Chapter 3

Vectors

Vectors

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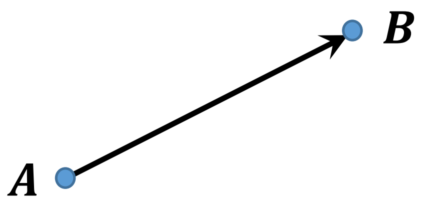
Vectors and Scalars

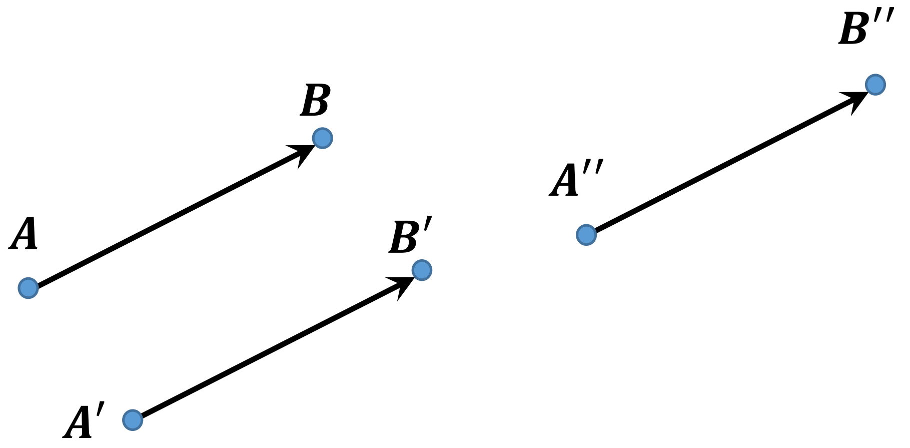
- In physics and engineering we deal with several vector quantities.
- Along a straight line we used a sign (+/-) to indicate direction. However, in three-dimensions, we need to use a vector.
- A **scalar quantity** has magnitude only. Examples are temperature, energy and mass. Scalars are specified by a number with a unit, such as 45°C and 55 kg. They obey the rules of arithmetic and ordinary algebra.
- A **vector quantity** has both direction and magnitude (5 m, north). Velocity, acceleration and electric field are examples of vector quantities. Vectors obey the rules of vector algebra.

Vectors and Scalars

- The simplest vector quantity is the displacement vector. We represent a displacement from A to B by an arrow. This arrow represent the vector graphically.



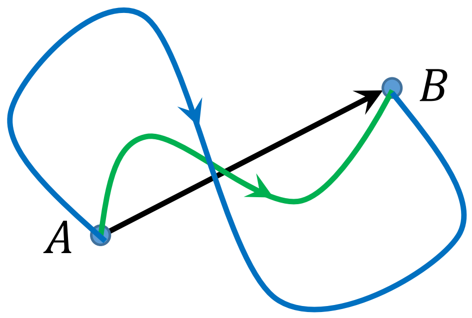
- A vector can be shifted without changing its value if its length and direction are not changed. These three vectors represent the same change of position!



Vectors and Scalars

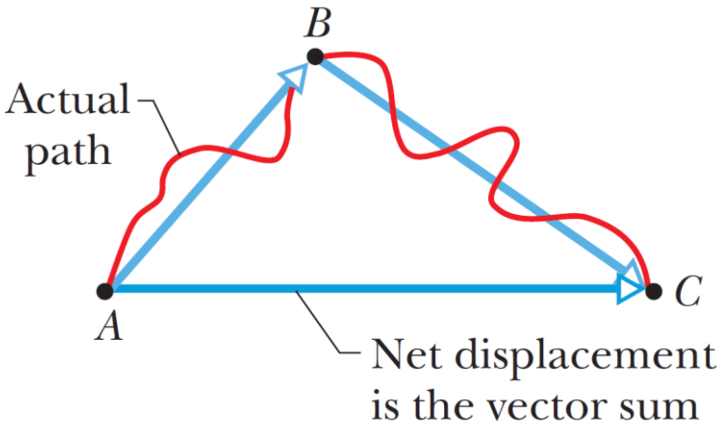
- The displacement vector tells us only about the initial and final positions.

  The three paths represent the same displacement vector.



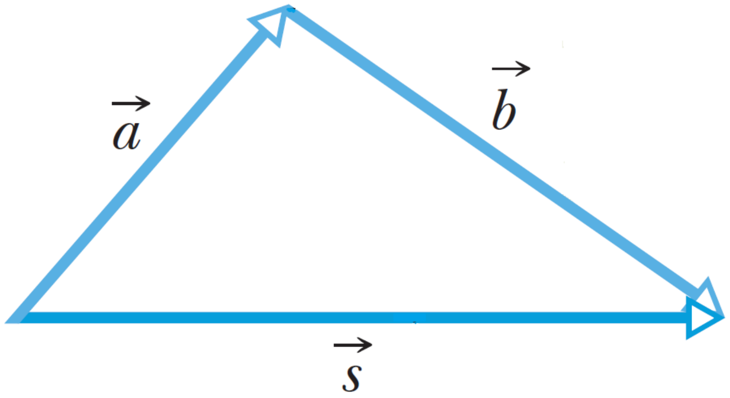
Adding Vectors Geometrically

- For a particle that moves from A to B then from B to C, we can represent its overall displacement with two successive displacement vectors: AB and BC. The net displacement is a single displacement from A to C. We call AC the **vector sum** or **resultant** of vectors AB and BC.



Adding Vectors Geometrically

- A vector will be labeled with an arrow over an *italic* symbol, e.g. $\vec{a}$. A vector magnitude will be labeled with an *italic* only, e.g. a.



- We write the relation among $\vec{a}$, $\vec{b}$ and $\vec{s}$ as

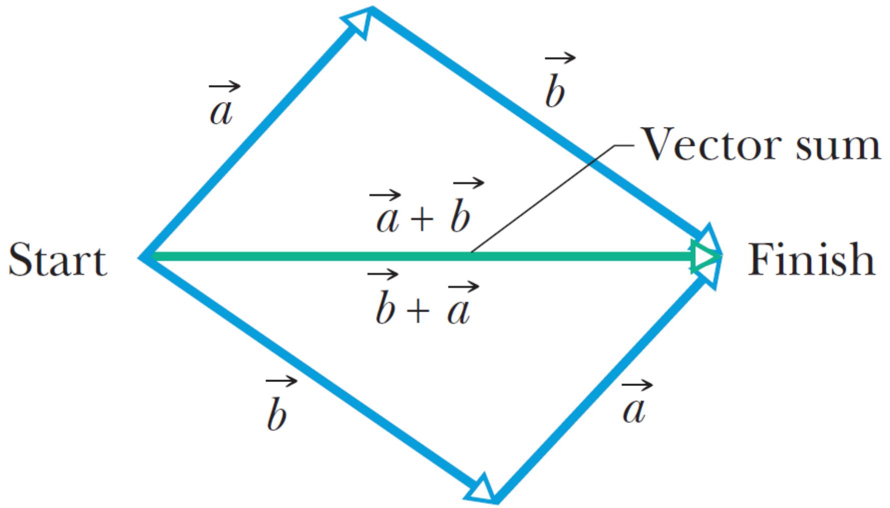
$$\vec{s} = \vec{a} + \vec{b}.$$

Adding Vectors Geometrically

- **Adding vectors geometrically:**
  1. We draw $\vec{a}$ at the right scale and angle.
  2. We draw $\vec{b}$ at the same scale and right angle with its tail at the head of $\vec{a}$.
  3. $\vec{s}$ is the vector that connects the tail of $\vec{a}$ to the head of $\vec{b}$.

