Chapter 13

GRAVITATION

1. Newton's Law of Gravitation

- Every body in the universe attracts every other body. This tendency of bodies to move toward one another is called **gravitation**.
- Newton proposed a force law that we call **Newton's law of gravitation**: Every particle attracts any other particle with a **gravitational force** of magnitude

$$F = G\frac{m_1 m_2}{r^2}.$$

Here m_1 and m_2 are the masses of the particles, r is the distance between them, and G is the **gravitational constant**, which has the value

$$G = 6.67 \times 10^{-11}\,\frac{\mathrm{N}\cdot\mathrm{m}^2}{\mathrm{kg}^2}\left(\text{or }\frac{\mathrm{m}^3}{\mathrm{kg}\cdot\mathrm{s}^2}\right).$$

1. Newton's Law of Gravitation

- The direction of $\vec{F}$ on particle 1 is toward particle 2. The force is said to be *attractive force* because every particle attracts the other toward it.
- We can describe $\vec{F}$ as being in the positive direction of an r axis extending radially from particle 1 to particle 2. We can also describe $\vec{F}$ using a radial unit vector $\hat{r}$ that is directed away from particle 1 along the r axis. The force on particle 1 is then

$$\vec{F} = G\frac{m_1 m_2}{r^2}\hat{r}.$$

- The gravitational force on particle 2 due to particle 1 has the same magnitude as the force on particle 1 but in the opposite direction. The two forces form a third-law force pair.

1. Newton's Law of Gravitation

- Newton's law of gravitation can also be applied to real, <u>extended</u> objects as long as their sizes are small relative to the distance between them.
- Newton also proved an important theorem called the **shell theorem**: A uniform spherical shell of matter attracts a particle that is outside the shell as if all the shell's mass were concentrated at its center.
- Earth can be thought of as a nest of such shells. Thus, Earth gravitationally behaves like a particle, located at the center of Earth with a mass equal to that of Earth.
- Suppose that Earth pulls down on an apple with a force of $1.0\ \mathrm{N}$. The apple must pull up on Earth with a force of magnitude $1.0\ \mathrm{N}$. Although the forces are equal, they produce very different accelerations. The acceleration of the apple near Earth's surface is $9.8\ \mathrm{m/s^2}$ and the acceleration of Earth, relative to the center of mass of the apple–Earth system, is about $1 \times 10^{-25}\ \mathrm{m/s^2}$.

1. Newton’s Law of Gravitation

CHECKPOINT 1

A particle is to be placed, in turn, outside four objects, each of mass m: (1) a large uniform solid sphere, (2) a large uniform spherical shell, (3) a small uniform solid sphere, and (4) a small uniform shell. In each situation, the distance between the particle and the center of the object is d. Rank the objects according to the magnitude of the gravitational force they exert on the particle, greatest first.

All tie.

2. Gravitation & the Principle of Superposition

- If we have a group of particles, we can find the net gravitational force on any of them using the **principle of superposition**.
- For n interacting particles, we can write the principle of superposition for gravitational forces on particle 1 as

$$\vec{F}_{1,\mathrm{net}} = \vec{F}_{12} + \vec{F}_{13} + \cdots + \vec{F}_{1n}.$$

Here $\vec{F}_{1n}$ is the force on particle 1 due to particle n. More compactly,

$$\vec{F}_{1,\mathrm{net}} = \sum_{i=2}^{n} \vec{F}_{1i}\,.$$

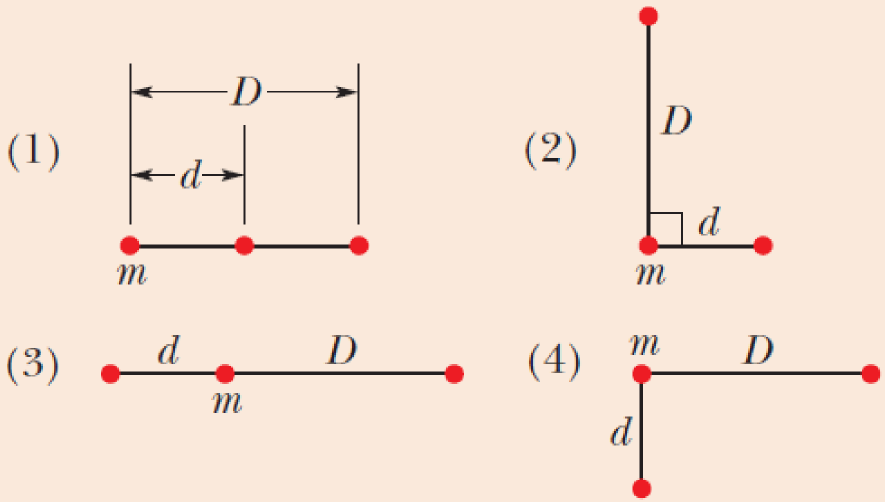
- The gravitational force on a particle from a real (extended) body is

$$\vec{F}_1 = \int d\vec{F}\,.$$

2. Gravitation & the Principle of Superposition

CHECKPOINT 2

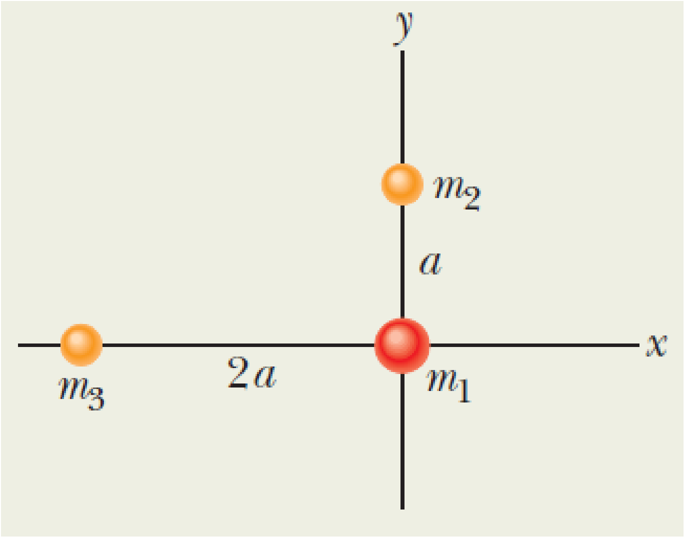
The figure shows four arrangements of three particles of equal masses. (a) Rank the arrangements according to the magnitude of the net gravitational force on the particle labeled m, greatest first. (b) In arrangement 2, is the direction of the net force closer to the line of length d or to the line of length D?



(a) 1, 2 & 4 tie, 3. (b) Closer to line d.

