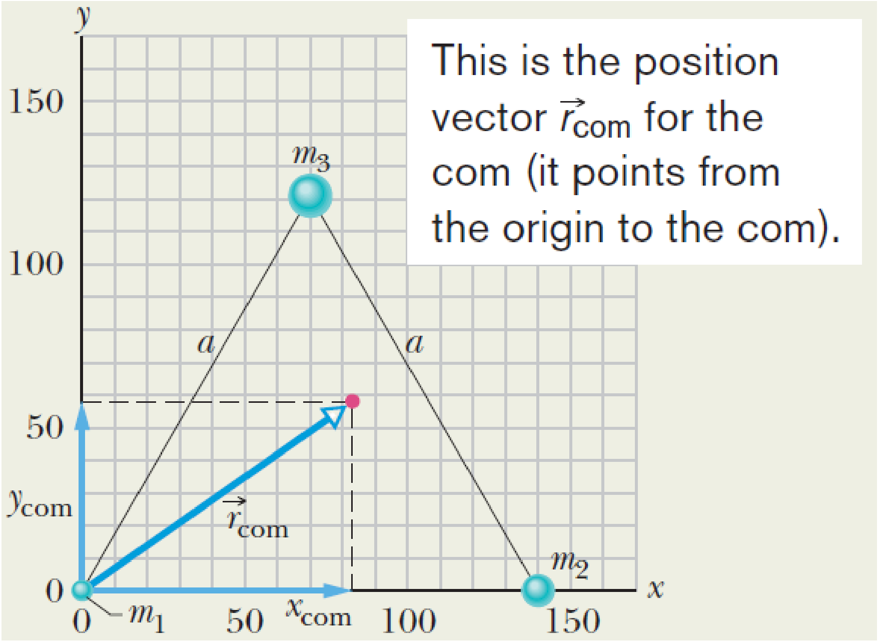
1. The Center of Mass

Example 2: Three particles of masses $m_1 = 1.2\text{ kg}$, $m_2 = 2.5\text{ kg}$, and $m_3 = 3.4\text{ kg}$ form an equilateral triangle of edge length $a = 140\text{ cm}$. Where is the center of mass of this system?

$$x_{\text{com}} = \frac{1}{M}\sum_{i=1}^{3} m_i x_i = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$

$$= \frac{(1.2\text{ kg})(0) + (2.5\text{ kg})(140\text{ cm}) + (3.4\text{ kg})(70\text{ cm})}{1.2\text{ kg} + 2.5\text{ kg} + 3.4\text{ kg}}$$

$$= 83\text{ cm}.$$

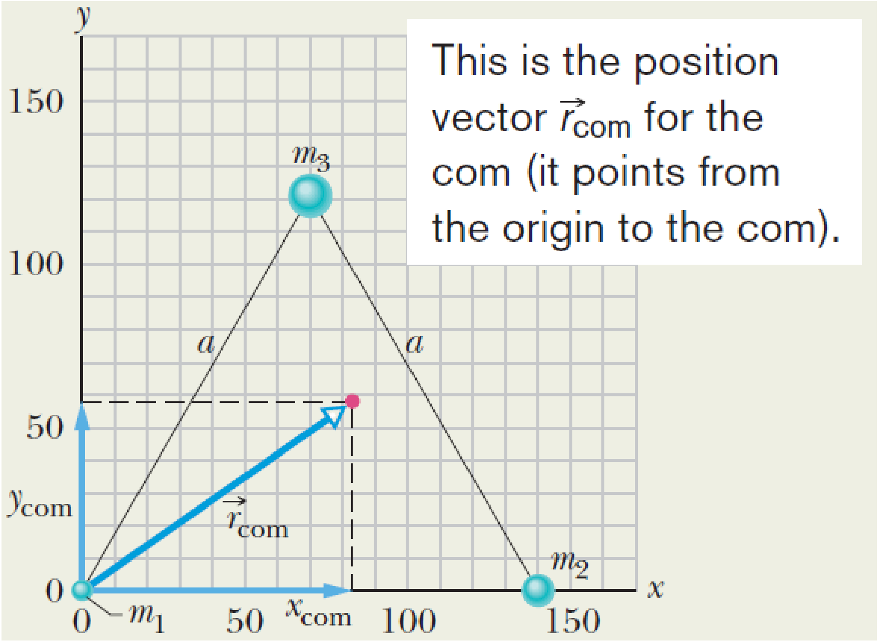


1. The Center of Mass

$$y_{com} = \frac{1}{M}\sum_{i=1}^{3} m_i y_i = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3}$$

$$= \frac{(1.2\text{ kg})(0) + (2.5\text{ kg})(0\text{ cm}) + (3.4\text{ kg})(120\text{ cm})}{1.2\text{ kg} + 2.5\text{ kg} + 3.4\text{ kg}}$$

$$= 58\text{ cm}.$$

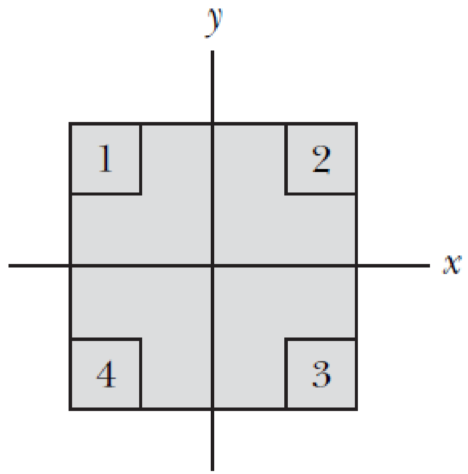


1. The Center of Mass

CHECKPOINT 1

The figure shows a uniform square plate from which four identical squares at the corners will be removed. (a) Where is the center of mass of the plate originally? Where is it after the removal of (b) square 1; (c) squares 1 and 2; (d) squares 1 and 3; (e) squares 1, 2, and 3; (f) all four squares? Answer in terms of quadrants, axes, or points (without calculation, of course).

(a) At the origin.
(b) In the 4th quadrant.
(c) On the y axis, blow the origin.
(d) At the origin.
(e) In this 3rd quadrant.
(f) At the origin.



2. Newton's 2nd Law for a System of Particles

- We now discuss how external forces can move the center of mass of a system.
- Consider a system of n particles. The motion of the center of mass of the system is governed by

$$\vec{F}_{\text{net}} = M\vec{a}_{\text{com}},$$

where:

1. $\vec{F}_{\text{net}}$ is the net force of all external forces acting on the system. Internal forces are not included.
2. M is the total mass of the system. We assume that the system is **closed**; no mass enters or leaves the system.
3. $\vec{a}_{\text{com}}$ is the acceleration of the center of mass of the system. The equation tells nothing about the motions or individual particles.

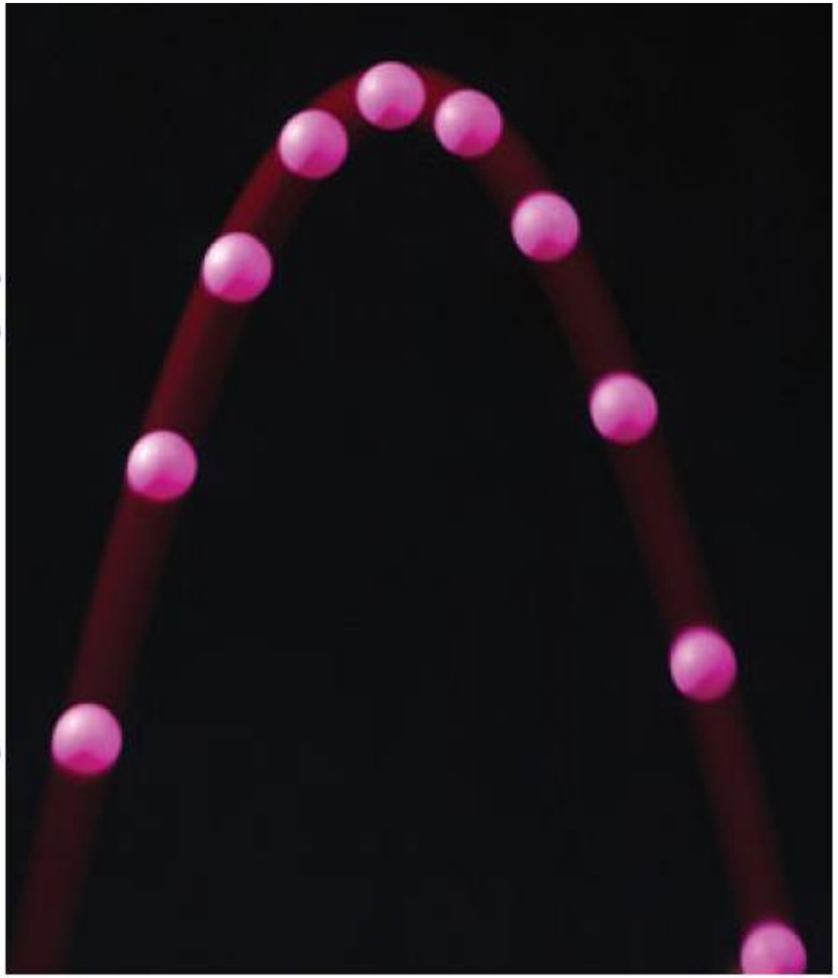
2. Newton's 2nd Law for a System of Particles

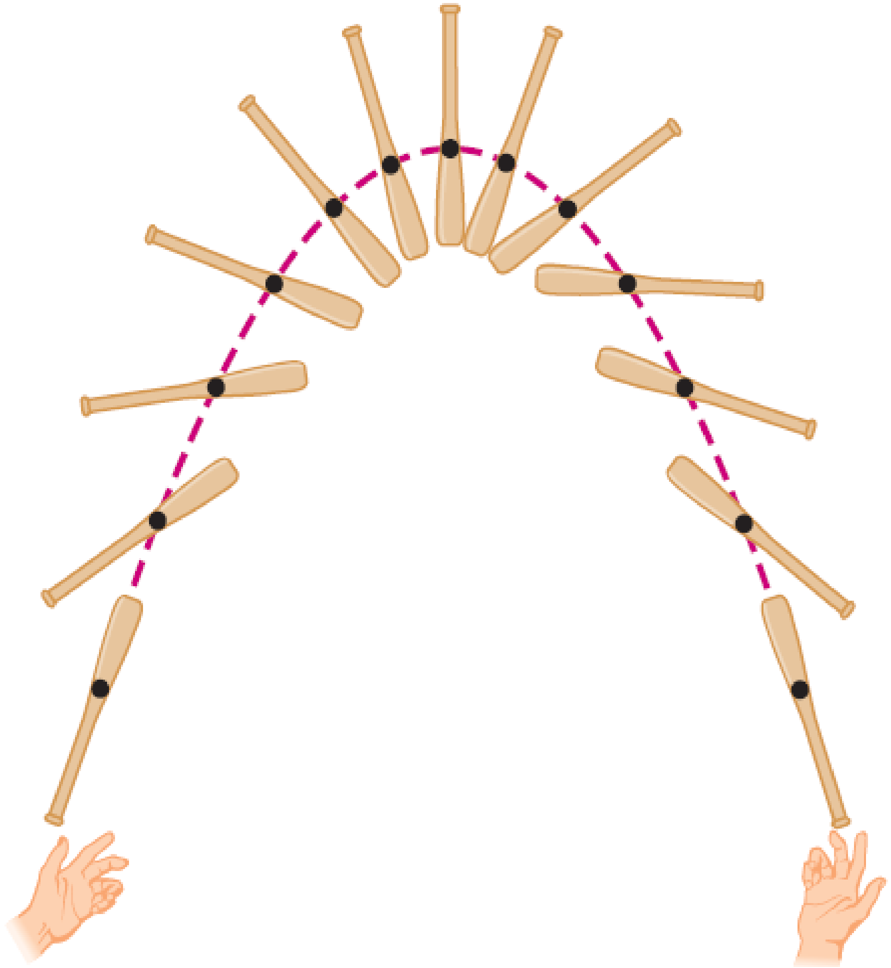
- In components,

$$F_{\text{net},x} = Ma_{\text{com},x}, \qquad F_{\text{net},y} = Ma_{\text{com},y}, \qquad F_{\text{net},z} = Ma_{\text{com},z}.$$

- Consider a system of two billiard balls, where one ball is moving toward the other which is at rest. Because $\vec{F}_{\text{net}} = 0$, $\vec{a}_{\text{com}} = 0$. The velocity of the center of mass does not change. The center of mass must continue moving forward before and after the collision, with the same speed and direction.
- $\vec{F}_{\text{net}} = M\vec{a}_{\text{com}}$, applies to solid bodies. It tells us that for a baseball bat in free fall, $\vec{a}_{\text{com}} = \vec{g}$. The center of mass of the bat moves as if the bat were a single particle.