- Vector addition properties:
  1. It is commutative:

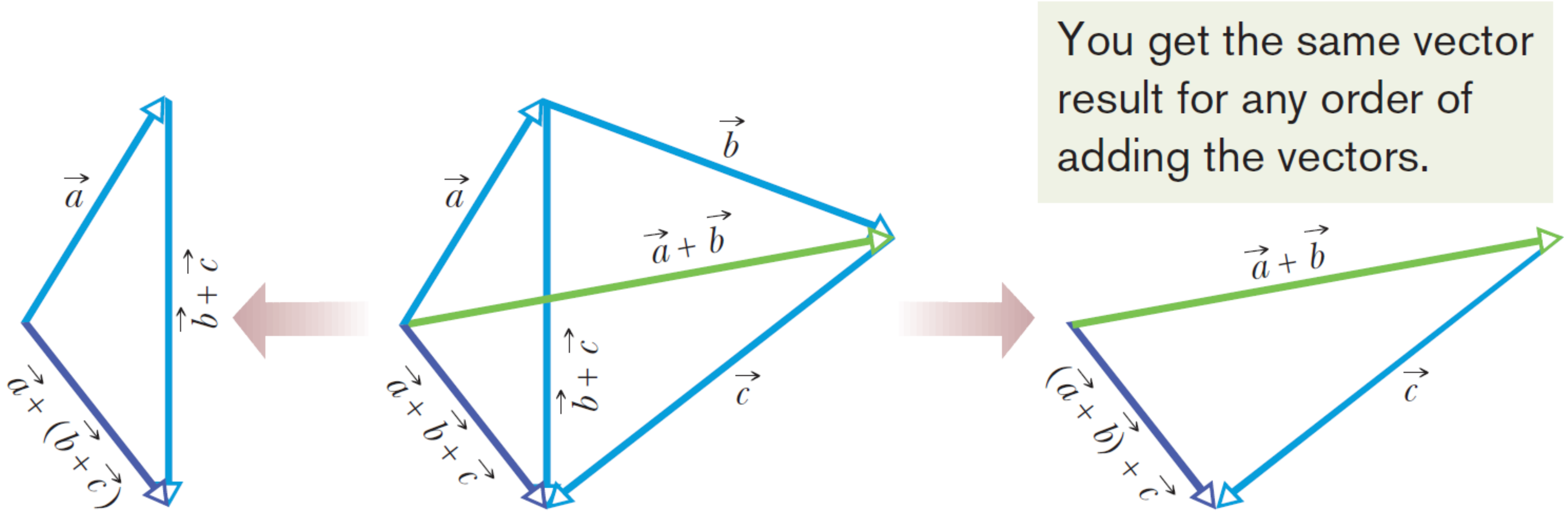
$$\vec{a} + \vec{b} = \vec{b} + \vec{a}.$$



Adding Vectors Geometrically

- Vector addition properties:
  2. It is associative:

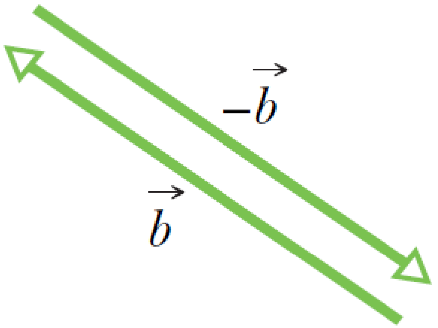
$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}).$$



Adding Vectors Geometrically

- **Vector Subtraction**

  The vector $-\vec{b}$ is the same as $\vec{b}$ but with opposite direction.



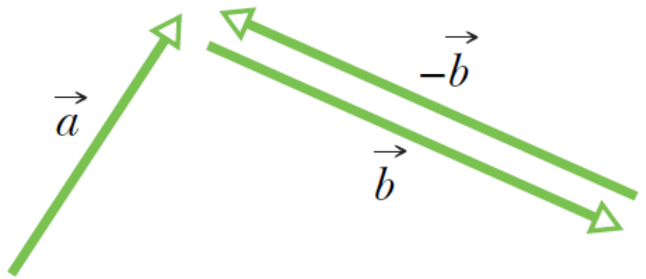
$$\vec{b} + (-\vec{b}) = 0.$$

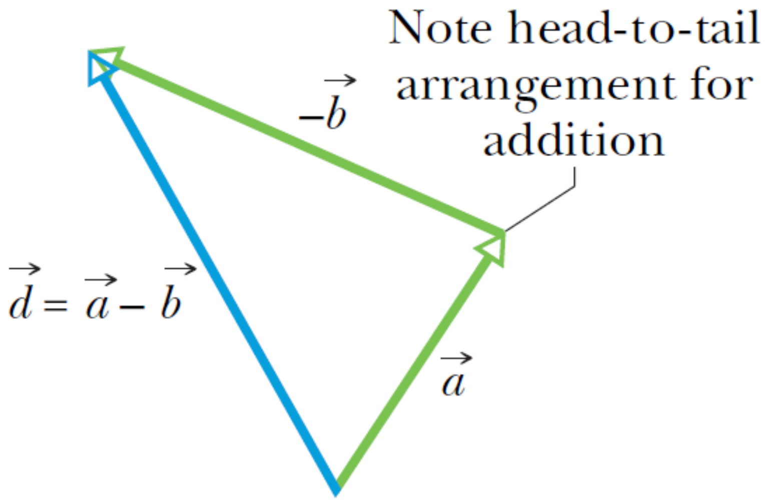
  Thus adding $-\vec{b}$ is the same as subtracting $\vec{b}$.

Adding Vectors Geometrically

- **Vector Subtraction**

$$\vec{d} = \vec{a} - \vec{b} = \vec{a} + (-\vec{b}).$$





Adding Vectors Geometrically

- In a vector equation, we can move terms from one side to the other with a sign change, just like in usual algebra. The equation above can be rewritten as:

$$\vec{d} + \vec{b} = \vec{a} \ \text{ or } \ \vec{a} = \vec{d} + \vec{b}.$$

Adding Vectors Geometrically

CHECKPOINT 1

The magnitudes of displacements $\vec{a}$ and $\vec{b}$ are 3 m and 4 m, respectively, and $\vec{c} = \vec{a} + \vec{b}$. Considering various orientations of $\vec{a}$ and $\vec{b}$, what is (a) the maximum possible magnitude for $\vec{c}$ and (b) the minimum possible magnitude?

(a) When $\vec{a}$ and $\vec{b}$ are in the same direction;

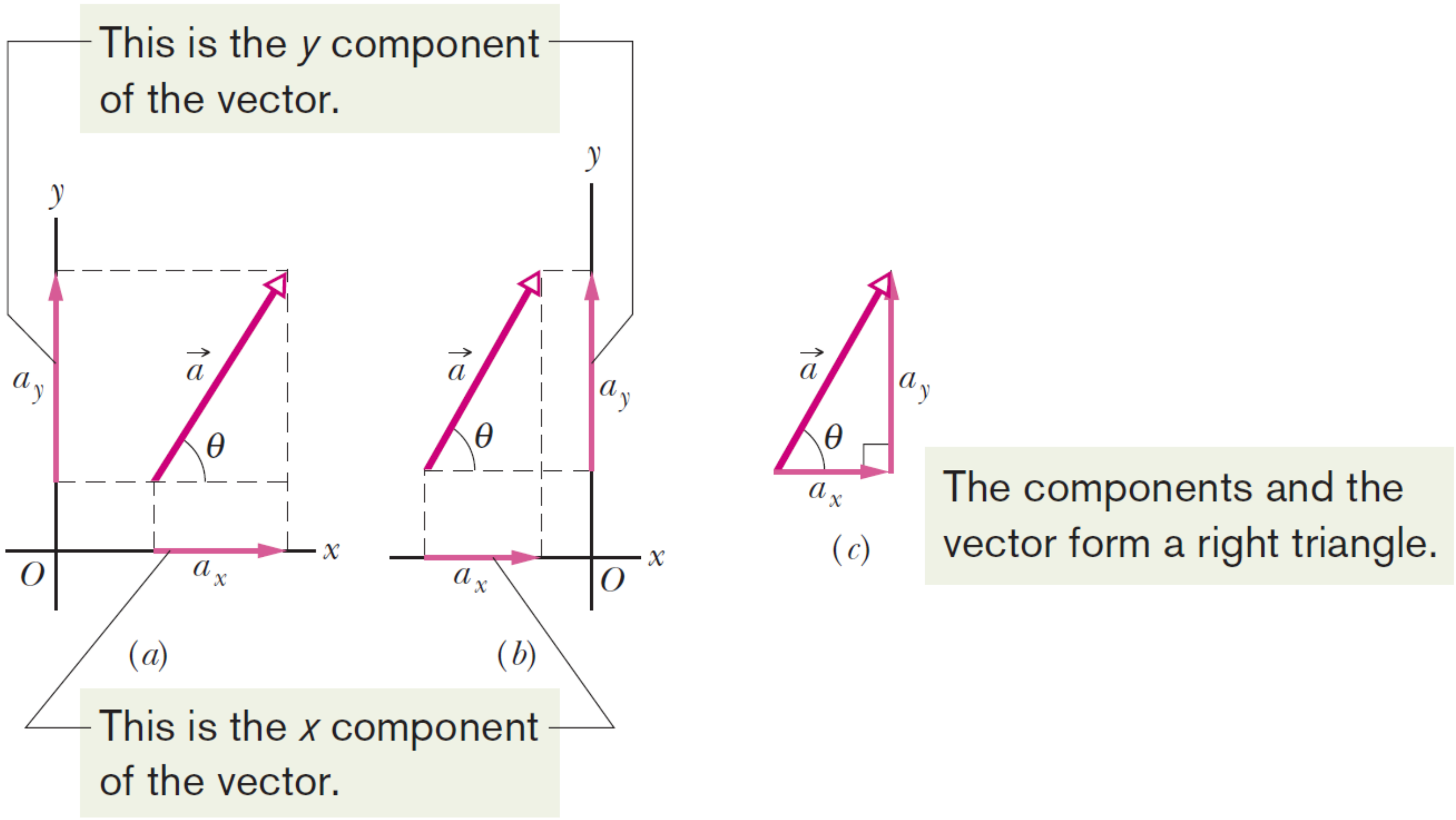
$$c = a + b = 3\text{ m} + 4\text{ m} = 7\text{ m}.$$

(b) When $\vec{a}$ and $\vec{b}$ are in opposite directions;

$$c = |a - b| = |3\text{ m} - 4\text{ m}| = 1\text{ m}.$$

Components of Vectors

- A **component** of a vector is the projection of the vector on an axis.