2. Gravitation & the Principle of Superposition

Example 1: The figure shows an arrangement of three particles, particle 1 of mass $m_1 = 6.0\text{ kg}$ and particles 2 and 3 of mass $m_2 = m_3 = 4.0\text{ kg}$, and distance $a = 2.0\text{ cm}$. What is the net gravitational force $\vec{F}_{1,\text{net}}$ on particle 1 due to the other particles?



$$F_{12} = G\frac{m_1 m_2}{a^2}$$

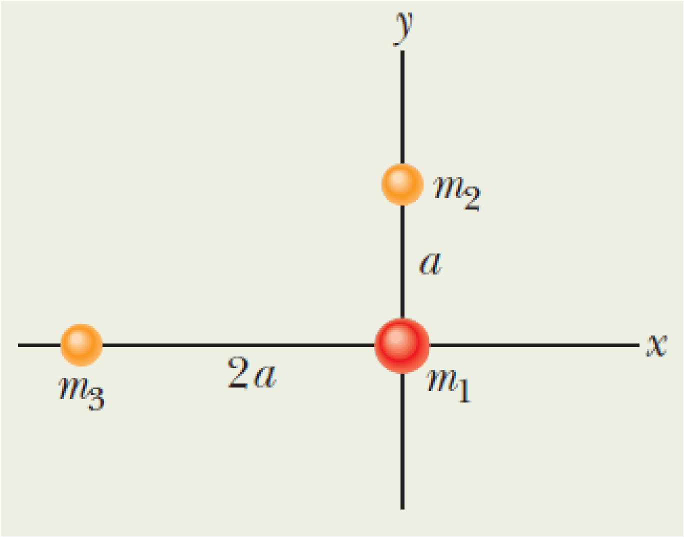
$$= (6.67 \times 10^{-11}\text{m}^3\text{kg} \cdot \text{s}^2)\frac{(6.0\text{ kg})(4.0\text{ kg})}{(0.020\text{ m})^2}$$

$$= 4.00 \times 10^{-6}\text{ N}.$$

Thus, $\vec{F}_{12} = (4.00 \times 10^{-6}\text{ N})\,\hat{\jmath}$.

2. Gravitation & the Principle of Superposition

$$F_{13} = G\frac{m_1 m_3}{(2a)^2}$$
$$= (6.67 \times 10^{-11} \mathrm{m^3 kg \cdot s^2}) \frac{(6.0\ \mathrm{kg})(4.0\ \mathrm{kg})}{(0.040\ \mathrm{m})^2}$$
$$= 1.00 \times 10^{-6}\ \mathrm{N}.$$



Thus, $\vec{F}_{13} = -(1.00 \times 10^{-6}\ \mathrm{N})\ \hat{\imath}$.

$$F_{1,\mathrm{net}} = \sqrt{(F_{12})^2 + (F_{13})^2}$$
$$= \sqrt{(4.00 \times 10^{-6}\ \mathrm{N})^2 + (1.00 \times 10^{-6}\ \mathrm{N})^2}$$
$$= 4.1 \times 10^{-6}\ \mathrm{N}.$$
$$\theta = \tan^{-1}(F_{12}/-F_{13}) = 104°.$$

3. Gravitation Near Earth's Surface

- Let us assume that Earth is a uniform sphere of mass M. The magnitude of the gravitational force from Earth on a particle of mass m, at distance r from Earth's center is

$$F = G\frac{Mm}{r^2}.$$

- If the particle is released, it will fall with gravitational acceleration $\vec{a}_g$. According to Newton's second law

$$F = ma_g.$$

Comparing these two expression we find

$$a_g = \frac{GM}{r^2}.$$

Table 13-1

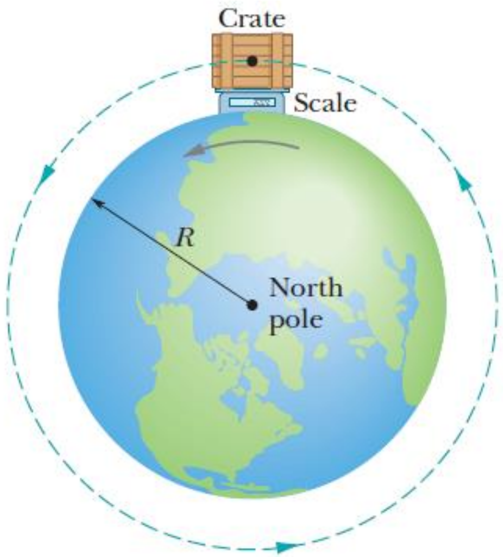
Variation of a_g with Altitude

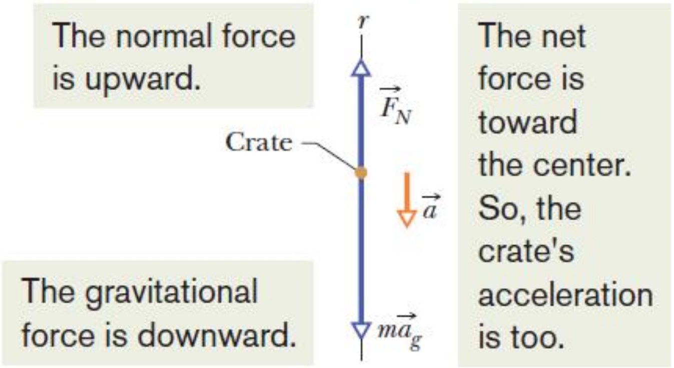
Altitude (km)	a_g (m/s^2)	Altitude Example
0	9.83	Mean Earth surface
8.8	9.80	Mt. Everest
36.6	9.71	Highest crewed balloon
400	8.70	Space shuttle orbit
35 700	0.225	Communications satellite

3. Gravitation Near Earth's Surface

- We assumed before that Earth is an inertial frame and hence assumed that the free-fall acceleration g is the same as the gravitational acceleration a_r. However, these two quantities are different for three reasons:
    1. Earth's mass is not uniformly distributed.
    2. Earth is not a perfect sphere.
    3. Earth is rotating.
- Let us examine the third effect. Consider the situation shown in the figure. The crate has a centripetal acceleration $a_r = \omega^2 R$. Newton's second law for components along the r axis can be written as

$$F_N - ma_g = m(-\omega^2 R).$$





3. Gravitation Near Earth's Surface

- F_N is equal to the weight read on the scale. Thus we substitute mg for F_N:

$$mg = ma_g - m(\omega^2 R).$$

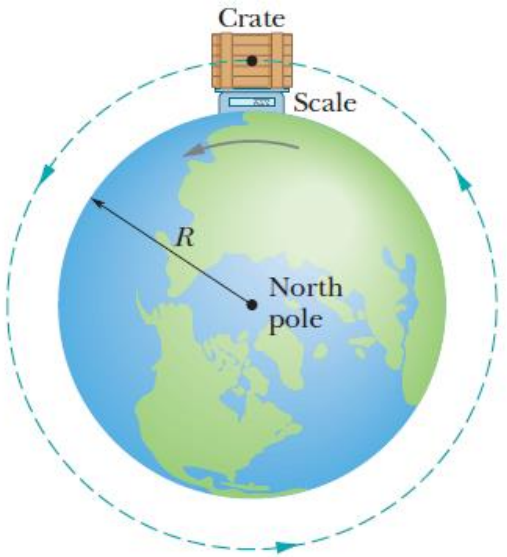
In words,

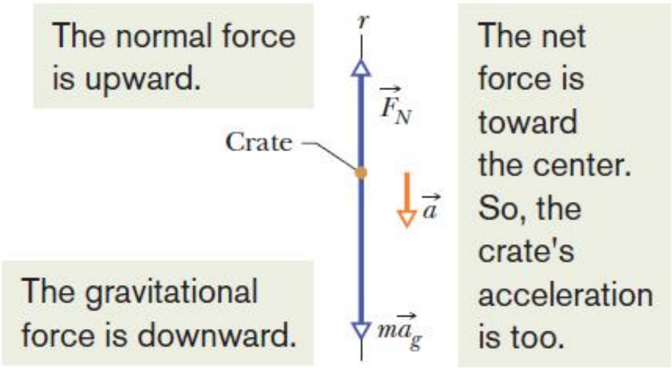
$$\begin{pmatrix}\text{measured}\\ \text{weight}\end{pmatrix} = \begin{pmatrix}\text{magnitude of}\\ \text{gravitational force}\end{pmatrix} - \begin{pmatrix}\text{mass times}\\ \text{centripetal acceleration}\end{pmatrix}.$$