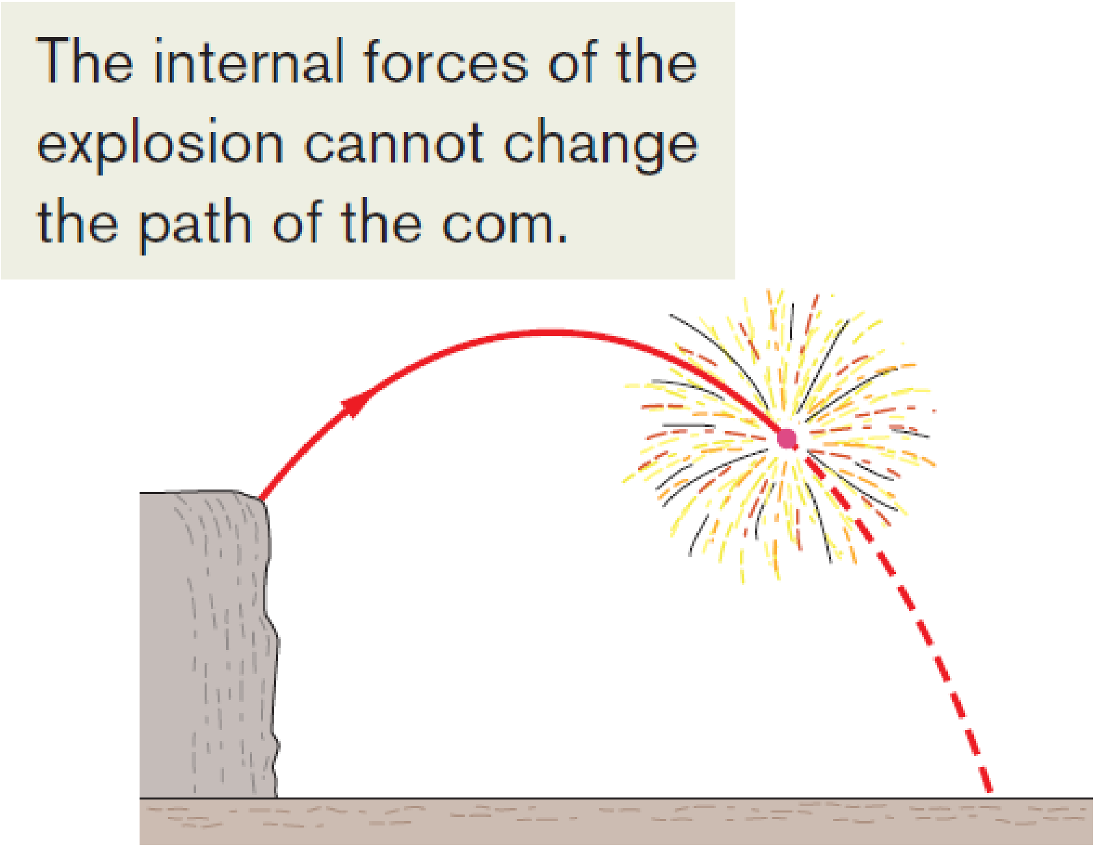
2. Newton's 2nd Law for a System of Particles





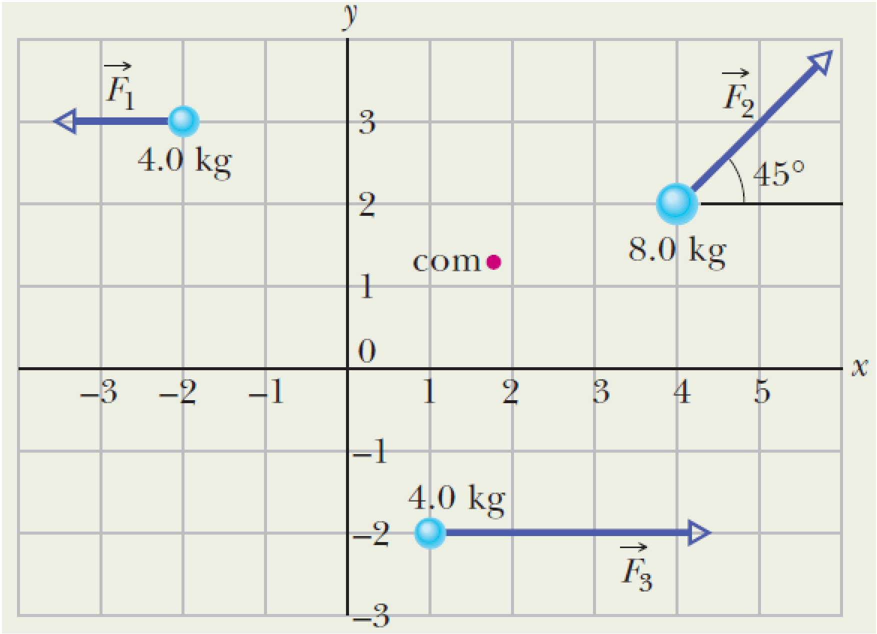
2. Newton’s 2nd Law for a System of Particles

- Another interesting example is the fireworks rocket. The center of mass of a fireworks rocket follows the same trajectory that the rocket would have followed if it had not exploded.



2. Newton's 2nd Law for a System of Particles

Example 3: The three particles in the figure are initially at rest. Each experiences an *external* force due to bodies outside the three-particle system. The directions are indicated, and the magnitudes are $F_1 = 6.0\text{ N}$, $F_2 = 12\text{ N}$, and $F_3 = 14\text{ N}$. What is the acceleration of the center of mass of the system, and in what direction does it move?



$$\vec{F}_{\text{net}} = M\vec{a}_{\text{com}},$$

or

$$\vec{a}_{\text{com}} = \frac{\vec{F}_{\text{net}}}{M} = \frac{\vec{F}_1 + \vec{F}_2 + \vec{F}_3}{M}.$$

2. Newton's 2nd Law for a System of Particles

$$a_{\text{com},x} = \frac{F_{1,x} + F_{2,x} + F_{3,x}}{M}$$

$$= \frac{-6.0 \text{ N} + 12 \, (\cos 45°) \text{ N} + 14 \text{ N}}{16 \text{ kg}}$$

$$= 1.03 \frac{m}{s^2}.$$

$$a_{\text{com},y} = \frac{F_{1,y} + F_{2,y} + F_{3,y}}{M}$$

$$= \frac{0 \; + 12 \, (\sin 45°) \text{ N} + 0}{16 \text{ kg}}$$

$$= 0.530 \frac{m}{s^2}.$$

$$a_{\text{com},x} = \frac{F_{1,x} + F_{2,x} + F_{3,x}}{M}$$

$$= \frac{-6.0 \text{ N} + 12 \, (\cos 45°) \text{ N} + 14 \text{ N}}{16 \text{ kg}}$$

$$= 1.03 \frac{m}{s^2}.$$

$$a_{\text{com},y} = \frac{F_{1,y} + F_{2,y} + F_{3,y}}{M}$$

$$= \frac{0 \; + 12 \, (\sin 45°) \text{ N} + 0}{16 \text{ kg}}$$

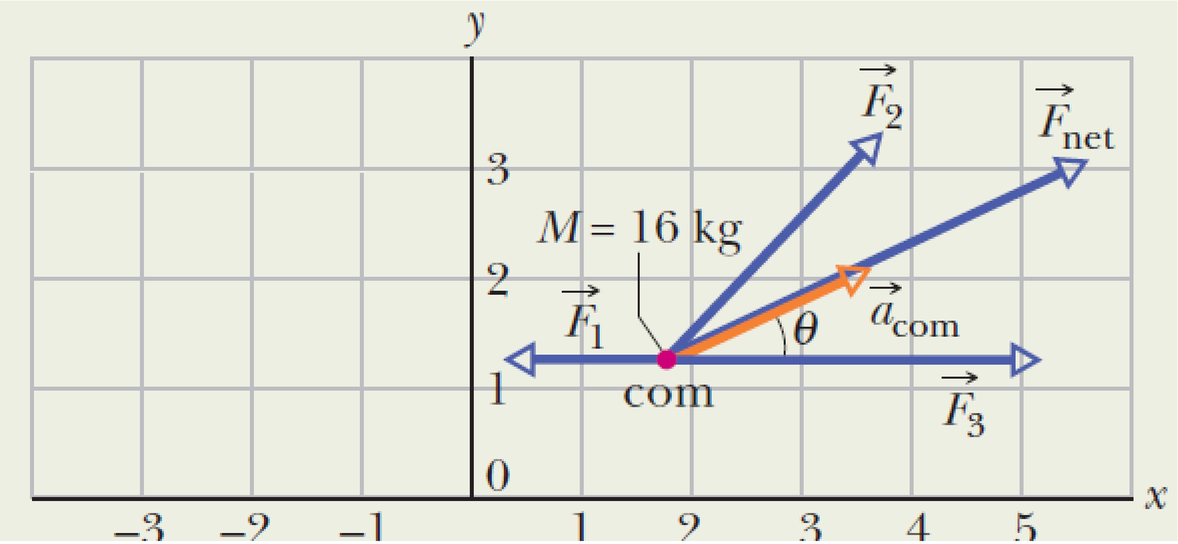
$$= 0.530 \frac{m}{s^2}.$$

2. Newton's 2nd Law for a System of Particles

$$a_{com} = \sqrt{\left(1.03\,\frac{m}{s^2}\right)^2 + \left(0.530\,\frac{m}{s^2}\right)^2}$$

$$\approx 1.2\,\frac{m}{s^2}.$$

$$\theta = \tan^{-1}\frac{0.530}{1.03} = 27°.$$



3. Linear Momentum

- In this section we return to the case of a single particle, in order to define two new quantities.
- The **linear momentum** (or momentum) of a particle of mass m and velocity $\vec{v}$ is a vector quantity $\vec{p}$, defined as

$$\vec{p} = m\vec{v}.$$

- $\vec{p}$ and $\vec{v}$ have the same direction. The SI unit for momentum is kilogram-meter per second ($\mathrm{kg} \cdot \mathrm{m/s}$).
- Newton expressed his second law of motion in terms of momentum:

  The time rate of momentum change of a particle is equal to the net force acting on the particle and is in the direction of that force.