Components of Vectors

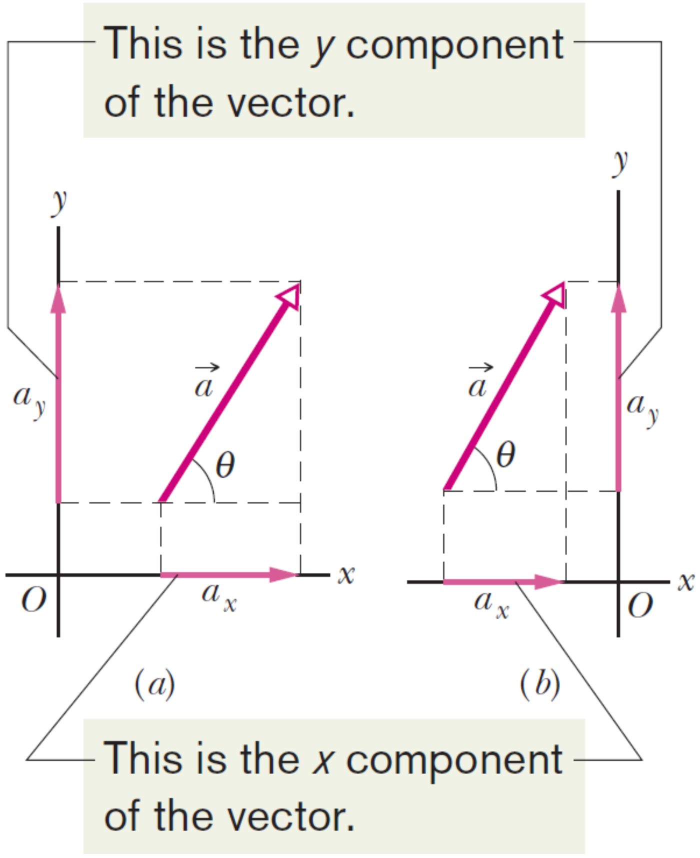
- The process of finding the components of a vector is called **resolving the vector**.
- From the figure we write

$$a_x = a\cos\theta \ \text{ and } \ a_y = a\sin\theta\,.$$

  - θ is the angle $\vec{a}$ makes with the positive *x*-axis.
- A vector $\vec{a}$ is determined completely either by
  1. a and θ, or
  2. a_x and a_y.

  The two are related by

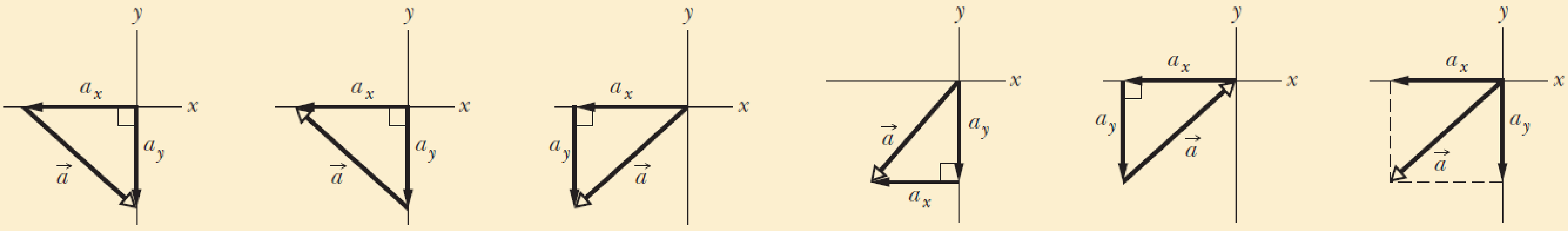
$$a = \sqrt{a_x^2 + a_y^2} \ \text{ and } \ \tan\theta = \frac{a_y}{a_x}.$$



Components of Vectors

Checkpoint 2

In the figure, which of the indicated methods for combining the x and y components of vector $\vec{a}$ are proper to determine that vector?

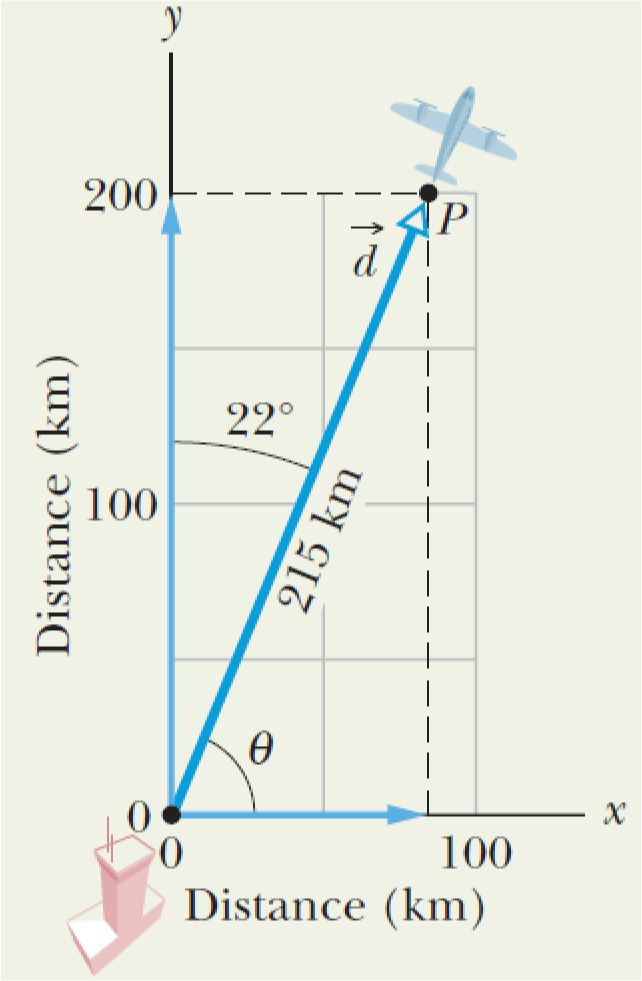


(a) (b) (c) (d) (e) (f)

c, d & f

Components of Vectors

Example 1: A small airplane leaves an airport on an overcast day and is later sighted 215 km away, in a direction making an angle of 22° east of due north. How far east and north is the airplane from the airport when sighted?



$$\theta = 90° - 22° = 68°$$

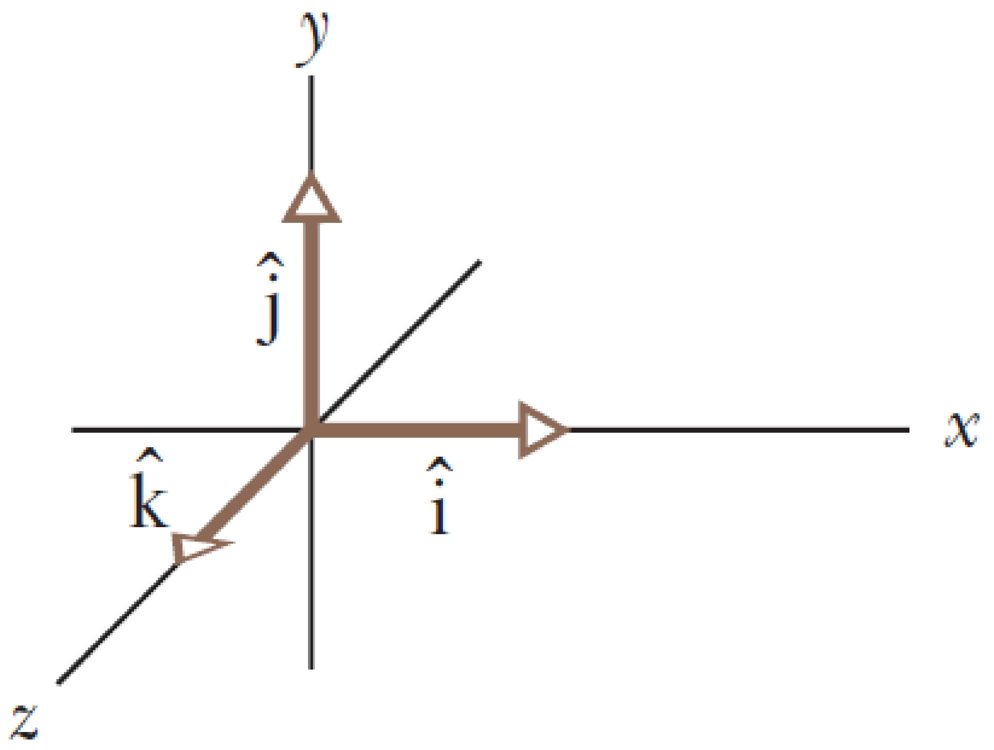
$$d_x = d\cos\theta = (215\text{ km})(\cos 68°) = 81\text{ km}.$$

$$d_y = d\cos\theta = (215\text{ km})(\sin 68°) = 199\text{ km} \approx 200\text{ km}.$$

The airplane is 81 km east and 200 km north.