Cancelling m from the equation above yields

$$g = a_g - \omega^2 R.$$

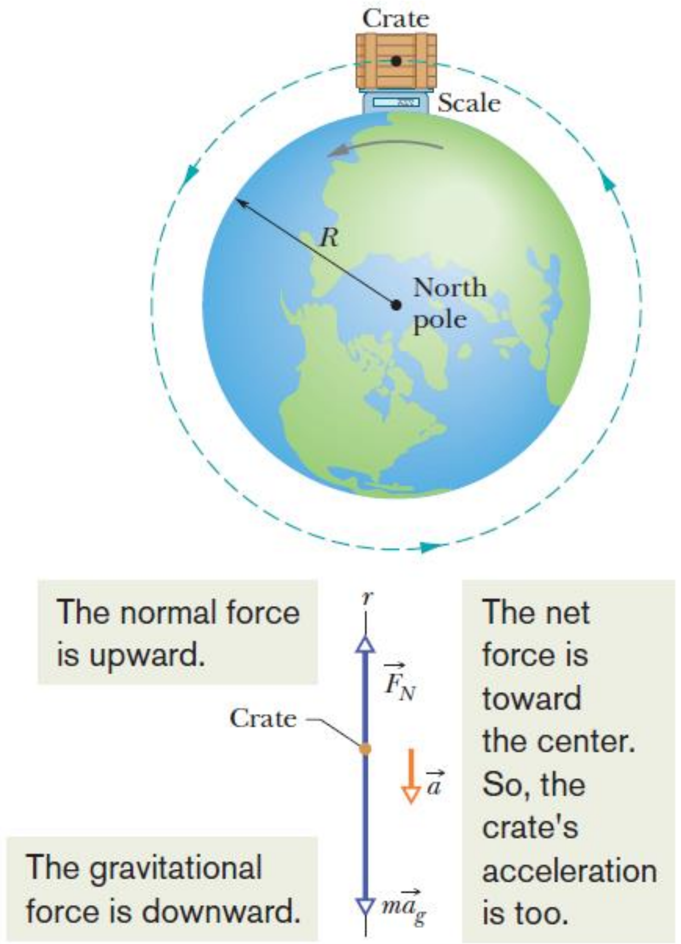
$$\begin{pmatrix}\text{free fall}\\ \text{acceleration}\end{pmatrix} = \begin{pmatrix}\text{gravitational}\\ \text{acceleration}\end{pmatrix} - \begin{pmatrix}\text{centripetal}\\ \text{acceleration}\end{pmatrix}.$$





3. Gravitation Near Earth's Surface

- The difference between g and a_g is greatest at the equator. Direct calculation reveals that $\omega^2 R$ is about $0.034\ \frac{\mathrm{m}}{\mathrm{s}^2}$. This fact justifies neglecting the difference between g and a_g.



4. Gravitation Inside Earth

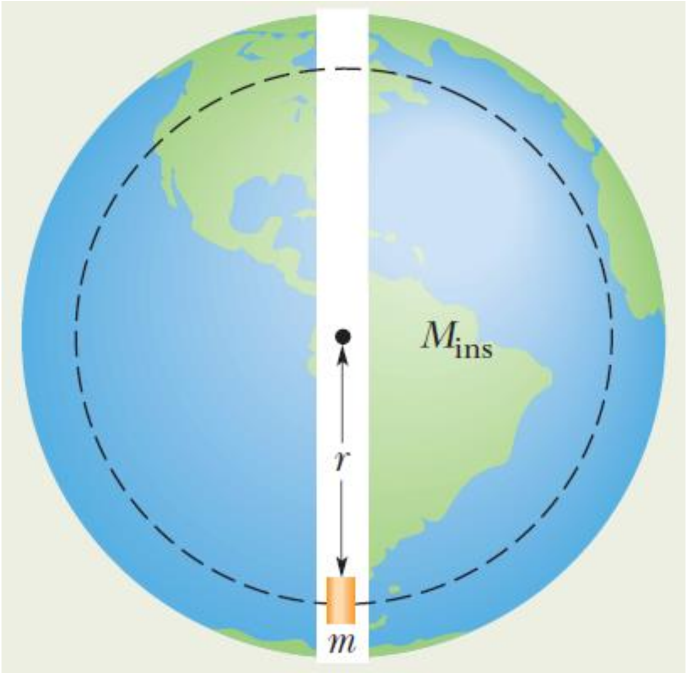
- A uniform shell of matter exerts no net gravitational force on a particle located inside it.
- If Earth's mass were uniformly distributed, the gravitational force acting on a particle would be maximum at the Earth's surface and would decrease as the particle moved away from the planet.
- If the particle were to move inward, the gravitational force would change in two ways:
    1. It would tend to increase as the particle would get closer to the Earth's center.
    2. It would decrease as the thickness of the shell of material lying outside the particle radial position would exert no force on the particle.

4. Gravitation Inside Earth

- For uniform Earth, the second effect would prevail and the force would decrease to zero at the center of Earth.
- However, for the real (nonuniform) Earth, the force on the particle actually increases as the particle begins to descend. The force reaches a maximum at a certain depth and then decreases as the particle descends farther.

4. Gravitation Inside Earth

Example 2: Find the gravitational force on a capsule of mass m when it reaches a distance r from Earth's center. Assume that Earth is a sphere of uniform density ρ (mass per unit volume).