3. Linear Momentum

- In equation form, Newton's second law reads

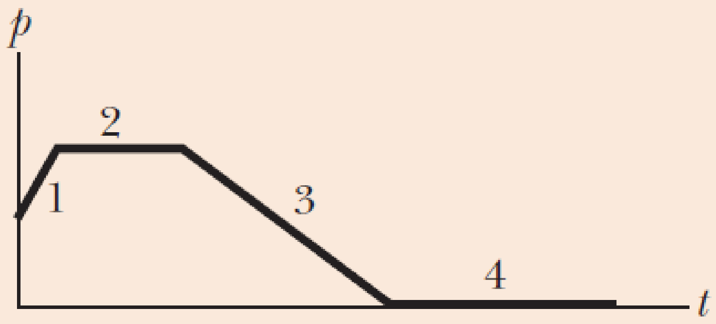
$$\vec{F}_{net} = \frac{d\vec{p}}{dt}.$$

- Note that $\vec{F}_{net} = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt} = m\frac{d(\vec{v})}{dt} = m\vec{a}$.
- In words, the net force $\vec{F}_{net}$ on a particle changes the linear momentum $\vec{p}$ of the particle. Conversely, the linear momentum can be changes only by a net force. If there is no net force, $\vec{p}$ cannot change.

3. Linear Momentum

CHECKPOINT 3

The figure gives the magnitude p of the linear momentum versus time t for a particle moving along an axis. A force directed along the axis acts on the particle. (a) Rank the four regions indicated according to the magnitude of the force, greatest first. (b) In which region is the particle slowing?



(a) 1, 3, 2 & 4 tie.

(b) 3.

4. Linear Momentum of a System of Particles

- Consider a system of n particles, each with its own mass, velocity and linear momentum. The particles may interact with each other, and external forces may act on them. The total linear momentum $\vec{P}$ of the system is

$$\begin{aligned}\vec{P} &= \vec{p}_1 + \vec{p}_2 + \cdots + \vec{p}_n \\ &= m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots + m_n\vec{v}_n,\end{aligned}$$

which can be written as

$$\vec{P} = M\vec{v}_{\mathrm{com}}.$$

- The linear momentum of a system of particles is equal to the product of the total mass of the system and velocity of the center of mass.

4. Linear Momentum of a System of Particles

- Differentiating the last relation with respect to time t yields

$$\frac{d\vec{P}}{dt} = M\frac{d\vec{v}_{\text{com}}}{dt} = M\vec{a}_{\text{com}},$$

or equivalently

$$\vec{F}_{\text{net}} = \frac{d\vec{P}}{dt}.$$

where $\vec{F}_{\text{net}}$ is the net external force acting on the system.

- In words, the net external force $\vec{F}_{net}$ acting on a system of particle changes the linear momentum $\vec{P}$ of the system. Conversely, the linear momentum of a system can be changes only by a net external force. If there is no net external force, $\vec{P}$ cannot change.

5. Collison and Impulse

- To change the momentum $\vec{p}$ of a particle-like object a net force $\vec{F}_{net}$ is required.
- We could arrange for the object to collide with another. In such a collision, the external force on the object is brief, large in magnitude, and suddenly changes the body’s momentum.
- We start studying collisions by a simple collision in which a moving particle-like object (a projectile) collides with some other body (a target.)

5. Collison and Impulse

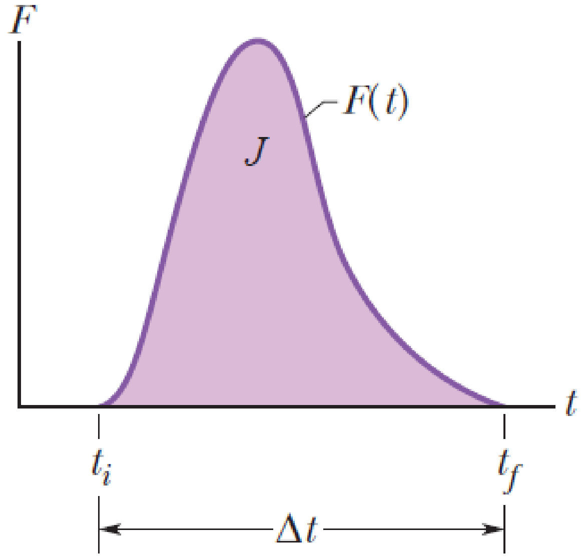
- <u>Single Collision</u>:

  Let the projectile be a ball and the target be a bat. During the brief collision, the ball experiences a force that is great enough to slow, stop, or even reverse its motion. The force $\vec{F}(t)$ varies during the collision and changes the ball's linear momentum $\vec{p}$. By Newton's second law ($\vec{F} = d\vec{p}/dt$), the change $d\vec{p}$ in the ball's momentum in time interval dt is

$$d\vec{p} = \vec{F}(t)dt.$$

  The net change in the ball's momentum due to the collision, from a time t_i to a time t_f is

$$\int_{t_i}^{t_f} d\vec{p} = \int_{t_i}^{t_f} \vec{F}(t)dt\,.$$

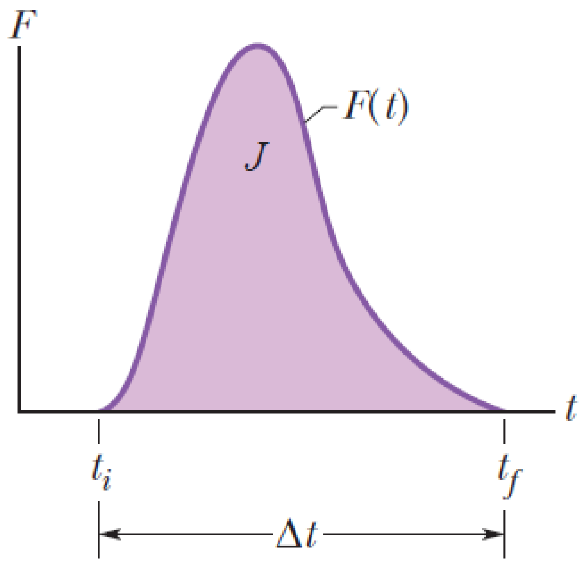


5. Collison and Impulse

- Single Collision:

  Let the left hand side of the last equation gives us the change in momentum $\vec{p}_f - \vec{p}_i = \Delta\vec{p}$. The right hand side, which is a measure of both the magnitude and the duration of the collision, is called the **impulse** $\vec{J}$ of the collision:

$$\vec{J} = \int_{t_i}^{t_f} \vec{F}(t)dt\,.$$



  Therefore, the change in an object's momentum is equal to the impulse on the object:

$$\Delta\vec{p} = \vec{J}.$$

5. Collison and Impulse

- Single Collision:

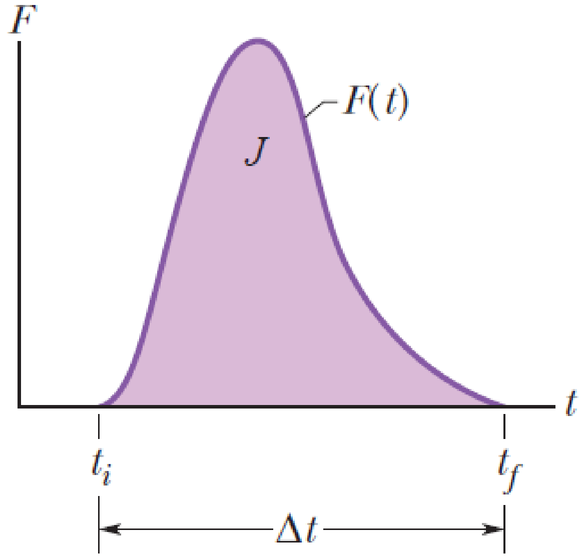
  $\Delta\vec{p} = \vec{J}$ is a vector equation. Its x component reads

  $$\Delta p_x = J_x,$$

  or

  $$p_{fx} - p_{ix} = \int_{t_i}^{t_f} F_x(t)dt\,.$$

  If we have a function for $\vec{F}(t)$, we can evaluate $\vec{J}$ by direct integration. If we have a plot $\vec{F}$ versus time t, we can find $\vec{J}$ by evaluating the *area* between the curve and the t axis.



The impulse in the collision is equal to the area under the curve.

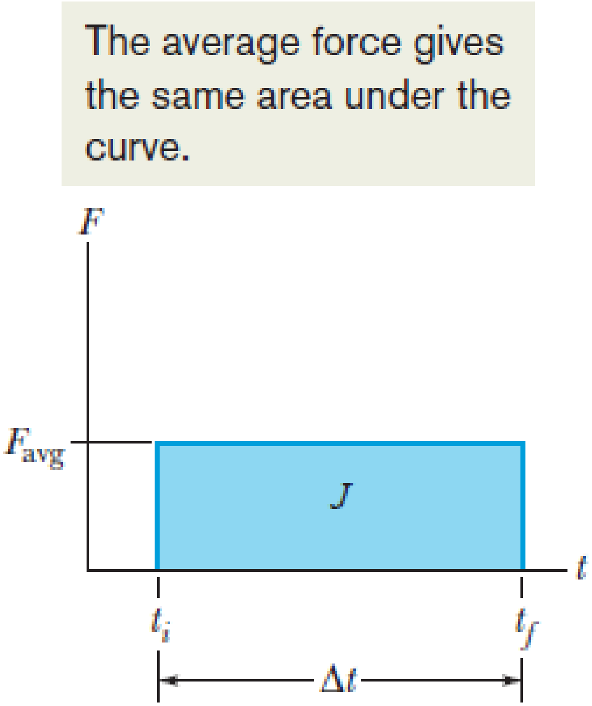
5. Collison and Impulse

- Single Collision:

  In many situations, we don't know how the force varies with time but we know the average magnitude F_{avg} of the force and the duration Δt $(= t_f - t_i)$ of the collision. We then can write the magnitude of the impulse as

$$J = F_{avg} \Delta t.$$

  We could have focused on the bat instead of the ball. By Newton's third law, the impulse on the bat has the same magnitude but the opposite direction as the impulse on the ball.



5. Collison and Impulse

CHECKPOINT 4

A paratrooper whose chute fails to open lands in snow; he is hurt slightly. Had he landed on bare ground, the stopping time would have been 10 times shorter and the collision lethal. Does the presence of the snow increase, decrease, or leave unchanged the values of (a) the paratrooper's change in momentum, (b) the impulse stopping the paratrooper, and (c) the force stopping the paratrooper?

(a) No change. In either case $\Delta p_y = p_{fy} - p_{iy} = 0 - mv_{iy}$.

(b) No change. In either case, $J_y = \Delta p_y$.

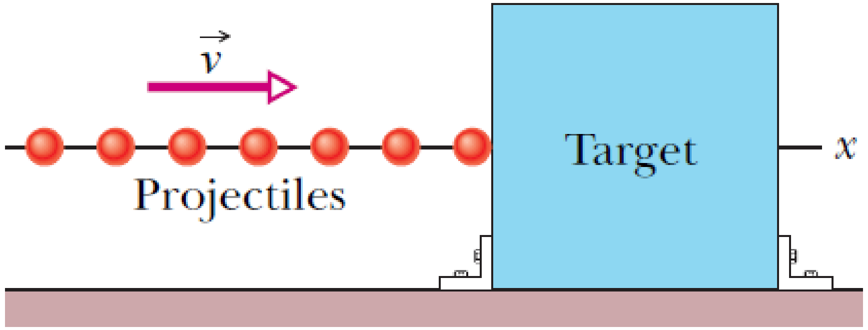
(c) Decrease. because $F_{avg} = J_y/\Delta t$ and Δt is 10 times longer.