3.5 Unit Vectors

- A unit vector is a dimensionless vector with magnitude of 1 in a particular direction.
- The units vectors in the positive directions of the x, y and z axes are labeled $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$, respectively.

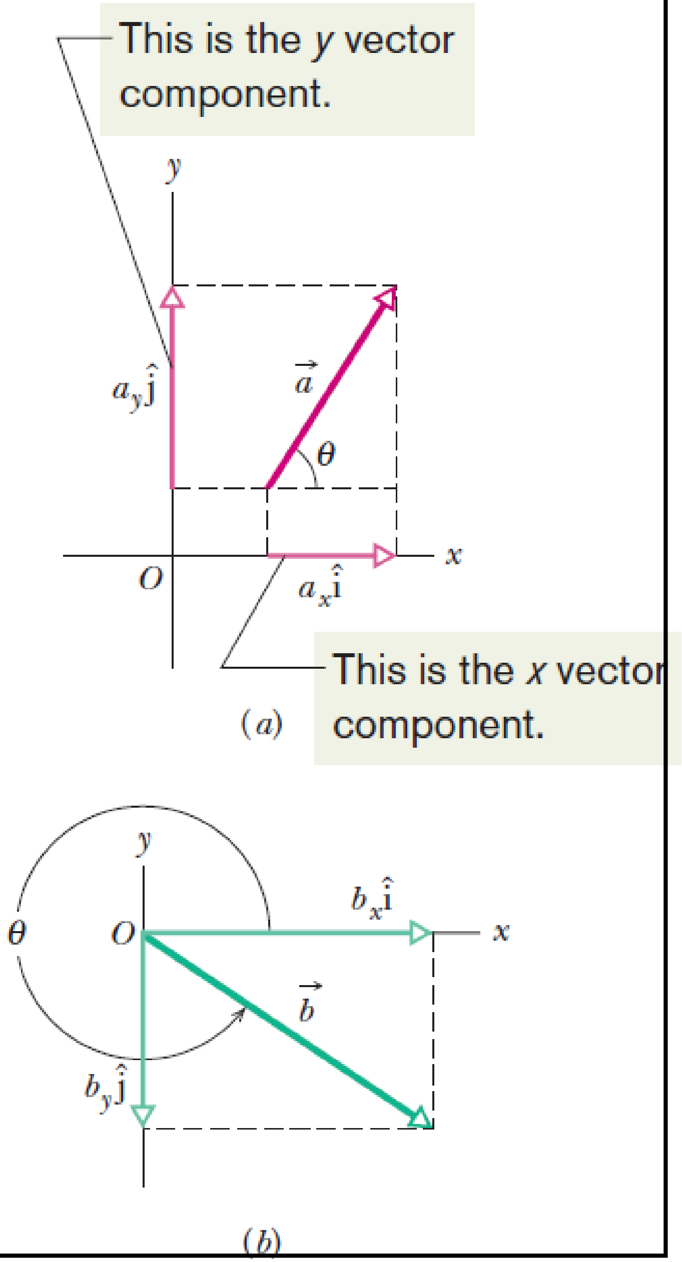


3.5 Unit Vectors

- Unit vectors are very helpful for expressing other vectors:

$$\vec{a} = a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}}.$$
$$\vec{b} = b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}}.$$

- $a_x\hat{\mathrm{i}}$ and $a_y\hat{\mathrm{j}}$ are called the **vector components** of $\vec{a}$, and a_x and a_y are called the **scalar components** of $\vec{a}$ (or simply **components.**)



3.6 Adding Vectors by Components

- We can add vectors axis by axis. Consider the vector sum

$$\vec{r} = \vec{a} + \vec{b}.$$

- If the two vectors are equal then their components are. We then write

$$\begin{aligned} r_x &= a_x + b_x, \\ r_y &= a_y + b_y, \\ r_z &= a_z + b_z. \end{aligned}$$

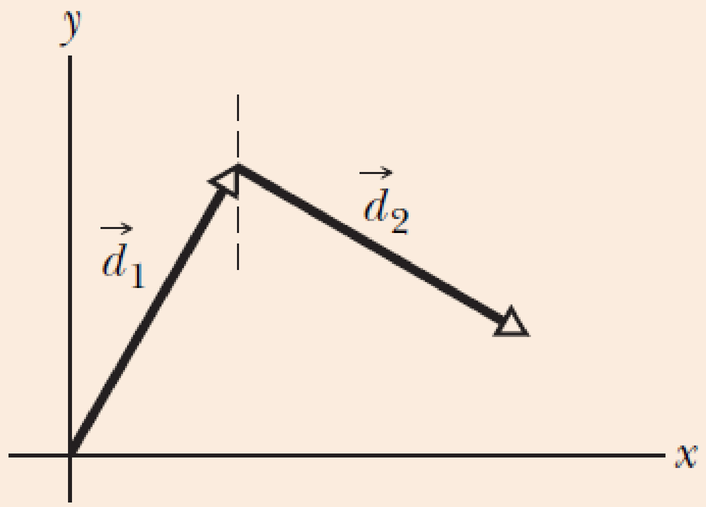
- To add vectors we
    1. Resolve them into their scalar components.
    2. Combine these scalar components axis by axis.
    3. Combine the components of $\vec{r}$ to get $\vec{r}$ itself.

The same applies to vector subtraction

3.6 Adding Vectors by Components

CHECKPOINT 3

(a) In the figure here, what are the signs of the x components of $\vec{d}_1$ and $\vec{d}_2$? (b) What are the signs of the y components of $\vec{d}_1$ and $\vec{d}_2$? (c) What are the signs of the x and y components of $\vec{d}_1 + \vec{d}_2$?



(a) $+, +$
(b) $+, -$
(c) $+, +$

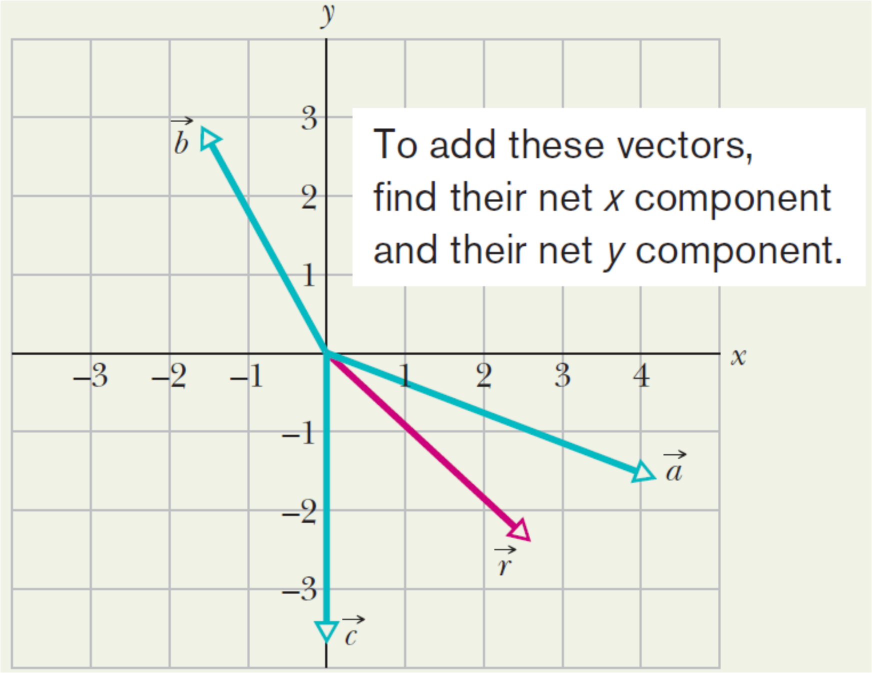
3.6 Adding Vectors by Components

Example 2: What is the sum $\vec{r}$ of the following three vectors:

$$\vec{a} = (4.2\text{ m})\hat{\imath} - (1.5\text{ m})\hat{\jmath},$$

$$\vec{b} = (-1.6\text{ m})\hat{\imath} + (2.9\text{ m})\hat{\jmath},$$

$$\vec{c} = (-3.7\text{ m})\hat{\jmath}.$$



We add components axis by axis:

$$r_x = a_x + b_x + c_x = 4.2\text{ m} - 1.6\text{ m} + 0$$
$$= 2.6\text{ m},$$

$$r_y = a_y + b_y + c_y = -1.5\text{ m} + 1.6\text{ m} - 3.7\text{ m}$$
$$= -2.3\text{ m}.$$