The portion of Earth outside the sphere of radius r does not produce net gravitational force. Only the portion of Earth inside the sphere of radius r produces net gravitational force. The inside mass M_{ins} is given by

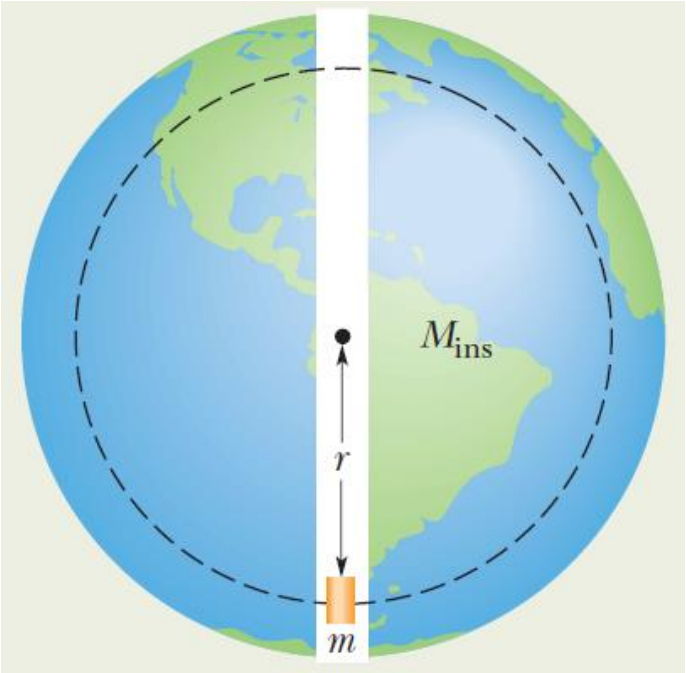
$$M_{\text{ins}} = \rho V_{\text{ins}} = \rho \frac{4\pi r^3}{3}.$$

4. Gravitation Inside Earth

The magnitude of the gravitation force on the capsule is

$$F = \frac{GmM_{\text{ins}}}{r^2} = \frac{Gm}{r^2}\left(\rho\frac{4\pi r^3}{3}\right) = \frac{4\pi Gm\rho}{3}r.$$

F is maximum at Earth's surface and zero at the center of Earth.



5. Gravitational Potential Energy

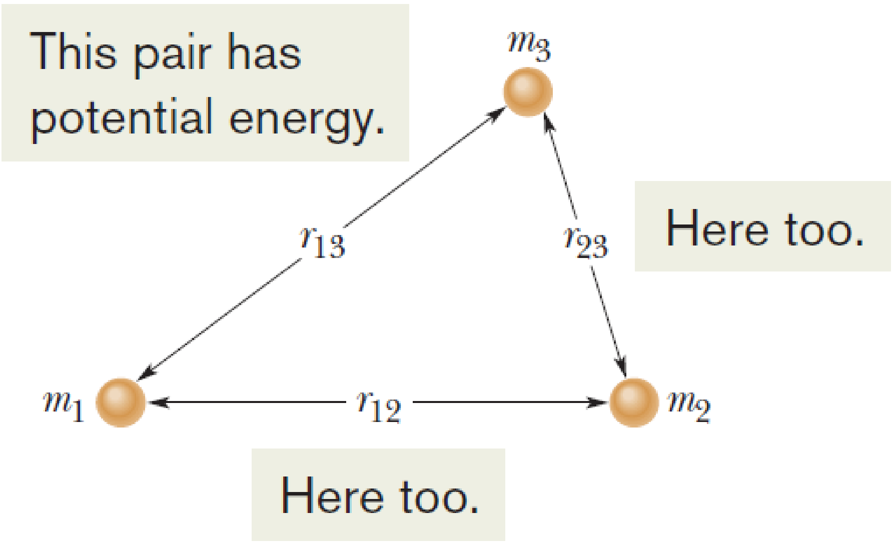
- In Ch. 8, we discussed the gravitational potential energy of particle-Earth system. We restricted our discussion to systems near Earth's surface so that g is constant. We then chose a reference configuration of the system as having a zero gravitational potential energy. The gravitational potential energy decreased when the separation between the particle and Earth decreased.
- Now we consider the gravitational potential energy U of two particles, of mass m and M, separated by a distance r. We choose the reference configuration that U become 0 as r approaches ∞. U is therefore negative for any finite r and becomes more negative as the particles become closer.
- We take the gravitational potential energy of a two-particle system to be

$$U = -\frac{GMm}{r}$$

5. Gravitational Potential Energy

- If we have more than two particles, we consider each pair of particles in turn, calculate the gravitational potential energy of that pair as if the other particles were not there and then algebraically sum the results.
- For example, the gravitational potential energy of the system shown in the figure is

$$U = -\left(\frac{Gm_1m_2}{r_{12}} + \frac{Gm_1m_3}{r_{13}} + \frac{Gm_2m_3}{r_{23}}\right).$$



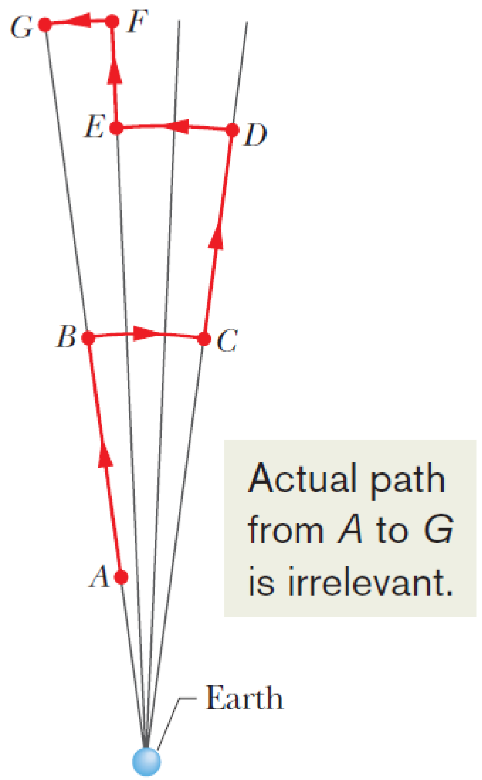
5. Gravitational Potential Energy

- <u>Path Independence:</u>

  The gravitational force is a conservative force. The work W done by the gravitational force on a particle moving from an initial point i to a final point f is independent of the path taken between the points. We know that

  $$\Delta U = -W.$$

  Because the work done by a conservative force is path independent, the change ΔU in gravitational potential energy is also path independent.



5. Gravitational Potential Energy

- <u>Potential Energy and Force:</u>

  In Ch. 8 we have seen that $F(x) = -dU(x)/dx$. For gravitational potential energy we have

$$F = -\frac{d}{dr}\left(-\frac{GMm}{r}\right) = -\frac{GMm}{r^2}.$$

  The minus sign indicates that the force on mass m points radially inward, toward mass M.

5. Gravitational Potential Energy

- <u>Escape Speed:</u>

  If you fire a projectile upward, usually it will slow, stop momentarily and return to Earth. There is a minimum initial speed that will cause the projectile to move upward forever, theoretically coming to rest only at infinity. This minimum initial speed is called the **escape speed**.

  Consider a projectile of mass m leaving the surface of a planet with escape speed v. The projectile has a kinetic energy $K = \frac{1}{2}mv^2$ and potential energy $U = -\frac{GMm}{R}$, where M and R are the planet's mass and radius, respectively. At infinity, the projectile stops and has no kinetic energy. It also has no potential energy there. Its total energy at infinity is 0.