5. Collison and Impulse

<u>Series of Collisions</u>:

Here we consider the average force F_{avg} on a body when it undergoes a series of identical, repeated collision.

Consider a stream of projectile bodies, each with mass m and linear momentum $\vec{p} = m\vec{v}$ along the x axis. Let n be the number of the projectiles that collide in a time interval Δt. The total change in linear momentum for n particles is $n\Delta p$, where Δp is the change in the momentum of a single particle due to a collision.



5. Collison and Impulse

Series of Collisions:

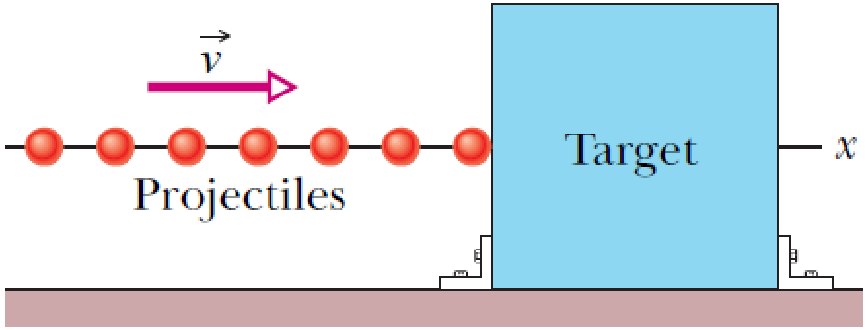
The resulting impulse $\vec{J}$ on the target during the time interval Δt along the x axis is

$$J = -n\Delta p.$$

The average force F_{avg} acting on the target during Δt is

$$F_{avg,x} = \frac{J}{\Delta t} = -\frac{n}{\Delta t}\Delta p = -\frac{n}{\Delta t}m\Delta v.$$

$\frac{n}{\Delta t}$ is the rate at which the projectile collides with a target.



5. Collison and Impulse

<u>Series of Collisions</u>:

If the projectiles stop after the collision, then $\Delta v = v_f - v_i = 0 - v = -v$. If instead, the projectiles bounce backward with the same speed v, then $\Delta v = v_f - v_i = -v - v = -2v$.

In time Δt, an amount of mass $\Delta m = nm$ collides with the target. The average force $F_{avg,x}$ becomes

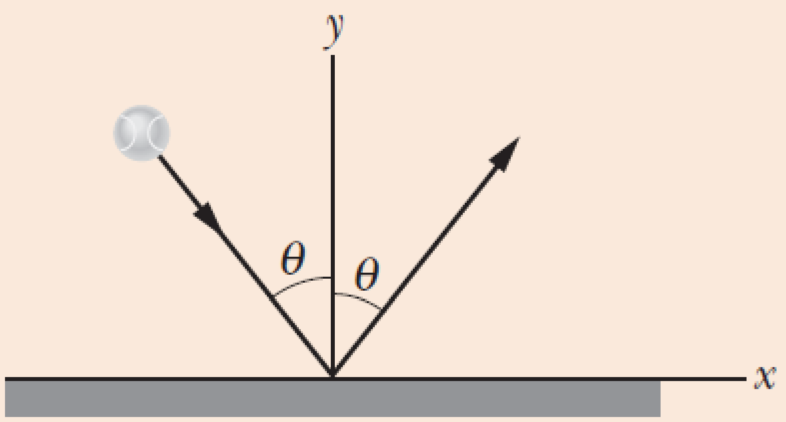
$$F_{avg,x} = -\frac{\Delta m}{\Delta t}\Delta v.$$

$\frac{\Delta m}{\Delta t}$ is the rate at which mass collides with the target.

5. Collison and Impulse

CHECKPOINT 5

The figure shows an overhead view of a ball bouncing from a vertical wall without any change in its speed. Consider the change $\Delta\vec{p}$ in the ball's linear momentum. (a) Is Δp_x positive, negative, or zero? (b) Is Δp_y positive, negative, or zero? (c) What is the direction of $\Delta\vec{p}$?



(a) Zero.

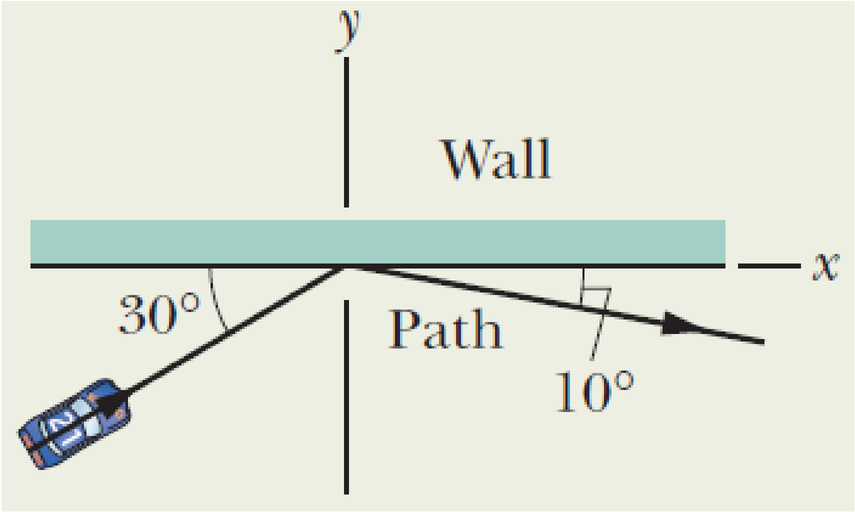
(b) Positive.

(c) Positive y direction.

5. Collison and Impulse

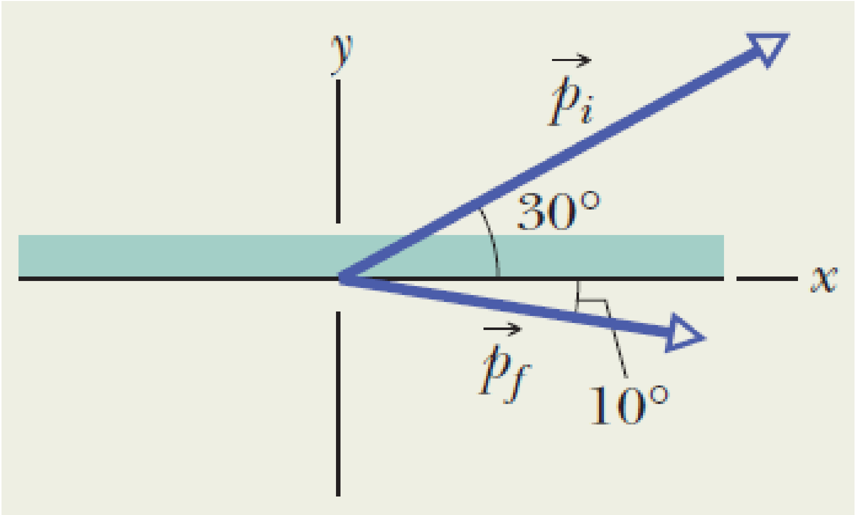
Example 4: The figure is an overhead view of the path taken by a race car driver as his car collides with the racetrack wall. Just before the collision, he is traveling at speed $v_i = 70\ \mathrm{m/s}$ along a straight line at $30°$ from the wall. Just after the collision, he is traveling at speed $v_f = 50\ \mathrm{m/s}$ along a straight line at $10°$ from the wall. His mass m is $80\ \mathrm{kg}$.

(a) What is the impulse $\vec{J}$ on the driver due to the collision?



5. Collison and Impulse

$$\vec{J} = \vec{p}_f - \vec{p}_i = m(\vec{v}_f - \vec{v}_i).$$



Along the x axis:

$$\begin{aligned} J_x &= m(v_{fx} - v_{ix}) \\ &= (80\ \mathrm{kg})[(50\ \mathrm{m/s})\cos(-10°) - (70\ \mathrm{m/s})\cos 30°] \\ &= -910\ \mathrm{kg} \cdot \frac{\mathrm{m}}{\mathrm{s}}. \end{aligned}$$

Along the y axis:

$$\begin{aligned} J_y &= m(v_{fy} - v_{iy}) \\ &= (80\ \mathrm{kg})[(50\ \mathrm{m/s})\sin(-10°) - (70\ \mathrm{m/s})\sin 30°] \\ &= -3495\ \mathrm{kg} \cdot \frac{\mathrm{m}}{\mathrm{s}}. \end{aligned}$$

5. Collison and Impulse

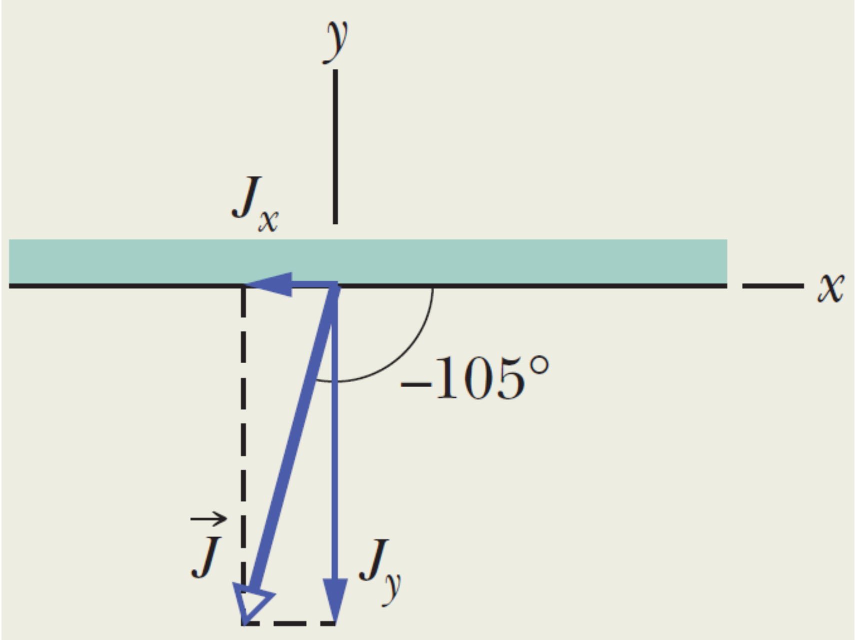
The impulse is then

$$\vec{J} = (-910\,\hat{\imath} - 3500\,\hat{\jmath})\ \mathrm{kg}\cdot\frac{\mathrm{m}}{\mathrm{s}},$$

and $\mathrm{J} = 3600\ \mathrm{kg}\cdot\frac{\mathrm{m}}{\mathrm{s}}$, at $105°$ below the x axis.

(b) The collision lasts for $14\ \mathrm{ms}$. What is the magnitude of the average force on the driver during the collision?

$$F_{avg} = \frac{J}{\Delta t} = \frac{3600\ \mathrm{kg}\cdot\frac{\mathrm{m}}{\mathrm{s}}}{14\ \mathrm{ms}} = 2.6\times 10^{5}\mathrm{N}.$$



6. Conservation of Linear Momentum

- When the net external force $\vec{F}_{net}$ (and impulse $\vec{J}$) acting on a closed, isolated system is zero, then $d\vec{P}/dt = 0$. We therefore write

$$\vec{P} = \text{constant.}$$

- If no net external force acts on a system of particles, the total liner momentum $\vec{P}$ of the system cannot change.
- This result is called the **law of conservation of linear momentum**. If can be also written as

$$\vec{P}_i = \vec{P}_f.$$

In words:

$$\begin{pmatrix}\text{total linear momentum}\\ \text{at some intial time } t_i\end{pmatrix} = \begin{pmatrix}\text{total linear momentum}\\ \text{at some final time } t_f\end{pmatrix}.$$

6. Conservation of Linear Momentum

- Each of the two vector equations in the previous slide is equivalent to three equations corresponding to the conservation of linear momentum in three mutually perpendicular directions (e.g. xyz).
- Depending on the forces acting on the system, linear momentum might be conserved in some of these three directions.