3.6 Adding Vectors by Components

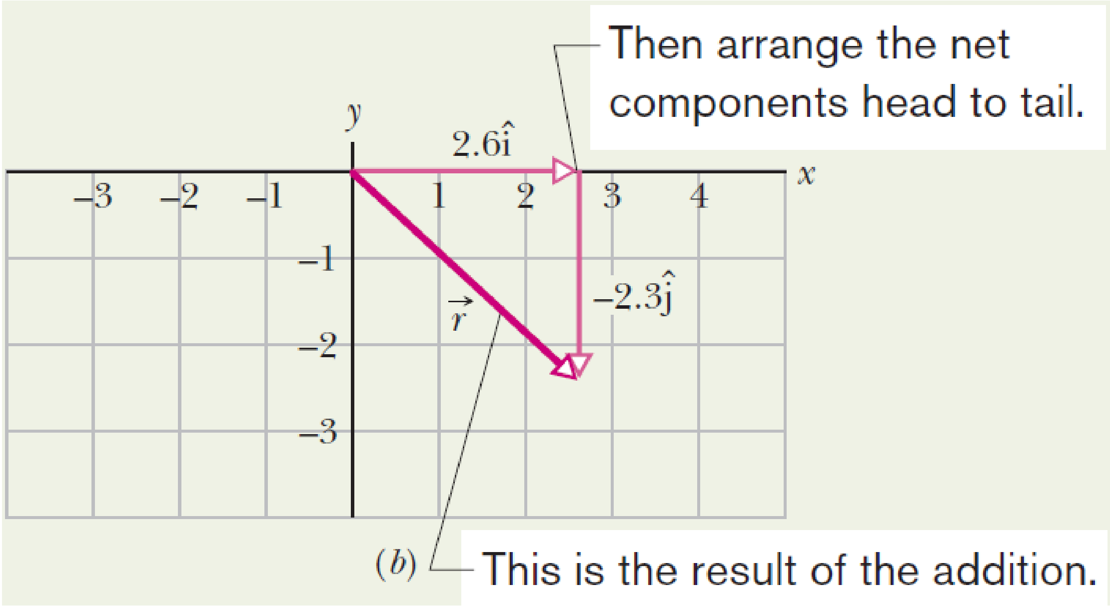
Combining the components of $\vec{r}$ we get

$$\vec{r} = (2.6\ \text{m})\hat{\imath} - (2.3\ \text{m})\hat{\jmath}.$$

We can also answer the question by giving the magnitude and the angle of $\vec{r}$:

$$r = \sqrt{(2.6\ \text{m})^2 + (2.3\ \text{m})^2} \approx 3.5\ \text{m}.$$

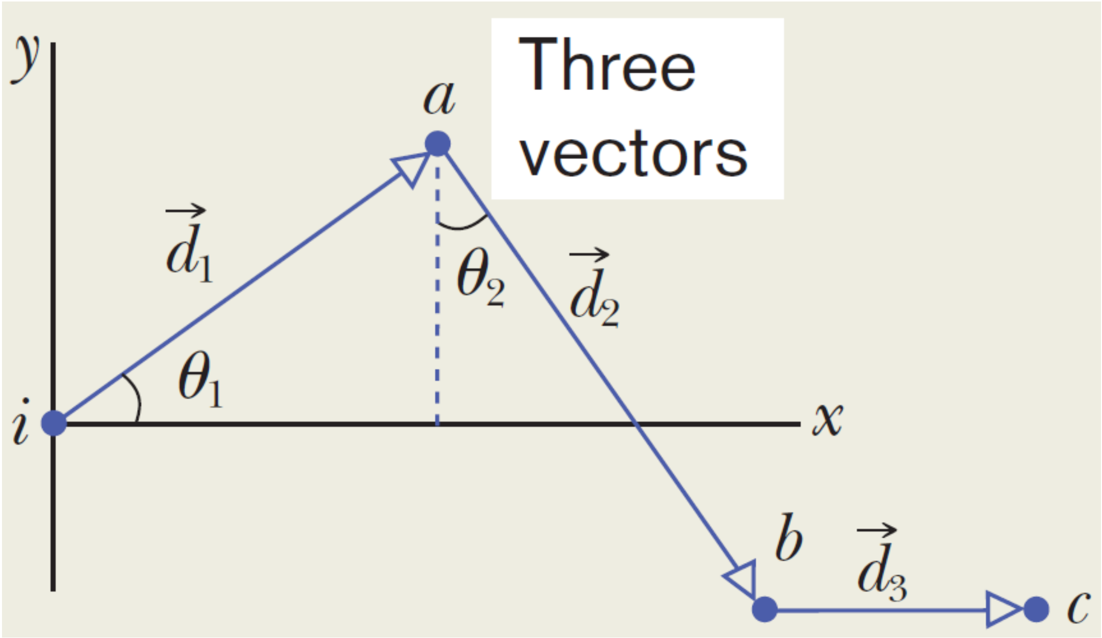
$$\theta = \tan^{-1}\frac{r_y}{r_x} = -41°.$$



3.6 Adding Vectors by Components

Example 3: Find the net displacement $\vec{d}_{\text{net}}$ of the three displacements $\vec{d}_1$, $\vec{d}_2$ and $\vec{d}_3$, where

$$d_1 = 6.00 \text{ m} \quad \theta_1 = 40.0°$$
$$d_2 = 8.00 \text{ m} \quad \theta_2 = 30.0°$$
$$d_3 = 5.00 \text{ m} \quad \theta_3 = 0°$$



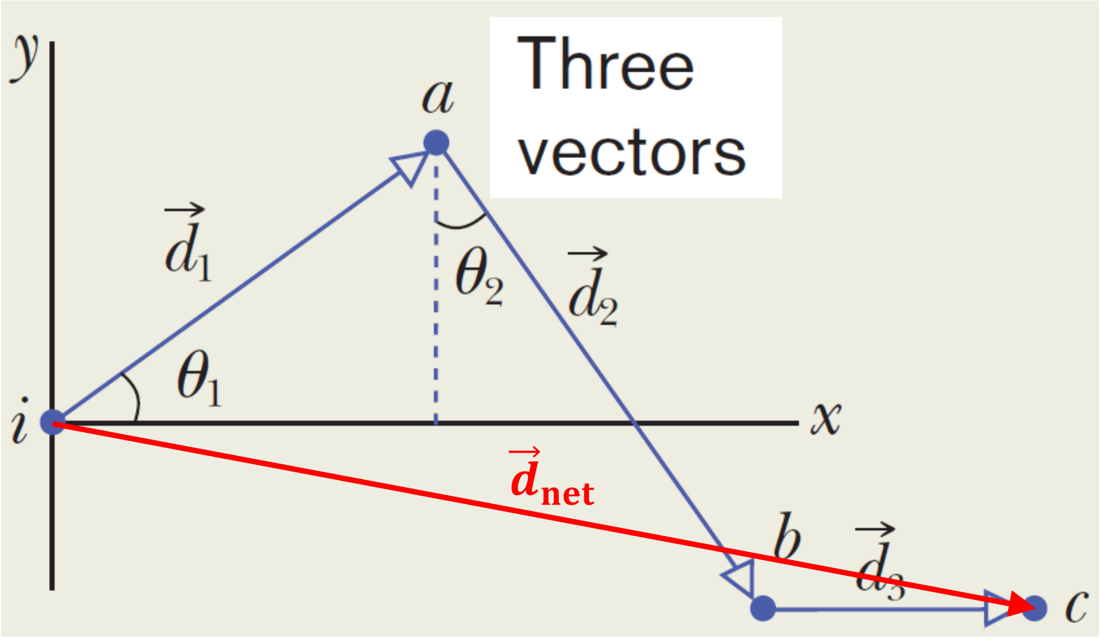
3.6 Adding Vectors by Components

The angle between $\vec{d}_2$ and the x-axis is -60.0.

$$d_{\text{net},x} = d_{1x} + d_{2x} + d_{3x}$$

$$d_{1x} = (6.0 \text{ m}) \cos 40.0° = 4.60 \text{ m}$$
$$d_{2x} = (8.0 \text{ m}) \cos -60.0° = 4.00 \text{ m}$$
$$d_{3x} = (5.0 \text{ m}) \cos 0 = 5.00 \text{ m}$$

$$d_{\text{net},x} = 4.60 \text{ m} + 4.00 \text{ m} + 5.00 \text{ m} = 13.6 \text{ m}.$$

