Chapter 12

Equilibrium and Elasticity

1. Equilibrium

- Consider these objects: (1) a book on a table (2) a hockey puck sliding with constant velocity across a frictionless surface (3) the rotating blades of a ceiling fan and (4) the wheel of a bicycle that is travelling along a straight path at constant speed.
- For each of these objects:
    1. The linear momentum $\vec{P}$ of its center of mass is constant.
    2. The angular momentum $\vec{L}$ about its center of mass, or about any other point, is constant.
- We say that such objects are in **equilibrium**. The two requirements for equilibrium are then

$$\vec{P} = \text{a constnat} \quad , \quad \vec{L} = \text{a constnat.}$$

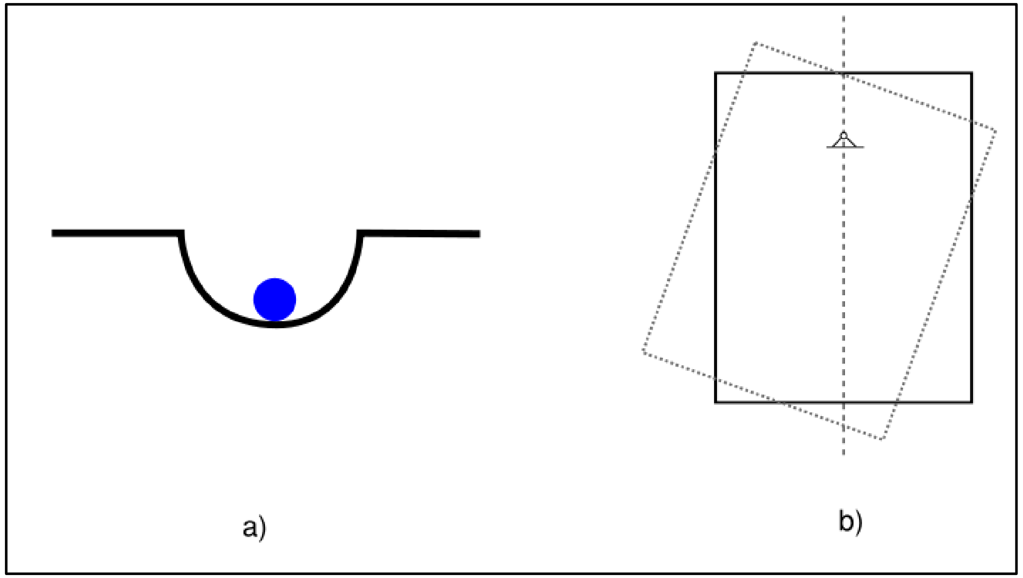
1. Equilibrium

- In this chapter we concentrate on situations in which the objects are not moving transnationally ($\vec{P} = 0$) or rotationally ($\vec{L} = 0$), in the reference frame in which we observer them. Such objects are in **static equilibrium**.

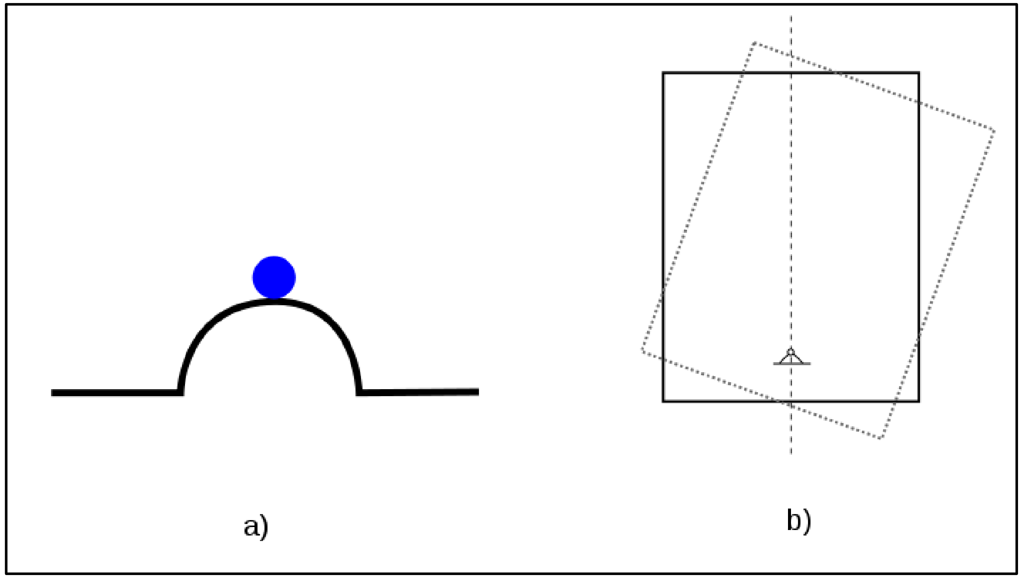
  Which of the four examples above are in static equilibrium?