  If the net external force on a closed system is zero along an axis, then the component of the linear momentum of the system along that axis cannot change.

6. Conservation of Linear Momentum

- Consider the example of tossing a stone. The gravitational force on the stone changes its linear momentum in the vertical direction. The other two horizontal components of the stone's linear momentum do not change.
- Note that internal forces can change the linear momentum of a portion of a system, but they cannot change the total momentum of the system.

6. Conservation of Linear Momentum

CHECKPOINT 6

An initially stationary device lying on a frictionless floor explodes into two pieces, which then slide across the floor. One piece slides in the positive direction of an x axis. (a) What is the sum of the momenta of the two pieces after the explosion? (b) Can the second piece move at an angle to the x axis? (c) What is the direction of the momentum of the second piece?

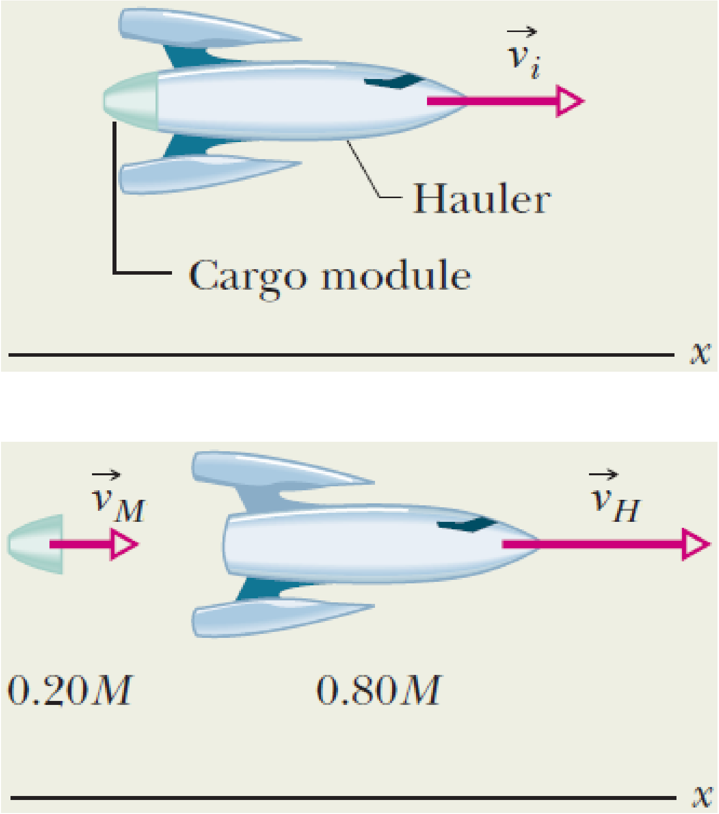
(a) Zero. $\vec{F}_{net} = 0$ and hence $\vec{P} = \text{constant} \ (= 0)$.

(b) No, because the momentum of the first piece is purely in the x axis.

(c) The negative x axis, by conservation of momentum.

6. Conservation of Linear Momentum

Example 5: The figure shows a space hauler and cargo module, of total mass M, traveling along an x axis in deep space. They have an initial velocity of magnitude 2100 km/h. With a small explosion, the hauler ejects the cargo module, of mass $0.20\,\text{M}$. The module then travels at 1700 km/h along the x axis. What then is the velocity of the hauler?



The system is closed and isolated:

$$\vec{P}_f = \vec{P}_i.$$

The initial momentum before the explosion is

$$\vec{P}_i = Mv_i.$$

6. Conservation of Linear Momentum

After the explosion, the total momentum of the hauler and the cargo module is

$$\vec{P}_f = m_H v_H + m_M v_M.$$

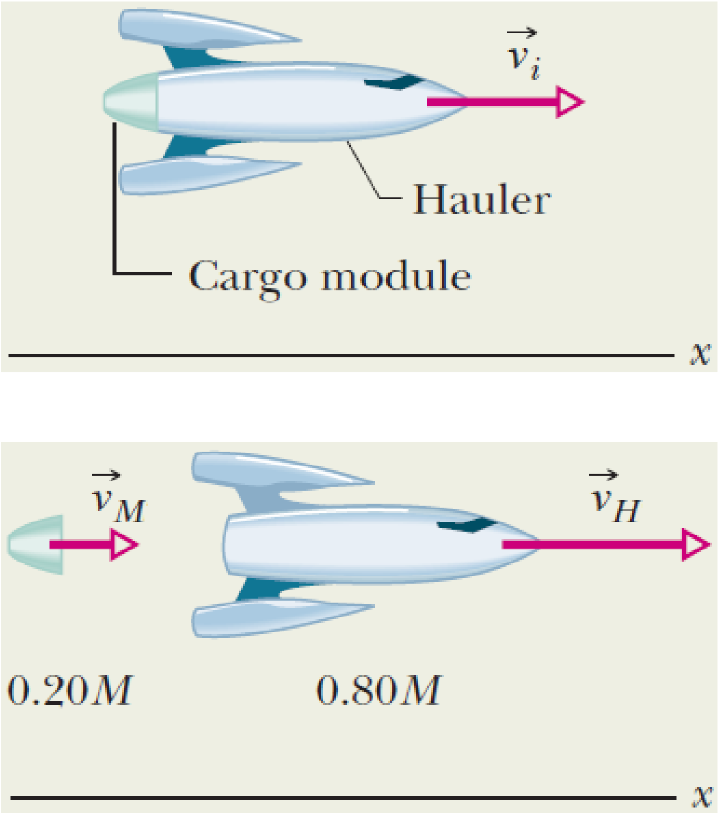
Equating the momenta before and after the collision we write

$$M v_i = m_H v_H + m_M v_M.$$

Solving for v_M and substituting we get that

$$v_M = \frac{M v_i - m_H v_H}{m_M}$$

$$= \frac{M(2100\text{ km/h}) - 0.20M(1700\text{ km/h})}{0.80M} = 2200\,\frac{\text{km}}{\text{h}}.$$

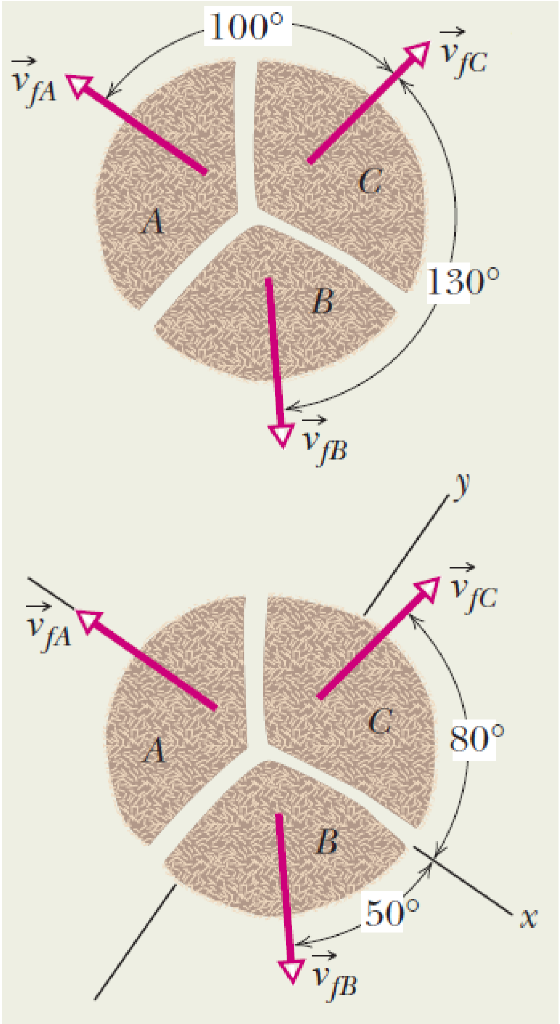


6. Conservation of Linear Momentum

Example 6: A firecracker placed inside a coconut of mass M, initially at rest on a frictionless floor, blows the coconut into three pieces that slide across the floor. An overhead view is shown in the figure. Piece C, with mass $0.30M$, has final speed $v_{fC} = 5.0\ \mathrm{m/s}$.

(a) What is the speed of piece B, with mass $0.20M$?

The system is closed and isolated and therefore $\vec{P}_f = \vec{P}_i$.
The coconut is initially at rest and hence $\vec{P}_i = 0$.



6. Conservation of Linear Momentum

Along the y axis:

$$P_{fy} = P_{fAy} + P_{fBy} + P_{fCy}$$

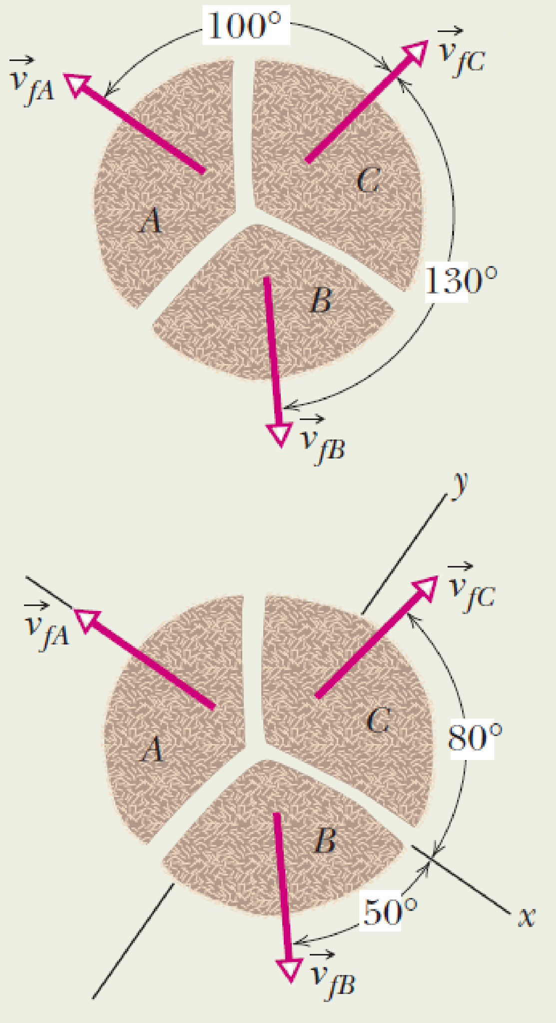
$$= P_{fA}\sin 180° + P_{fB}\sin(-50°) + P_{fC}\sin 80°$$

$$= m_B v_{fB}\sin(-50°) + m_C v_{fC}\sin 80° = 0.$$

Solving for v_{fB} and substituting we get

$$v_{fB} = -\frac{m_C v_{fC}\sin 80°}{m_B\sin(-50°)} = -\frac{(0.3M)\left(5.0\,\frac{\text{m}}{\text{s}}\right)\sin 80°}{(0.2M)\sin(-50°)}$$

$$= 9.6\text{ m/s}.$$



6. Conservation of Linear Momentum

(b) What is the speed of piece A?