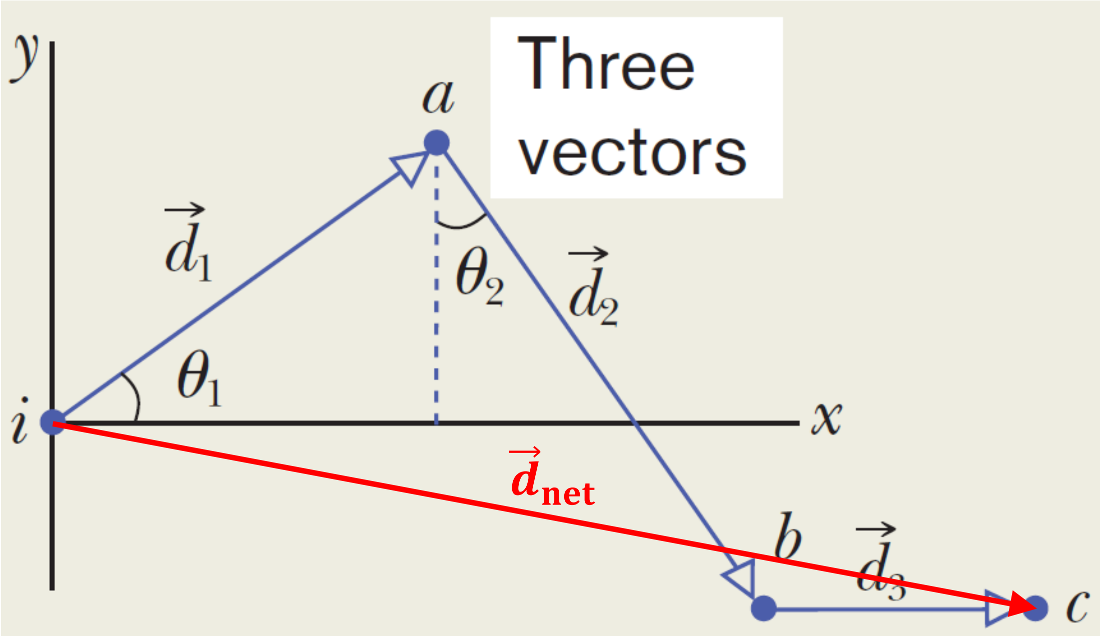


3.6 Adding Vectors by Components

$$d_{\text{net},y} = d_{1y} + d_{2y} + d_{3y}$$

$$\begin{aligned} d_{1y} &= (6.0\text{ m})\sin 40.0^\circ &&= 3.86\text{ m} \\ d_{2y} &= (8.0\text{ m})\sin -60.0^\circ &&= -6.93\text{ m} \\ d_{3y} &= (5.0\text{ m})\sin 0 &&= 0.00\text{ m} \end{aligned}$$

$$\begin{aligned} d_{\text{net},x} &= 3.86\text{ m} - 6.93\text{ m} + 0\text{ m} \\ &= -3.07\text{ m.} \end{aligned}$$



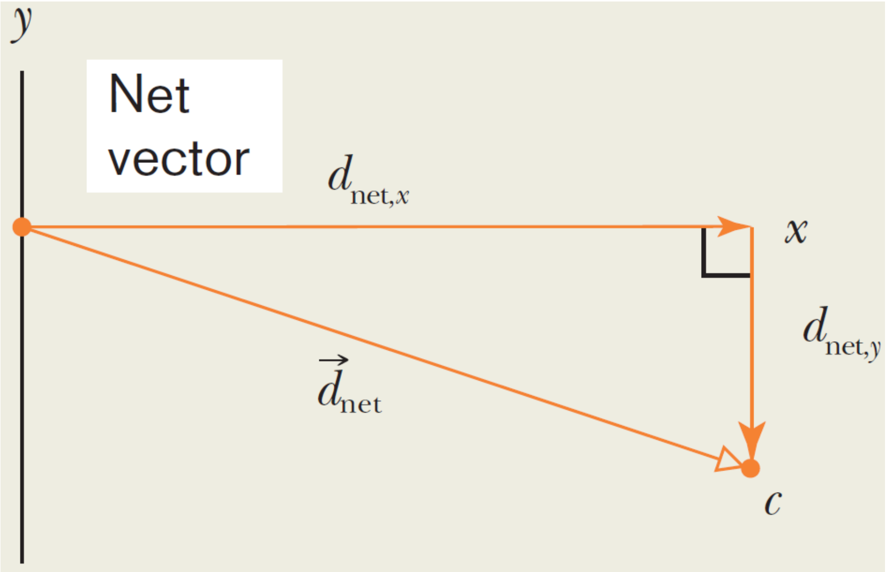
3.6 Adding Vectors by Components

$$\vec{d}_{\text{net}} = (13.6\text{ m})\hat{\imath} - (3.07\text{ m})\hat{\jmath}.$$

The magnitude and the angle $\vec{d}_{\text{net}}$ are, respectively,

$$d_{\text{net}} = \sqrt{(13.6\text{ m})^2 + (-3.07\text{ m})^2} = 13.9\text{ m},$$

$$\theta = \tan^{-1}\frac{-3.07\text{ m}}{13.6\text{ m}} = -12.7°$$



3.8 Multiplying Vectors

- There are three ways in which vectors are multiplied:
   1. **Multiplying a Vector by a Scalar:**

   When a vector $\vec{a}$ multiplied by a scalar s, the resulting vector has magnitude $|s|a$.

   $s\vec{a}$ is in the direction of $\vec{a}$ if $s > 0$ and in the opposite direction of $\vec{a}$ when $s < 0$.

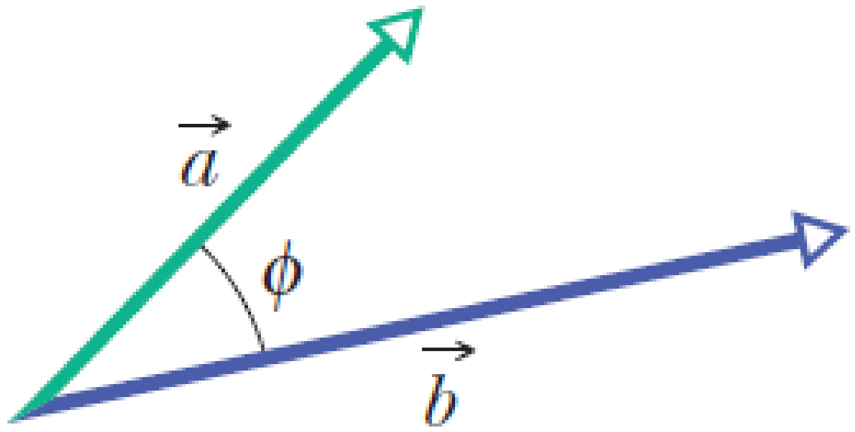
3.8 Multiplying Vectors

2. **Multiplying a Vector by a Vector:** The Scalar Product (Dot Product)

The scalar product of the vectors $\vec{a}$ and $\vec{b}$ is written as $\vec{a} \cdot \vec{b}$ and is defined to be

$$\vec{a} \cdot \vec{b} = ab\cos\phi\,.$$

Note that $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$.

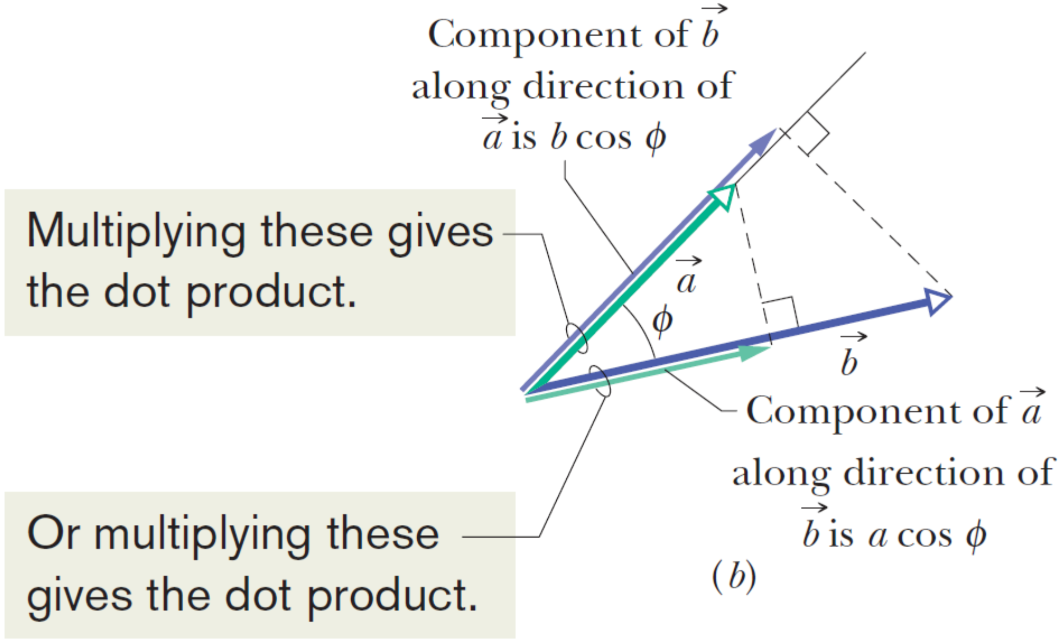


3.8 Multiplying Vectors

The scalar product is the product of the magnitude of one vector and the scalar component of the second vector along the direction of the first vector.