- If a body returns to a state of static equilibrium after having been displaced from that state by a force, the body is said to be in **stable equilibrium**. If the force displacing the body and ends the equilibrium, the body is in **unstable equilibrium**.

1. Equilibrium



Stable equilibrium



Unstable equilibrium

From Wikipedia

2. The Requirements of Equilibrium

- If a body is in translational equilibrium ($\vec{P} = \text{a constnat}$) then by Newton's second law
$$\vec{F}_{\text{net}} = 0.$$
- If a body is in rotational equilibrium ($\vec{L} = \text{a constnat}$) then by Newton's second law (in the angular form)
$$\vec{\tau}_{\text{net}} = 0.$$
- The two requirements for a body to be in equilibrium are then:
  1. The vector sum of all external forces that act on the body must be zero.
  2. The vector sum of all external torques that act on the body, measured about any possible point, must be zero.

2. The Requirements of Equilibrium

- In rectangular components, the two equilibrium conditions are:

$$F_{\mathrm{net},x} = 0, \qquad F_{\mathrm{net},y} = 0, \qquad F_{\mathrm{net},z} = 0,$$

  and

$$\tau_{\mathrm{net},x} = 0, \qquad \tau_{\mathrm{net},y} = 0, \qquad \tau_{\mathrm{net},z} = 0.$$

- We will consider simplified situations in which the forces that act on the body lie in the xy plane. In this case the equilibrium conditions reduce to

$$F_{\mathrm{net},x} = 0, \qquad F_{\mathrm{net},y} = 0,$$

  and

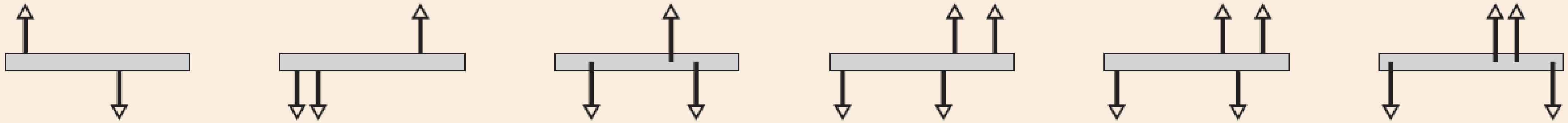
$$\tau_{\mathrm{net},z} = 0.$$

  Why $\tau_{\mathrm{net},z}$ in particular?

2. The Requirements of Equilibrium

CHECKPOINT 1

The figure gives six overhead views of a uniform rod on which two or more forces act perpendicularly to the rod. If the magnitudes of the forces are adjusted properly (but kept nonzero), in which situations can the rod be in static equilibrium?



(*a*) (*b*) (*c*) (*d*) (*e*) (*f*)

(c)

(e)

(f)