5. Gravitational Potential Energy

- <u>Escape Speed:</u>

  By the principle of conservation of energy, the projectile's total energy is 0 too at the planet's surface. Therefore, we write

$$K + U = \frac{1}{2}mv^2 + \left(-\frac{GMm}{R}\right) = 0.$$

  Solving for v we get

$$v = \sqrt{\frac{2GM}{R}}.$$

5. Gravitational Potential Energy

Body	Mass (kg)	Radius (m)	Escape Speed (km/s)
Ceres[a]	1.17×10^{21}	3.8×10^{5}	0.64
Earth's moon[a]	7.36×10^{22}	1.74×10^{6}	2.38
Earth	5.98×10^{24}	6.37×10^{6}	11.2
Jupiter	1.90×10^{27}	7.15×10^{7}	59.5
Sun	1.99×10^{30}	6.96×10^{8}	618
Sirius B[b]	2×10^{30}	1×10^{7}	5200
Neutron star[c]	2×10^{30}	1×10^{4}	2×10^{5}

5. Gravitational Potential Energy

Example 3: A black hole has a "no return" surface called the *event horizon*. Nothing, not even light, can escape from that surface or anywhere inside the black hole. If Earth were compressed into a black hole, what would be its radius?

Substituting the speed of light c in the expression for the escape speed we write

$$c = \sqrt{\frac{2GM}{R}}.$$

Solving for R and substituting we get

$$R = \frac{2GM}{c^2} = \frac{2(6.67 \times 10^{-11}\ \mathrm{m^3/kg \cdot s^2})(5.98 \times 10^{24}\ \mathrm{kg})}{(3.00 \times 10^{8}\ \mathrm{m/s})^2} = 8.89\ \mathrm{mm}!$$

5. Gravitational Potential Energy

CHECKPOINT 3

You move a ball of mass m away from a sphere of mass M. (a) Does the gravitational potential energy of the system of ball and sphere increase or decrease? (b) Is positive work or negative work done by the gravitational force between the ball and the sphere?

(a) Increases.

(b) Negative.

5. Gravitational Potential Energy

Example 4: An asteroid, headed directly toward Earth, has a speed of $12\ \mathrm{km/s}$ relative to the planet when the asteroid is 10 Earth radii from Earth's center. Neglecting the effects of Earth's atmosphere on the asteroid, find the asteroid's speed v_f when it reaches Earth's surface.

Conservation of mechanical energy is written as

$$K_f + U_f = K_i + U_i,$$

or

$$\frac{1}{2}mv_f^2 - \frac{GMm}{R_E} = \frac{1}{2}mv_i^2 - \frac{GMm}{10R_E}.$$

5. Gravitational Potential Energy

Solving for v_f and substituting yields

$$v_f = \sqrt{v_i^2 + \frac{GM}{R_E}\left(1 - \frac{1}{10}\right)}$$

$$= \sqrt{(12 \times 10^3\ \mathrm{m/s})^2 + \frac{(6.67 \times 10^{-11}\ \mathrm{m^3/kg \cdot s^2})(5.98 \times 10^{24}\ \mathrm{kg})}{6.37 \times 10^6\ \mathrm{m}}\left(1 - \frac{1}{10}\right)}$$

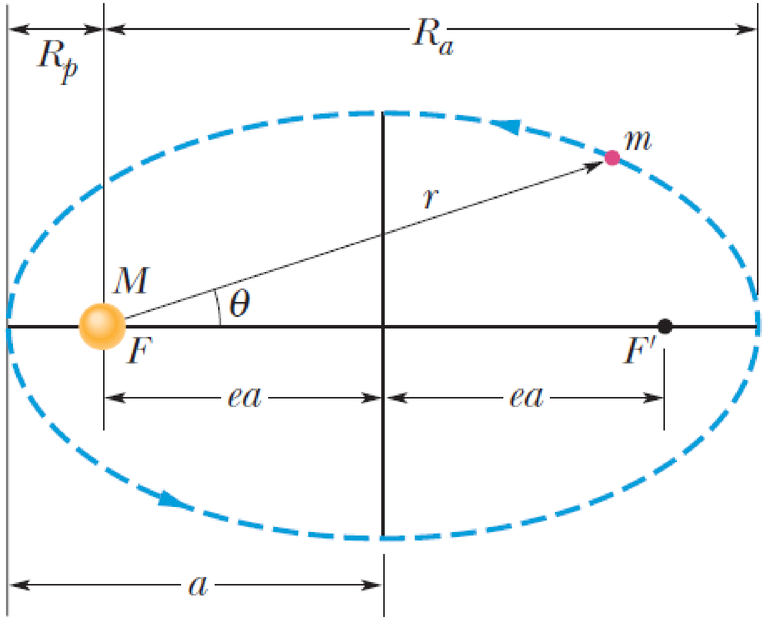
$$= 16\ \mathrm{km/s}.$$

6. Planets and Satellites: Kepler’s Laws

- In this section we discuss the three Kepler's laws of planetary motion.

1. THE LAW OF ORBITS: All planets move in elliptical orbits, with the Sun at one focus.

   The orbit in the figure is described by giving its **semimajor axis** a and its **eccentricity** e. It is defined so that ea is the distance from the center of the ellipse to either focus F or F'. For a circle $e = 0$. For Earth’s orbit $e = 0.0167$.



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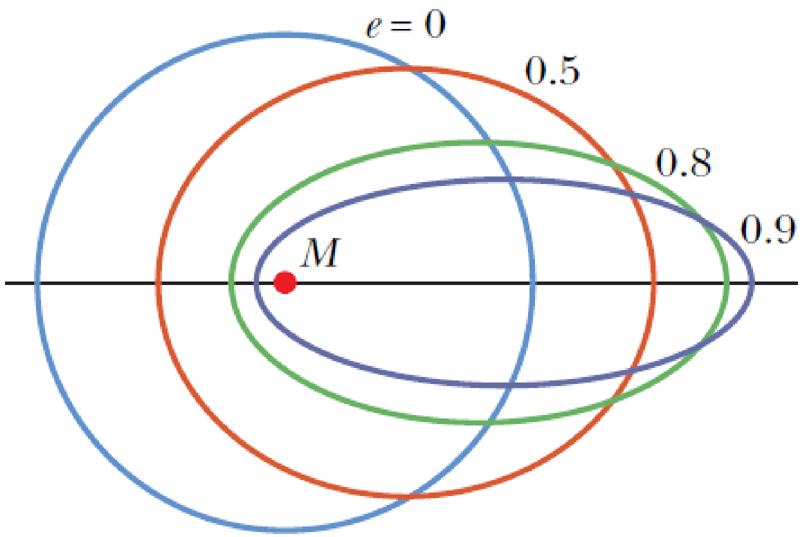
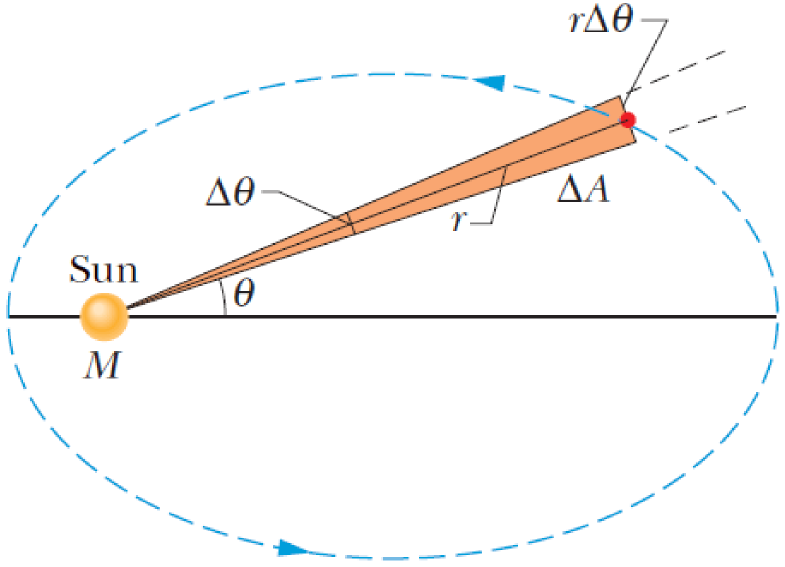


6. Planets and Satellites: Kepler's Laws

2. <u>THE LAW OF AREAS</u>: A line that connects a planet to the Sun sweeps out equal areas in the plane of the planet's orbit in equal time internal; the rate dA/dt at which it sweeps out area A is constant.