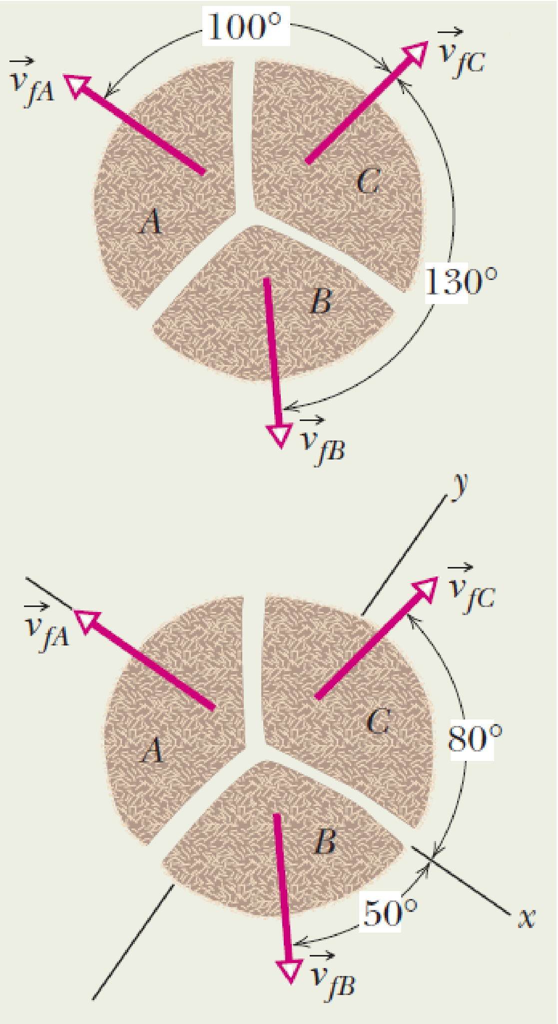
Along the x axis:

$$\begin{aligned} P_{fx} &= P_{fAx} + P_{fBx} + P_{fCx} \\ &= P_{fA}\cos 180° + P_{fB}\cos(-50°) + P_{fC}\cos 80° \\ &= -m_A v_{fA} + m_B v_{fB}\cos(50°) + m_C v_{fC}\cos 80° \\ &= 0. \end{aligned}$$

Solving for v_{fB} and substituting we get

$$v_{fA} = \frac{m_B v_{fB}\cos 50° + m_C v_{fC}\cos 80°}{m_A}$$

$$= \frac{(0.2M)\left(9.6\,\frac{\mathrm{m}}{\mathrm{s}}\right)\cos 50° + (0.3M)\left(5.0\,\frac{\mathrm{m}}{\mathrm{s}}\right)\cos 80°}{0.5M} = 3.0\,\frac{\mathrm{m}}{\mathrm{s}}.$$



7. Momentum and Kinetic Energy in Collisions

- In the remaining of this chapter, we focus on all colliding particles in an isolated closed system, instead of focusing on a single particle.
- We discussed a rule about the system: The total momentum of the system is conserved.
- This rule is very powerful because it enables us to determine the results of a collision without knowing the details of the collision.
- We will be interested in the total kinetic energy of a system of two colliding particles. If that total energy happens to be unchanged by the collision, then we say that the kinetic energy of the system is conserved. Such a collision is called an **elastic collision**.

7. Momentum and Kinetic Energy in Collisions

- In everyday collisions, some of the kinetic energy is always transferred to other forms of energy. Thus the kinetic energy of the system is not conserved and the collision is called an **inelastic collision**.
- In some situations, the loss in kinetic energy of a system due to a collision is ‘small’ and we can approximate the collision to be elastic.
- The greatest loss in kinetic energy occurs if the colliding bodies stick together, in which case the collision is called a **completely inelastic collision**.

8. Inelastic Collisions in One Dimension

- <u>One-Dimensional Inelastic Collision</u>:

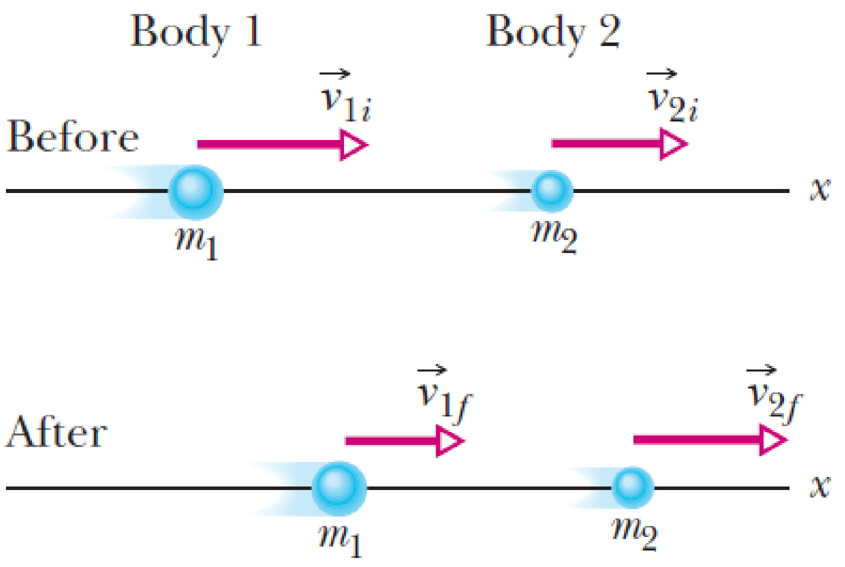
  Consider the two-body system shown in the figure. The velocities before the collision (subscript i) and after the collision (subscript f) are indicated. The system is closed and isolated. Therefore,

$$\vec{P}_i = \vec{P}_f,$$

or
$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}.$$

Using $p = mv$, we can write this relation as

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}.$$



8. Inelastic Collisions in One Dimension

- <u>One-Dimensional Completely Inelastic Collision</u>:

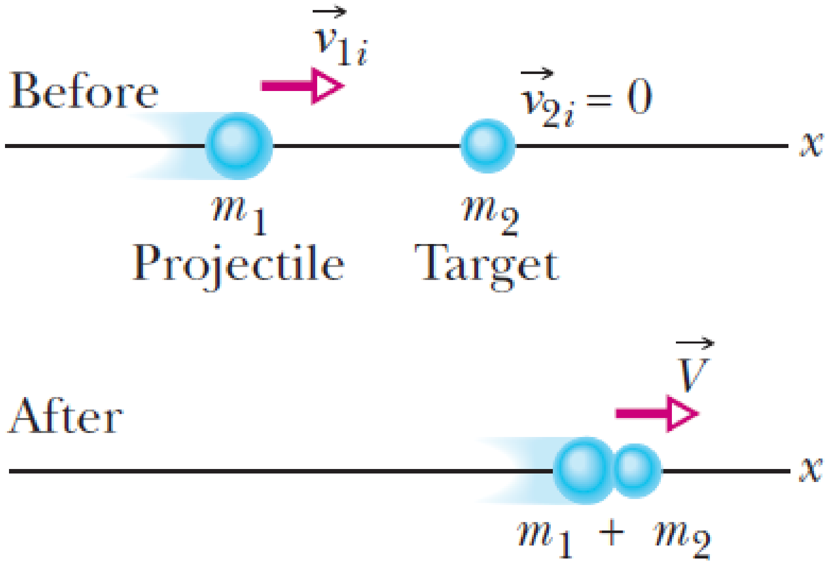
  Consider the situation shown in the figure. After the collision, the two particles stick and move together with velocity V. We therefore write

$$m_1 v_{1i} = (m_1 + m_2)V,$$

or
$$V = \frac{m_1}{m_1+m_2} v_{1i}.$$

When the second particle is moving too, V becomes

$$V = \frac{m_1 v_{1i} + m_2 v_{2i}}{m_1 + m_2}.$$



8. Inelastic Collisions in One Dimension

- Velocity of the Center of Mass:

  In a closed, isolated system, the velocity $\vec{v}_{\mathrm{com}}$ of the center of mass cannot be changed by a collision. We can write a relation between $\vec{v}_{\mathrm{com}}$ and the total momentum $\vec{P}$ of the two-body system. We know that

$$\vec{P} = M\vec{v}_{\mathrm{com}} = (m_1 + m_2)\vec{v}_{\mathrm{com}},$$

  which gives that

$$\vec{v}_{\mathrm{com}} = \frac{\vec{P}}{m_1 + m_2} = \frac{\vec{p}_{1i} + \vec{p}_{2i}}{m_1 + m_2}.$$

8. Inelastic Collisions in One Dimension

CHECKPOINT 7

Body 1 and body 2 are in a completely inelastic one-dimensional collision. What is their final momentum if their initial momenta are, respectively, (a) 10 kg · m/s and 0; (b) 10 kg · m/s and 4 kg · m/s; (c) 10 kg · m/s and −4 kg · m/s?

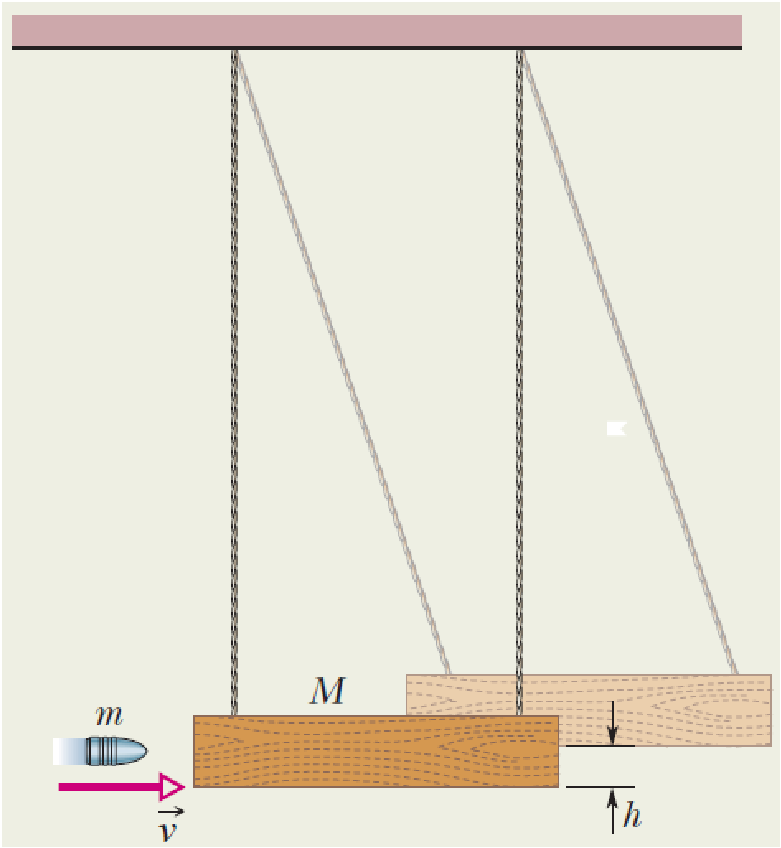
(a) $10\ \mathrm{kg} \cdot \frac{\mathrm{m}}{\mathrm{s}}$.

(b) $14\ \mathrm{kg} \cdot \frac{\mathrm{m}}{\mathrm{s}}$.

(c) $6\ \mathrm{kg} \cdot \frac{\mathrm{m}}{\mathrm{s}}$.

8. Inelastic Collisions in One Dimension

Example 6: The *ballistic pendulum* was used to measure the speeds of bullets before electronic timing devices were developed. The version shown in the figure consists of a large block of wood of mass $M = 5.4\ \mathrm{kg}$, hanging from two long cords. A bullet of mass $m = 9.5\ \mathrm{g}$ is fired into the block, coming quickly to rest. The *block-bullet* then swing upward, their center of mass rising a vertical distance $h = 6.3\ \mathrm{cm}$ before the pendulum comes momentarily to rest at the end of its arc. What is the speed of the bullet just prior to the collision?