If the angle between two vectors is 0°, the component of one vector along the other is maximum, and so also is the dot product of the vectors.

If, instead, is 90°, the component of one vector along the other is zero, and so is the dot product.



3.8 Multiplying Vectors

In unit vector notation

$$\begin{aligned}\vec{a}\cdot\vec{b} &= \left(a_x\hat{\imath}+a_y\hat{\jmath}+a_z\hat{\mathrm{k}}\right)\cdot\left(b_x\hat{\imath}+b_y\hat{\jmath}+b_z\hat{\mathrm{k}}\right)\\ &= a_xb_x(\hat{\imath}\cdot\hat{\imath})+a_xb_y(\hat{\imath}\cdot\hat{\jmath})+a_xb_z\left(\hat{\imath}\cdot\hat{\mathrm{k}}\right)\\ &\quad + a_yb_x(\hat{\jmath}\cdot\hat{\imath})+a_yb_y(\hat{\jmath}\cdot\hat{\jmath})+a_yb_z\left(\hat{\jmath}\cdot\hat{\mathrm{k}}\right)\\ &\quad + a_zb_x\left(\hat{\mathrm{k}}\cdot\hat{\imath}\right)+a_zb_y\left(\hat{\mathrm{k}}\cdot\hat{\jmath}\right)+a_zb_z\left(\hat{\mathrm{k}}\cdot\hat{\mathrm{k}}\right)\\ &= a_xb_x+a_yb_y+a_zb_z.\end{aligned}$$

We used that

$$\hat{\imath}\cdot\hat{\imath}=\hat{\jmath}\cdot\hat{\jmath}=\hat{\mathrm{k}}\cdot\hat{\mathrm{k}}=1\times 1\cos 0=1,$$

and that

$$\hat{\imath}\cdot\hat{\jmath}=\hat{\jmath}\cdot\hat{\imath}=\hat{\imath}\cdot\hat{\mathrm{k}}=\hat{\mathrm{k}}\cdot\hat{\imath}=\hat{\jmath}\cdot\hat{\mathrm{k}}=\hat{\mathrm{k}}\cdot\hat{\jmath}=\cos 90°=0.$$

3.8 Multiplying Vectors

CHECKPOINT 4

Vectors $\vec{C}$ and $\vec{D}$ have magnitudes of 3 units and 4 units, respectively. What is the angle between the directions of $\vec{C}$ and $\vec{D}$ if $\vec{C} \cdot \vec{D}$ equals (a) zero, (b) 12 units, and (c) -12 units?

(a) When the angle between $\vec{C}$ and $\vec{D}$ is $90°$

$$\vec{C} \cdot \vec{D} = (3)(4)\cos 90° = 0.$$

(b) When the angle between $\vec{C}$ and $\vec{D}$ is 0

$$\vec{C} \cdot \vec{D} = (3)(4)\cos 0 = 12.$$

(c) When the angle between $\vec{C}$ and $\vec{D}$ is $180°$

$$\vec{C} \cdot \vec{D} = (3)(4)\cos 180° = -12.$$

3.8 Multiplying Vectors

Example 4: What is the angle ϕ between $\vec{a} = 3.0\hat{\imath} - 4.0\hat{\jmath}$ and $\vec{b} = -2.0\hat{\imath} + 3.0\hat{k}$?

We know that

$$\cos\phi = \frac{\vec{a}\cdot\vec{b}}{ab}$$

$$a = \sqrt{(3.0)^2 + (-4.0)^2 + 0} = 5.00$$

$$b = \sqrt{(-2.0)^2 + 0 + (3.0)^2} = 3.61$$

$$\vec{a}\cdot\vec{b} = (3.0\hat{\imath} - 4.0\hat{\jmath})\left(-2.0\hat{\imath} + 3.0\hat{k}\right) = -6.0.$$

$$\phi = \cos^{-1}\frac{-6.0}{(5.00)(3.61)} = 109° \approx 110°$$

3.8 Multiplying Vectors

3. **Multiplying a Vector by a Vector:** The Vector Product (Cross Product)

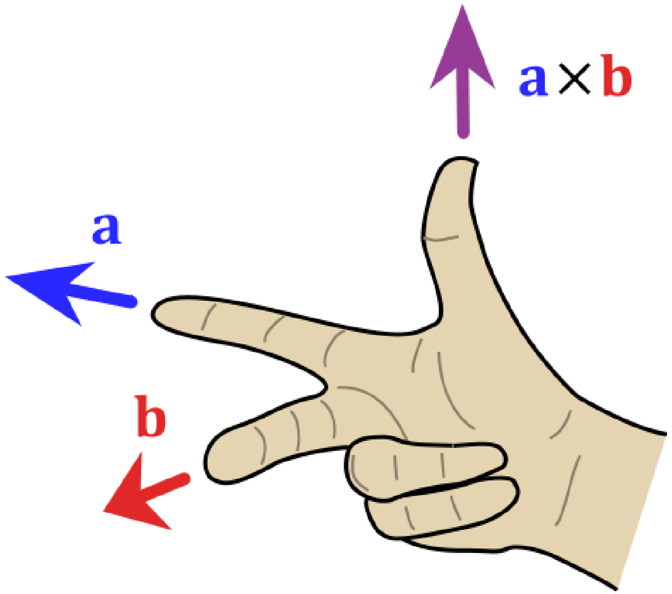
The vector product of the vectors $\vec{a}$ and $\vec{b}$ is written as $\vec{a} \times \vec{b}$. It has the magnitude

$$\left|\vec{a} \times \vec{b}\right| = ab \sin \phi \,,$$

where ϕ is the smaller angle between the directions of $\vec{a}$ and $\vec{b}$.

$\vec{a} \times \vec{b}$ is perpendicular to the plane containing $\vec{a}$ and $\vec{b}$. We use the **right hand rule** to find its direction.

Note that $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$.



From Wikipedia

3.8 Multiplying Vectors

If $\vec{a}$ and $\vec{b}$ are parallel or antiparallel, $\vec{a} \times \vec{b} = 0$. The magnitude of $\vec{a} \times \vec{b}$, which can be written as $\left|\vec{a} \times \vec{b}\right|$, is maximum when $\vec{a}$ and $\vec{b}$ are perpendicular to each other.

In unit vector notation,

$$\begin{aligned}
\vec{a} \times \vec{b} &= \left(a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}}\right) \times \left(b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}} + b_z\hat{\mathrm{k}}\right) \\
&= a_xb_x(\hat{\mathrm{i}} \times \hat{\mathrm{i}}) + a_xb_y(\hat{\mathrm{i}} \times \hat{\mathrm{j}}) + a_xb_z\left(\hat{\mathrm{i}} \times \hat{\mathrm{k}}\right) + a_yb_x(\hat{\mathrm{j}} \times \hat{\mathrm{i}}) + a_yb_y(\hat{\mathrm{j}} \times \hat{\mathrm{j}}) \\
&\quad + a_yb_z\left(\hat{\mathrm{j}} \times \hat{\mathrm{k}}\right) + a_zb_x\left(\hat{\mathrm{k}} \times \hat{\mathrm{i}}\right) + a_zb_y\left(\hat{\mathrm{k}} \times \hat{\mathrm{j}}\right) + a_zb_z\left(\hat{\mathrm{k}} \times \hat{\mathrm{k}}\right) \\
&= \left(a_yb_z - b_ya_z\right)\hat{\mathrm{i}} + (a_zb_x - b_za_x)\hat{\mathrm{j}} + \left(a_xb_z - b_xa_y\right)\hat{\mathrm{k}}.
\end{aligned}$$

3.8 Multiplying Vectors

We used that

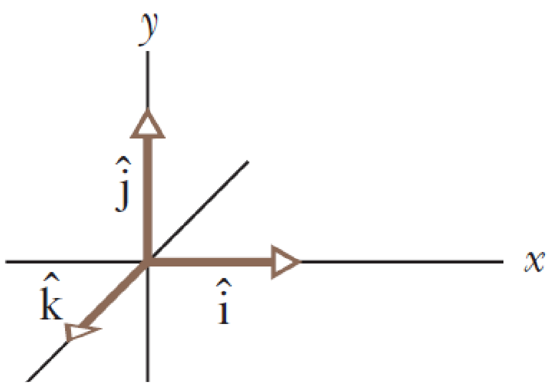
$$\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = (1)(1)\sin 0 = 0,$$