   This law tells us that the planet moves most slowly when it is farthest from the Sun and most rapidly when it is nearest to the Sun.

   This law is equivalent to the law of conservation of angular momentum.



6. Planets and Satellites: Kepler's Laws

From the figure, the area swept out by the line connecting the Sun and the planet in time Δt is $\Delta A \approx \frac{1}{2} r^2 \Delta\theta$. The instantaneous rate of area swept out is

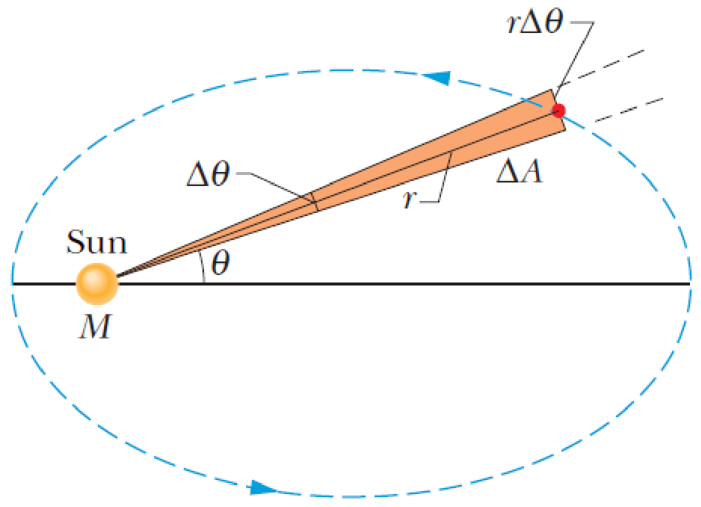
$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \omega.$$

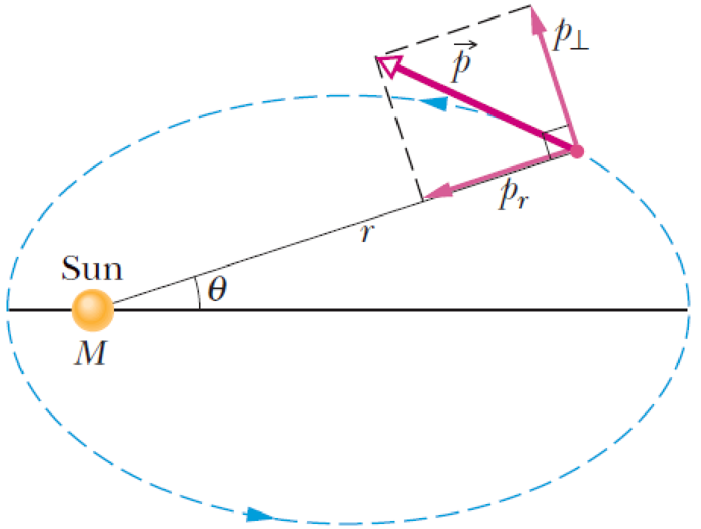
The magnitude of the angular momentum $\vec{L}$ of the planet is given by

$$L = rp_\perp = r(mv_\perp) = r(m\omega r) = mr^2\omega.$$

Combining the above results we have

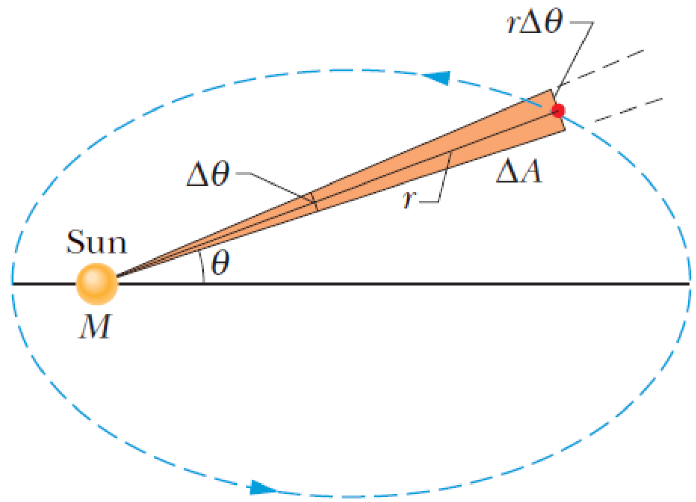
$$\frac{dA}{dt} = \frac{L}{2m}.$$

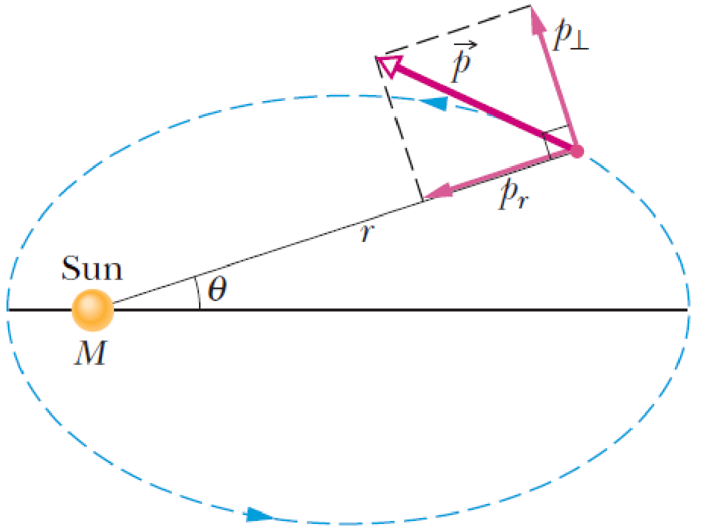




6. Planets and Satellites: Kepler's Laws

- The constancy of dA/dt, according to Kepler, is equivalent to the constancy of $\vec{L}$. Kepler's second law is therefore equivalent to the law of conservation of angular momentum.