8. Inelastic Collisions in One Dimension

We relate the speed V of the bullet-block just after the completely inelastic collision, to the initial bullet's speed v by

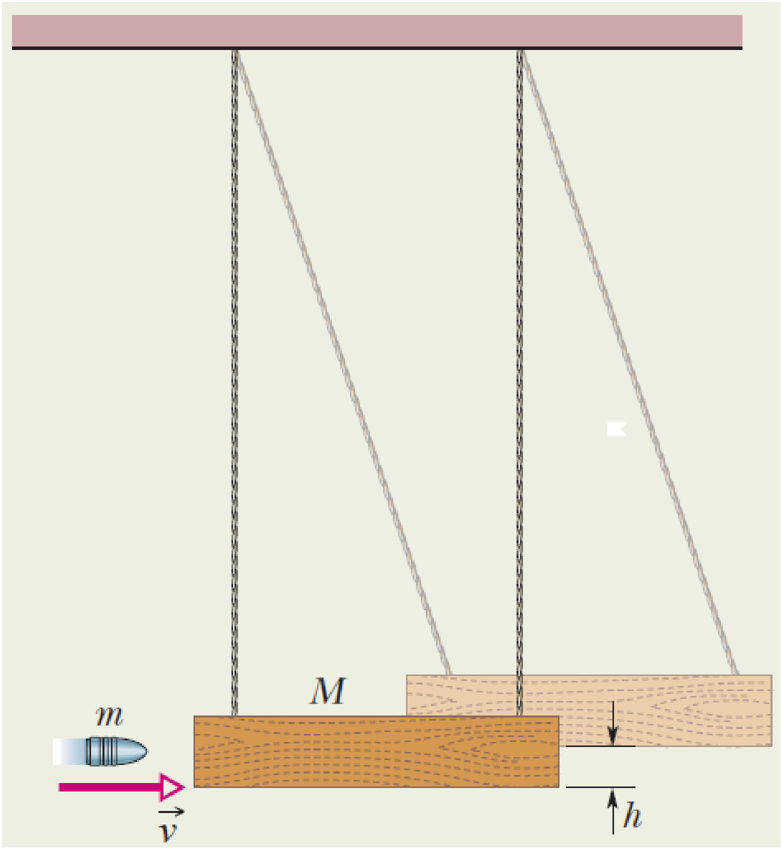
$$mv = (m + M)V,$$

or
$$v = \frac{m+M}{m}V.$$

We can also relate the rise h of the bullet-block to its speed V just after the collision, by

$$(m + M)gh = \frac{1}{2}(m + M)V^2,$$

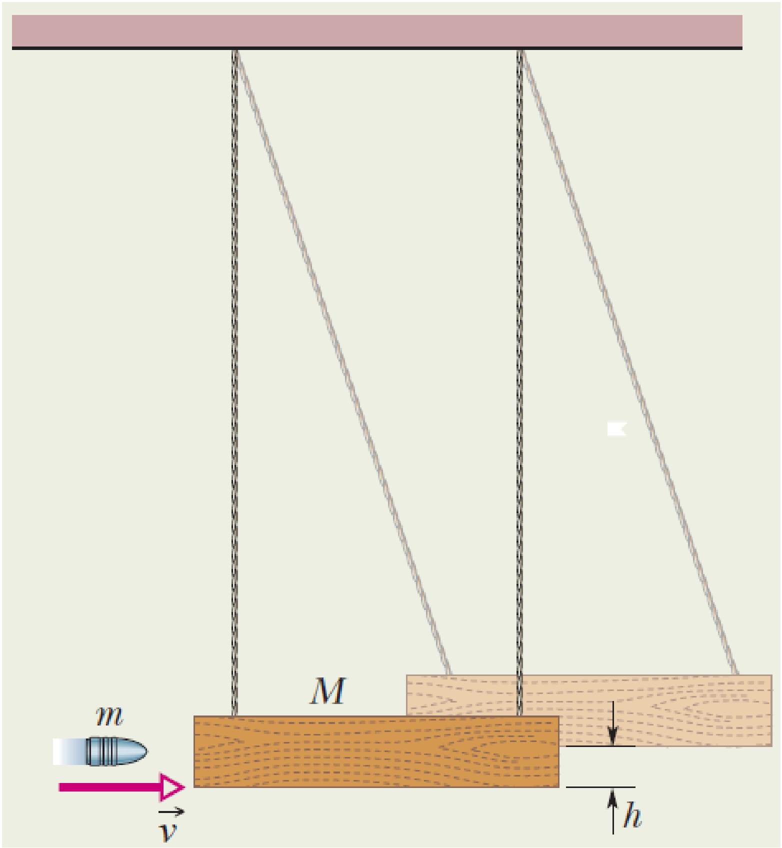
or
$$V = \sqrt{2gh}.$$



8. Inelastic Collisions in One Dimension

The bullet's speed v becomes

$$v = \frac{m+M}{m}\sqrt{2gh}$$
$$= \frac{9.5 \times 10^{-3}\ \text{kg} + 5.4\ \text{kg}}{9.5 \times 10^{-3}\ \text{kg}}\sqrt{2(9.8\ \text{m/s}^2)(0.063\ \text{m})}$$
$$= 630\,\frac{\text{m}}{\text{s}}.$$



9. Elastic Collisions in One Dimension

- Although everyday collisions are inelastic, we still can approximate some of them as being elastic. We can approximate that the total kinetic energy of the colliding bodies is conserved:

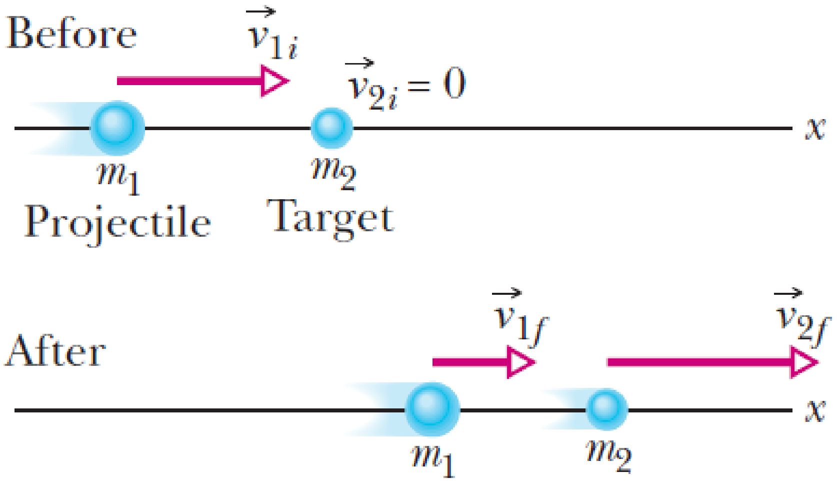
$$\begin{pmatrix}\text{total kinetic energy}\\ \text{before the collisions}\end{pmatrix} = \begin{pmatrix}\text{total kinetic energy}\\ \text{after the collisions}\end{pmatrix}.$$

- This does not mean that the kinetic energy of *each* colliding body cannot change. In an elastic collision, the kinetic energy of each colliding body may change, but the total kinetic energy of the system does not change.

9. Elastic Collisions in One Dimension

- <u>Stationary Target</u>:

  Consider the situation shown in the figure. Assuming that this two-body system is closed and isolated, the net linear momentum of the system is conserved:



$$m_1 v_{1i} = m_1 v_{1f} + m_2 v_{2f}.$$

  The total kinetic energy of the system is conserved:

$$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2.$$

9. Elastic Collisions in One Dimension

- Stationary Target:

  If we know the two masses and v_{1i}, we can write the final velocities in terms of these three quantities as

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i}, \qquad v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i}.$$

  Note that v_{2f} is always positive. v_{1f} is positive when $m_1 > m_2$; the projectile moves forward. v_{1f} is negative when $m_1 < m_2$; the projectile rebounds.

  Let us consider a few special cases:

9. Elastic Collisions in One Dimension

1. <u>Equal masses</u>: If $m_1 = m_2$,

$$v_{1f} = 0 \text{ and } v_{2f} = v_{1i}.$$

The projectile stops completely, transferring all of its kinetic energy to the target.

2. <u>A massive target</u>: If $m_2 \gg m_1$,

$$v_{1f} \approx -v_{1i} \text{ and } v_{2f} = \frac{2m_1}{m_2} v_{1i}.$$

The projectile bounces back with essentially the same initial speed. The target moves forward at a low speed.

9. Elastic Collisions in One Dimension

3. A massive projectile: If $m_1 \gg m_2$,

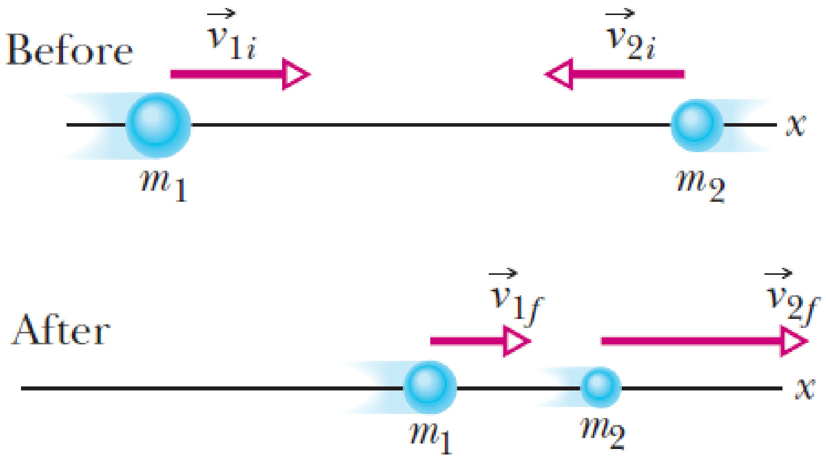
$$v_{1f} = v_{1i} \text{ and } v_{2f} = 2v_{1i}.$$

The projectile keeps on going with essentially the same speed. The target moves forward at *twice* the projectile's speed.

9. Elastic Collisions in One Dimension

- <u>Moving Target</u>:

  Now we examine the situation in which both bodies are initially moving, The conservation of linear momentum and kinetic energy are written, respectively, as



$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}.$$

and

$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2.$$

9. Elastic Collisions in One Dimension

- Moving Target:

  We then solve for the final velocities to get

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i},$$
$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2i}.$$

9. Elastic Collisions in One Dimension

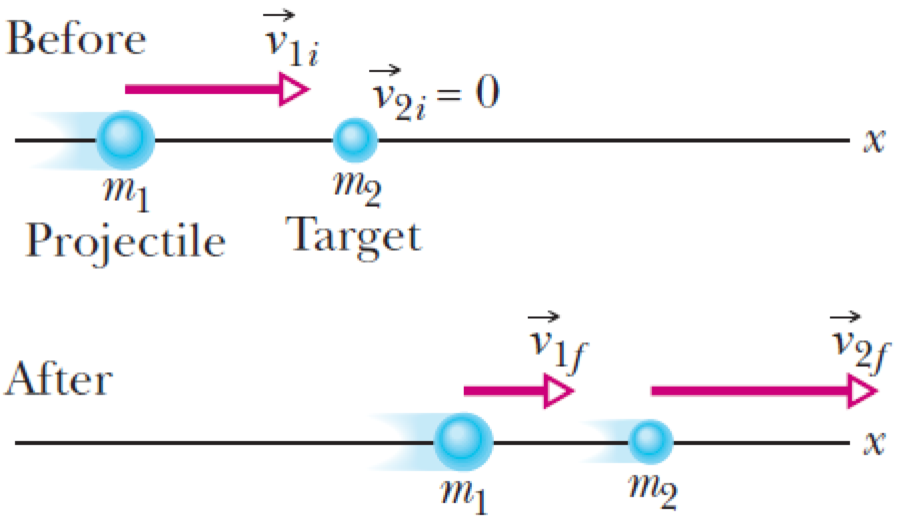
CHECKPOINT 8

What is the final linear momentum of the target in Fig. 9-18 if the initial linear momentum of the projectile is 6 kg · m/s and the final linear momentum of the projectile is (a) 2 kg · m/s and (b) −2 kg · m/s? (c) What is the final kinetic energy of the target if the initial and final kinetic energies of the projectile are, respectively, 5 J and 2 J?