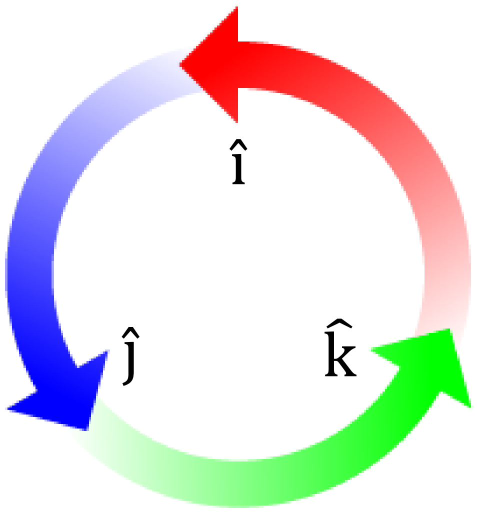
and that

$$\hat{\imath} \times \hat{\jmath} = \hat{k},$$
$$\hat{\jmath} \times \hat{\imath} = -\hat{k},$$

$$\hat{\jmath} \times \hat{k} = \hat{\imath},$$
$$\hat{k} \times \hat{\jmath} = -\hat{\imath},$$

$$\hat{k} \times \hat{\imath} = \hat{\jmath},$$
$$\hat{\imath} \times \hat{k} = -\hat{\jmath}.$$





3.8 Multiplying Vectors

Also, $\vec{a} \times \vec{b}$ can be expressed as

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\mathrm{i}} & \hat{\mathrm{j}} & \hat{\mathrm{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}.$$

This approach is much easier to work with!

3.8 Multiplying Vectors

Example 5: If $\vec{a} = 3\hat{\imath} - 4\hat{\jmath}$ and $\vec{b} = -2\hat{\imath} + 3\hat{k}$, find $\vec{c} = \vec{a} \times \vec{b}$.

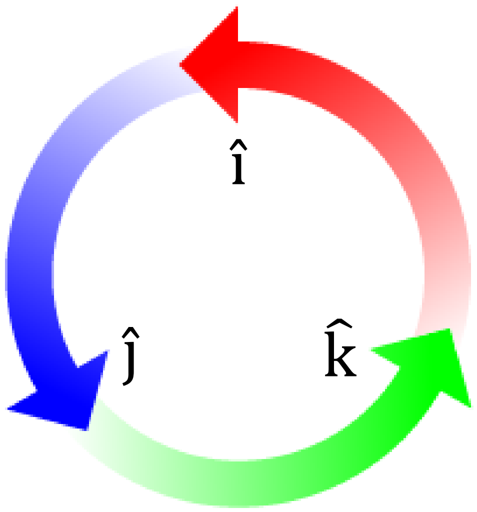
$\vec{c} = (3\hat{\imath} - 4\hat{\jmath}) \times \left(-2\hat{\imath} + 3\hat{k}\right)$

$= (3\hat{\imath}) \times (-2\hat{\imath}) + (3\hat{\imath}) \times \left(3\hat{k}\right) + (-4\hat{\jmath}) \times (-2\hat{\imath}) + (-4\hat{\jmath}) \times \left(3\hat{k}\right)$

$= -6(0) + 9(-\hat{\jmath}) + 8\left(-\hat{k}\right) - 12(\hat{\imath})$

$= -12\hat{\imath} - 9\hat{\jmath} - 8\hat{k}$

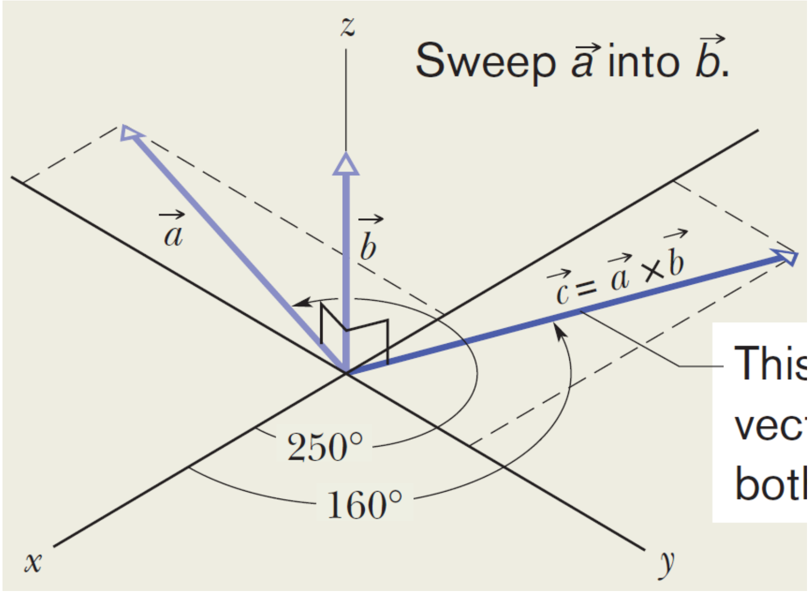
$\vec{c}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.



Exercise: Find $\vec{c}$ using the determinant in the previous slide. Also, check that $\vec{c} \cdot \vec{a} = \vec{c} \cdot \vec{b} = 0$.

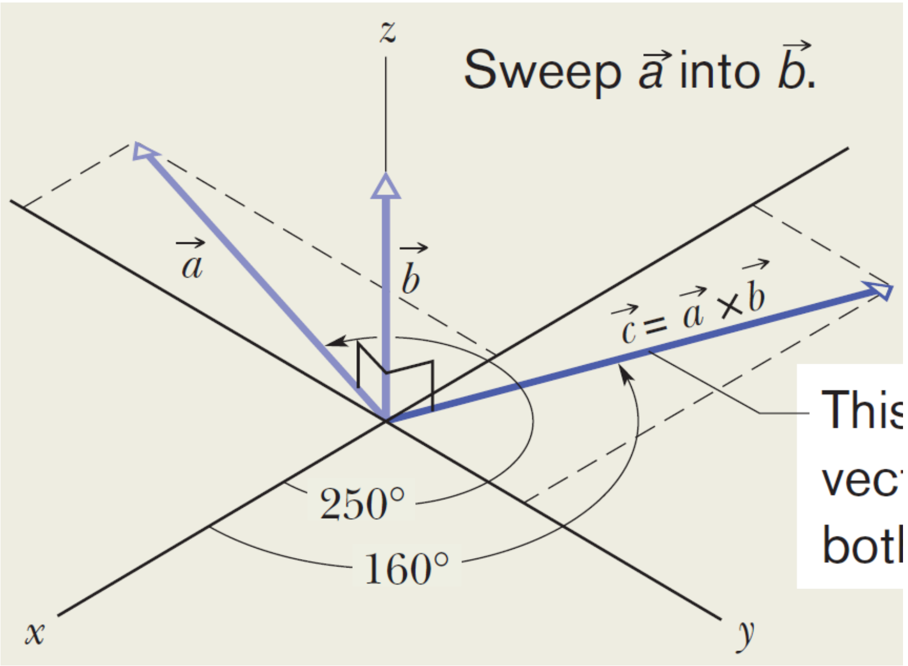
3.8 Multiplying Vectors

Example 6: In the figure , vector $\vec{a}$ lies in the xy plane, has a magnitude of 18 units, and points in a direction 250° from the positive direction of the x axis. Also, vector $\vec{b}$ has a magnitude of 12 units and points in the positive direction of the z axis. What is the vector product $\vec{a} \times \vec{b}$?



3.8 Multiplying Vectors

Example 6: In the figure , vector $\vec{a}$ lies in the xy plane, has a magnitude of 18 units, and points in a direction 250° from the positive direction of the x axis. Also, vector $\vec{b}$ has a magnitude of 12 units and points in the positive direction of the z axis. What is the vector product $\vec{c} = \vec{a} \times \vec{b}$?



$$c = ab \sin\phi = (18)(12)\sin 90° = 216.$$

$\vec{c}$ is perpendicular to the plane containing both $\vec{a}$ and $\vec{b}$. The angle it makes with the positive x axis is

$$250° - 90° = 160°$$

3.8 Multiplying Vectors

CHECKPOINT 5

Vectors $\vec{C}$ and $\vec{D}$ have magnitudes of 3 units and 4 units, respectively. What is the angle between the directions of $\vec{C}$ and $\vec{D}$ if the magnitude of the vector product $\vec{C} \times \vec{D}$ is (a) zero and (b) 12 units?

(a) When the angle between $\vec{C}$ and $\vec{D}$ is 0 or 180°

$$\left|\vec{C} \times \vec{D}\right| = (3)(4)\sin 0 = (3)(4)\sin 180° = 0.$$

(b) When the angle between $\vec{C}$ and $\vec{D}$ is 90°

$$\left|\vec{C} \times \vec{D}\right| = (3)(4)\sin 90° = 12.$$