6. Planets and Satellites: Kepler's Laws

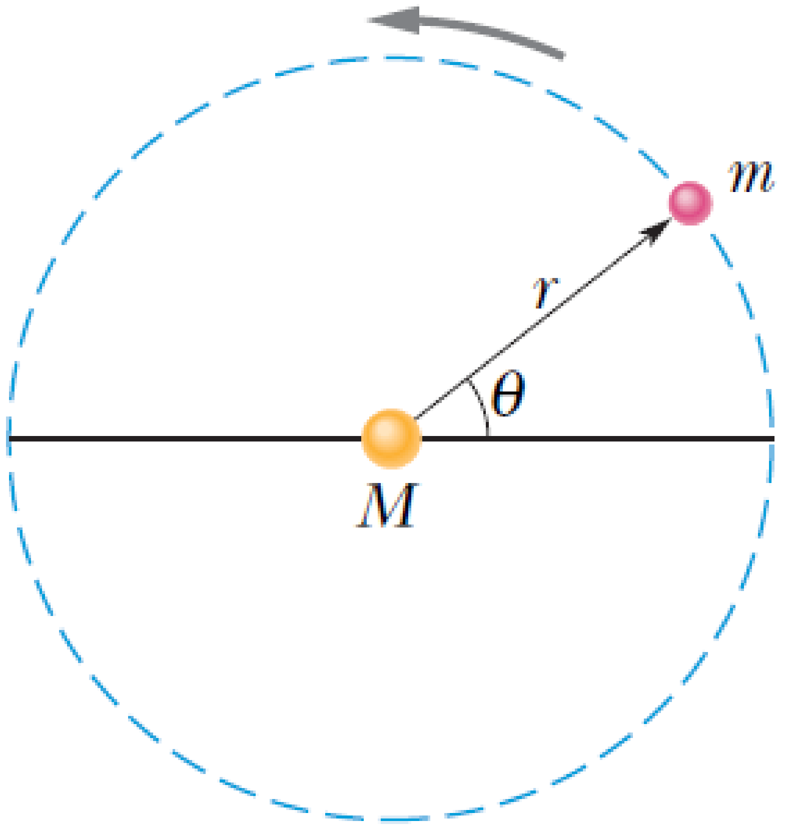
3. <u>THE LAW OF PERIODS</u>: The square of the period of any planet is proportional to the cube of the semimajor axis of its orbit.

   To see this we apply Newton's second law to the orbiting planet in the figure and write

$$\frac{GMm}{r^2} = m(\omega^2 r).$$

   Using $\omega = 2\pi/T$ we find that

$$T^2 = \left(\frac{4\pi^2}{GM}\right) r^3.$$



6. Planets and Satellites: Kepler's Laws

$$T^2 = \left(\frac{4\pi^2}{GM}\right) r^3.$$

The law of periods holds for elliptical orbits, provided that we replace r with a, the semimajor axis of the ellipse.

This law predicts that T^2/a^3 is the same for every object orbiting around a given massive body.

Kepler's Law of Periods for the Solar System

Planet	Semimajor Axis a (10^{10} m)	Period T (y)	T^2/a^3 (10^{-34} y^2/m^3)
Mercury	5.79	0.241	2.99
Venus	10.8	0.615	3.00
Earth	15.0	1.00	2.96
Mars	22.8	1.88	2.98
Jupiter	77.8	11.9	3.01
Saturn	143	29.5	2.98
Uranus	287	84.0	2.98
Neptune	450	165	2.99
Pluto	590	248	2.99

6. Planets and Satellites: Kepler's Laws

CHECKPOINT 4

Satellite 1 is in a certain circular orbit around a planet, while satellite 2 is in a larger circular orbit. Which satellite has (a) the longer period and (b) the greater speed?

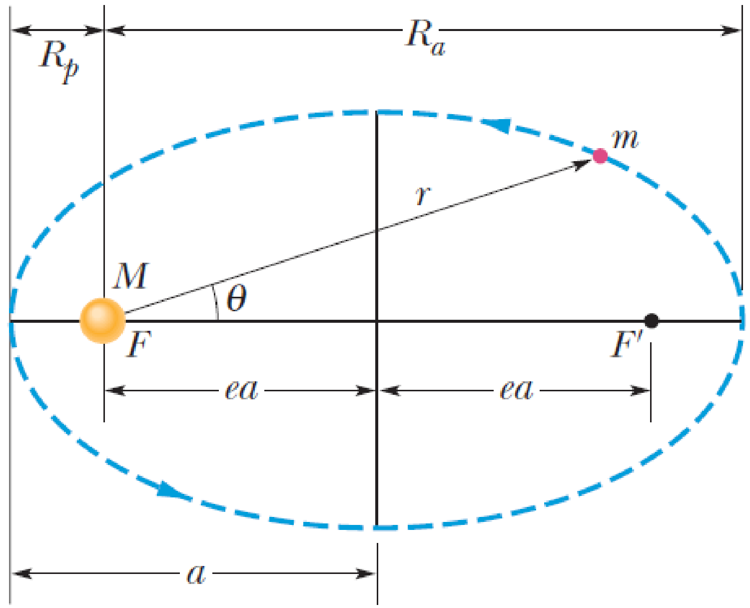
(a) Satellite 2.

(b) Satellite 1.

6. Planets and Satellites: Kepler's Laws

Example 5: Comet Halley orbits the Sun with a period of 76 years and, in 1986, had a distance of closest approach to the Sun, its *perihelion distance* R_p, of 8.9×10^{10} m. This is between the orbits of Mercury and Venus.

(a) What is the comet's farthest distance from the Sun, which is called its *aphelion distance* R_a?



From the figure, we see that $R_p + R_a = 2a$. We can find a using

$$a = \left(\frac{GMT^2}{4\pi^2}\right)^{1/3} = 2.7 \times 10^{12} \text{ m},$$

where we have used $M = 1.99 \times 10^{30}$ kg and $T = 2.4 \times 10^9$ s.

6. Planets and Satellites: Kepler's Laws

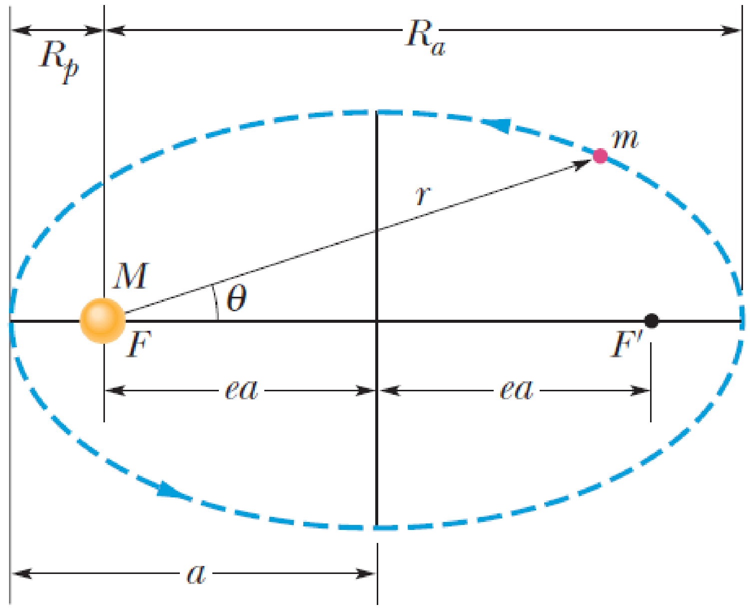
We now have

$$\begin{aligned} R_a &= 2a - R_p \\ &= 2(2.7 \times 10^{12}\ \mathrm{m}) - 8.9 \times 10^{10}\ \mathrm{m} \\ &= 5.3 \times 10^{12}\ \mathrm{m}. \end{aligned}$$

(b) What is the eccentricity of the orbit of comet Halley?