(a) 4 kg · m/s.

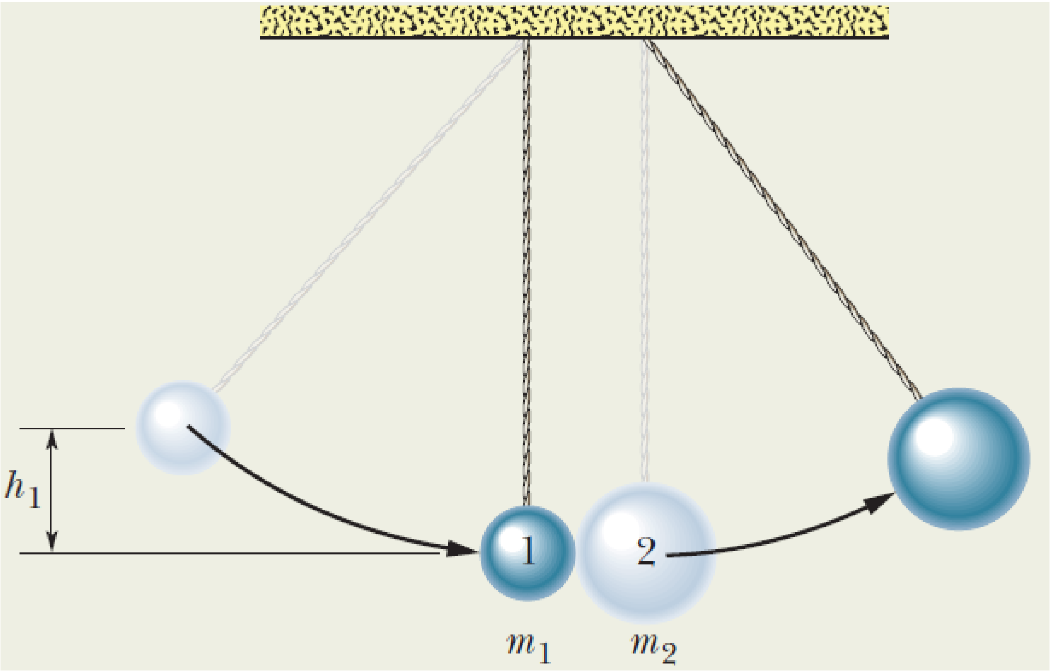
(b) 8 kg · m/s.

(c) 3 J.



9. Elastic Collisions in One Dimension

Example 7: Two metal spheres, suspended by vertical cords, initially just touch, as shown in the figure. Sphere 1, with mass $m_1 = 30$ g, is pulled to the left to height $h_1 = 8.0$ cm, and then released from rest. After swinging down, it undergoes an elastic collision with sphere 2, whose mass $m_2 = 75$ g. What is the velocity v_{1f} of sphere 1 just after the collision?



First, we need to find the speed of sphere 1 just before it collides with sphere 2. We have that

$$\frac{1}{2} m_1 v_{1i}^2 = m_1 g h_1.$$

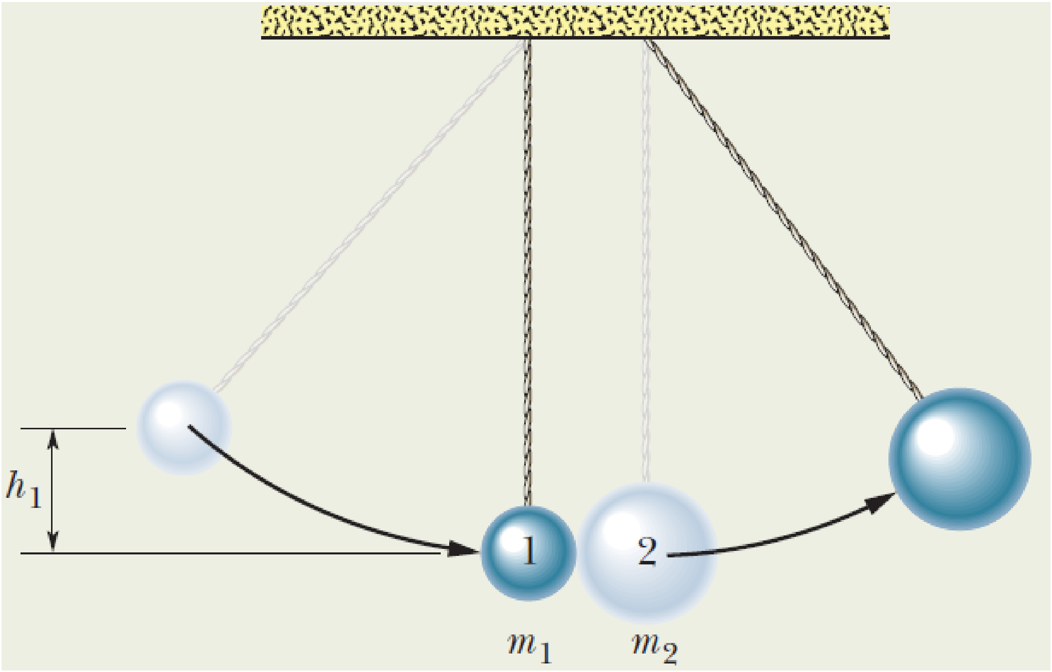
9. Elastic Collisions in One Dimension

We find that

$$v_{1i} = \sqrt{2(9.8\ \mathrm{m/s^2})(0.080\ \mathrm{m})} = 1.25\ \mathrm{m/s}.$$

The final velocity of sphere 1 just after the elastic collision is given by

$$\begin{aligned} v_{1f} &= \frac{m_1 - m_2}{m_1 + m_2} v_{1i} \\ &= \frac{30\ \mathrm{g} - 75\ \mathrm{g}}{30\ \mathrm{g} + 75\ \mathrm{g}}(1.25\ \mathrm{m/s}) \\ &= -0.54\ \frac{\mathrm{m}}{\mathrm{s}}. \end{aligned}$$



10. Collisions in Two Dimensions

When a collision is not head-on, the bodies do not end up travelling along their initial axis. The conservation of linear momentum imposes that

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}.$$

If the collision is elastic then

$$K_{1i} + K_{2i} = K_{1f} + K_{2f}.$$

10. Collisions in Two Dimensions

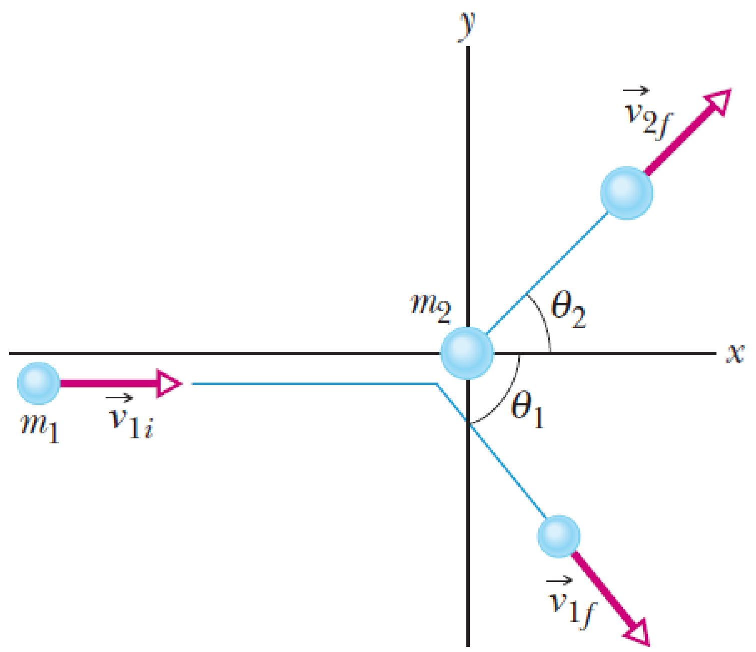
Consider the glancing collision shown in the figure ($v_{2i} = 0$). Conservation of momentum for components along the x axis and y axis read, respectively,

$$m_1 v_{1i} = m_1 v_{1f} \cos\theta_1 + m_2 v_{2f} \cos\theta_2 \,,$$

$$0 = m_1 v_{1f} \sin\theta_1 + m_2 v_{2f} \sin\theta_2 \,.$$

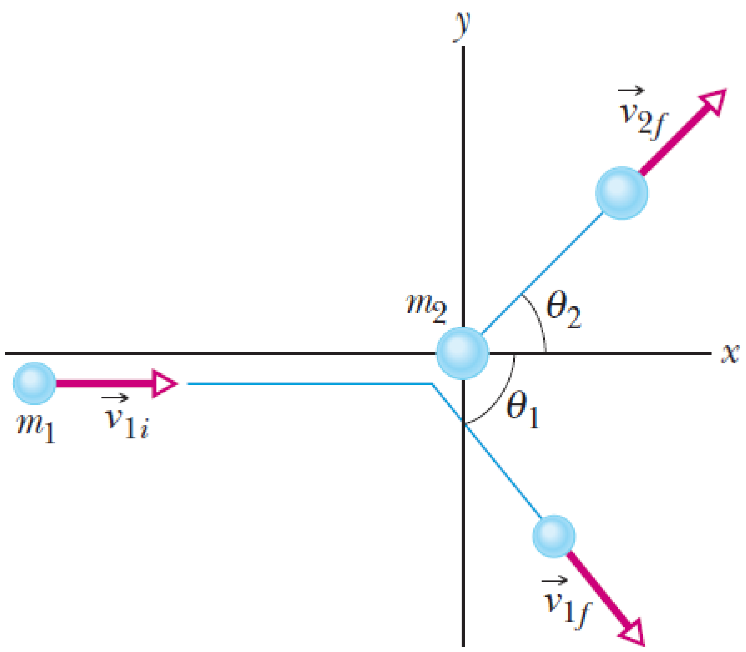
The expression for the conservation of kinetic energy becomes

$$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 .$$



10. Collisions in Two Dimensions

These three equations contain 7 unknowns; 2 masses; 3 velocities; and 2 angles. If we have 4 of these variables we can solve for the remaining three.



10. Collisions in Two Dimensions

CHECKPOINT 9

In Fig. 9-21, suppose that the projectile has an initial momentum of 6 kg·m/s, a final x component of momentum of 4 kg·m/s, and a final y component of momentum of −3 kg·m/s. For the target, what then are (a) the final x component of momentum and (b) the final y component of momentum?

$$\vec{p}_{1i} = \vec{p}_{1f} + \vec{p}_{2f}.$$

(a) $p_{2fx} = p_{1ix} - p_{1fx} = 6\ \mathrm{kg \cdot m/s} - 4\ \mathrm{kg \cdot m/s} = 2\ \mathrm{kg \cdot m/s}.$

(b) $p_{2fy} = p_{1iy} - p_{1fy} = 0\ \mathrm{kg \cdot m/s} - (-3\ \mathrm{kg \cdot m/s}) = 3\ \mathrm{kg \cdot m/s}.$