From the figure, we see that $R_p + ea = a$. Therefore,

$$\begin{aligned} e &= \frac{a - R_p}{a} = 1 - \frac{R_p}{a} \\ &= 1 - \frac{8.9 \times 10^{10}\ \mathrm{m}}{2.7 \times 10^{12}\ \mathrm{m}} \\ &= 0.97. \end{aligned}$$



7. Satellites: Orbits and Energy

- As a satellite orbits Earth in an elliptical path, both its speed, which fixes its kinetic energy K, and its distance from the center of Earth, which fixes its gravitational potential energy U, fluctuate with fixed periods. The mechanical energy E of the satellite, however, remains constant.
- The potential energy of the system is

$$U = -\frac{GMm}{r},$$

where M and m are the masses of Earth and satellite, respectively. For satellite in a circular orbit, Newton's second law reads

$$\frac{GMm}{r^2} = m\frac{v^2}{r}.$$

7. Satellites: Orbits and Energy

- The kinetic energy of the satellite is then

$$K = \frac{1}{2}mv^2 = \frac{GMm}{2r},$$

or

$$K = -\frac{U}{2}.$$

- The total mechanical energy of the orbiting satellite is

$$E = K + U = \frac{GMm}{2r} - \frac{GMm}{r},$$

or

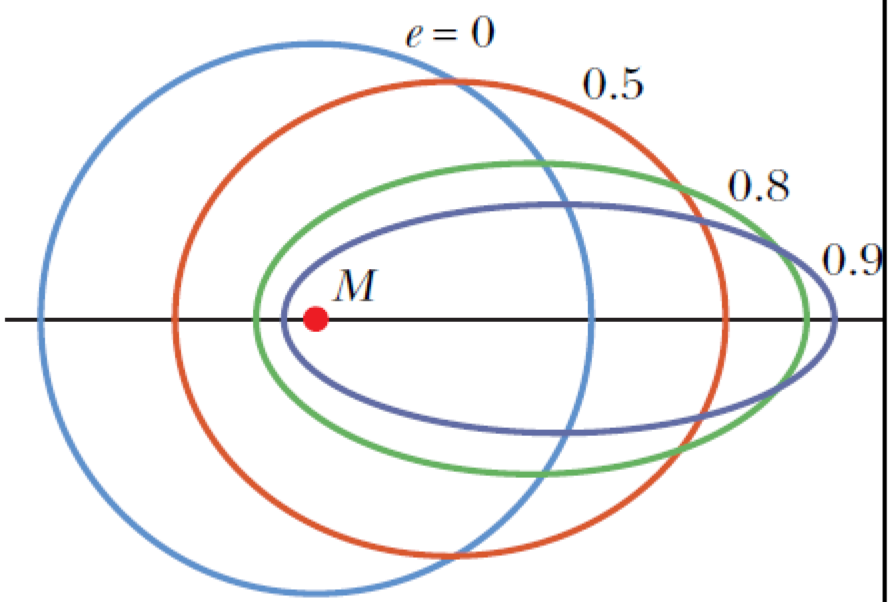
$$E = -\frac{GMm}{2r} = -K.$$

7. Satellites: Orbits and Energy

- $E = -K$ tells us that for a satellite in a circular orbit, the total mechanical energy E is the negative of the kinetic energy K.
- For a satellite in an elliptical orbit of semimajor axis a the expression for E becomes

$$E = -\frac{GMm}{2a}.$$

  The energy of an orbiting satellite is determined by the semimajor axis only. The energy does not depend of eccentricity e. For example, the same satellite in the four different orbits shown in the figure have the same energy.



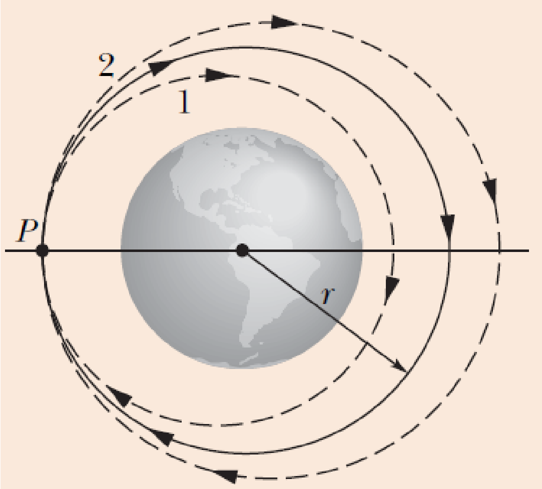
7. Satellites: Orbits and Energy

CHECKPOINT 5

In the figure here, a space shuttle is initially in a circular orbit of radius r about Earth. At point P, the pilot briefly fires a forward-pointing thruster to decrease the shuttle's kinetic energy K and mechanical energy E. (a) Which of the dashed elliptical orbits shown in the figure will the shuttle then take? (b) Is the orbital period T of the shuttle (the time to return to P) then greater than, less than, or the same as in the circular orbit?

(a) 1.

(b) Less.



7. Satellites: Orbits and Energy

Example 6: A playful astronaut releases a bowling ball, of mass $m = 7.20\ \mathrm{kg}$, into circular orbit about Earth at an altitude h of $350\ \mathrm{m}$.

(a) What is the mechanical energy E of the ball in its orbit?

The orbital radius is $r = R_E + h = 6370\ \mathrm{km} + 350\ \mathrm{km} = 6.72 \times 10^6\ \mathrm{m}$. The ball's mechanical energy is

$$E = -\frac{GMm}{2r}$$

$$= \frac{(6.67 \times 10^{-11}\ \mathrm{m^3/kg \cdot s^2})(5.98 \times 10^{24}\ \mathrm{kg})(7.20\ \mathrm{kg})}{2(6.72 \times 10^6\ \mathrm{m})}$$

$$= -214\ \mathrm{MJ}.$$

7. Satellites: Orbits and Energy

(b) What is the mechanical energy E_0 of the ball on the launch pad (before it, the astronaut, and the spacecraft are launched)? From there to the orbit, what is the change ΔE in the ball's mechanical energy?

On the launch pad, the energy E_0 of the ball is $K_0 + U_0$. K_0 is tiny compared to U_0 and therefore we approximate E_0 by U_0. We then write

$$E_0 \approx U_0 = -\frac{GMm}{R_E} = -\frac{(6.67 \times 10^{-11}\ \mathrm{m^3/kg \cdot s^2})(5.98 \times 10^{24}\ \mathrm{kg})(7.20\ \mathrm{kg})}{6.37 \times 10^6\ \mathrm{m}}$$

$$= -451\ \mathrm{MJ}.$$

The *increase* in the mechanical energy of the ball from launch pad to orbit is

$$\Delta E = E - E_0 = -214\ \mathrm{MJ} - (-451\ \mathrm{MJ}) = 237\ \mathrm{MJ}.$$