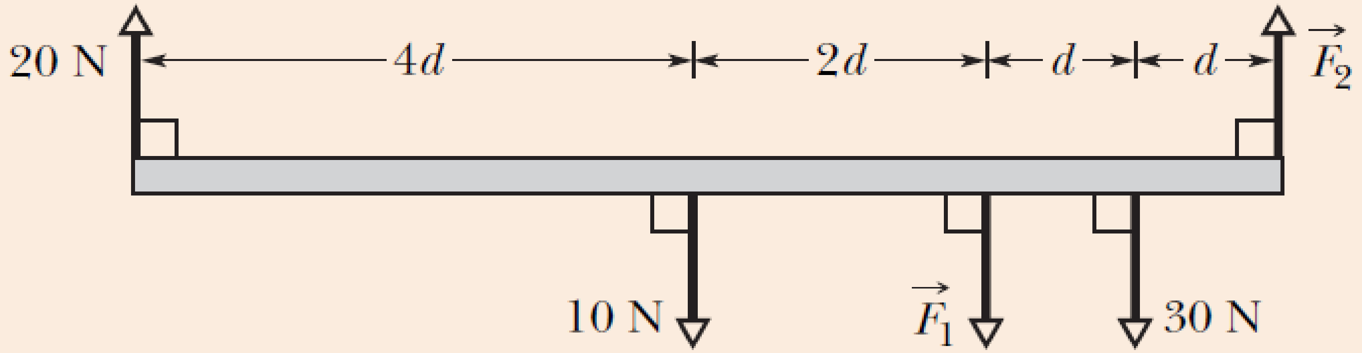
3. The Center of Gravity

- The gravitational force $\vec{F}_g$ on an extended body is the vector sum of the individual gravitational forces acting on every element (atom) of the body.
- Alternatively, we can consider a single point, called the **center of gravity** (cog) of the body, on which the gravitational force $\vec{F}_g$ effectively act.
- If $\vec{g}$ is constant over a body, then the center of gravity coincides with the body's center of mass. (See your textbook for the proof.)

4. Some Examples of Static Equilibrium

CHECKPOINT 2

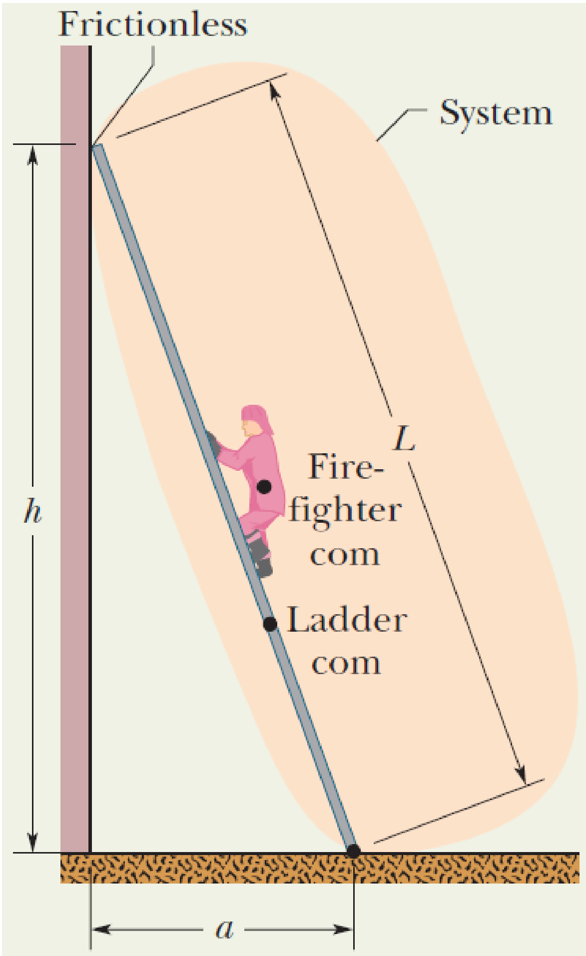
The figure gives an overhead view of a uniform rod in static equilibrium. (a) Can you find the magnitudes of unknown forces $\vec{F}_1$ and $\vec{F}_2$ by balancing the forces? (b) If you wish to find the magnitude of force $\vec{F}_2$ by using a balance of torques equation, where should you place a rotation axis to eliminate $\vec{F}_1$ from the equation? (c) The magnitude of $\vec{F}_2$ turns out to be 65 N. What then is the magnitude of $\vec{F}_1$?



(a) No. (b) At position of $\vec{F}_1$. (c) 45 N.

4. Some Examples of Static Equilibrium

Example 1: a ladder of length $L = 12\ \mathrm{m}$ and mass $m = 45\ \mathrm{kg}$ leans against a slick wall (that is, there is no friction between the ladder and the wall). The ladder's upper end is at height $h = 9.3\ \mathrm{m}$ above the pavement on which the lower end is supported (the pavement is not frictionless). The ladder's center of mass is $L/3$ from the lower end, along the length of the ladder. A firefighter of mass $M = 72\ \mathrm{kg}$ climbs the ladder until his center of mass is $L/2$ from the lower end. What then are the magnitudes of the forces on the ladder from the wall and the pavement?



4. Some Examples of Static Equilibrium

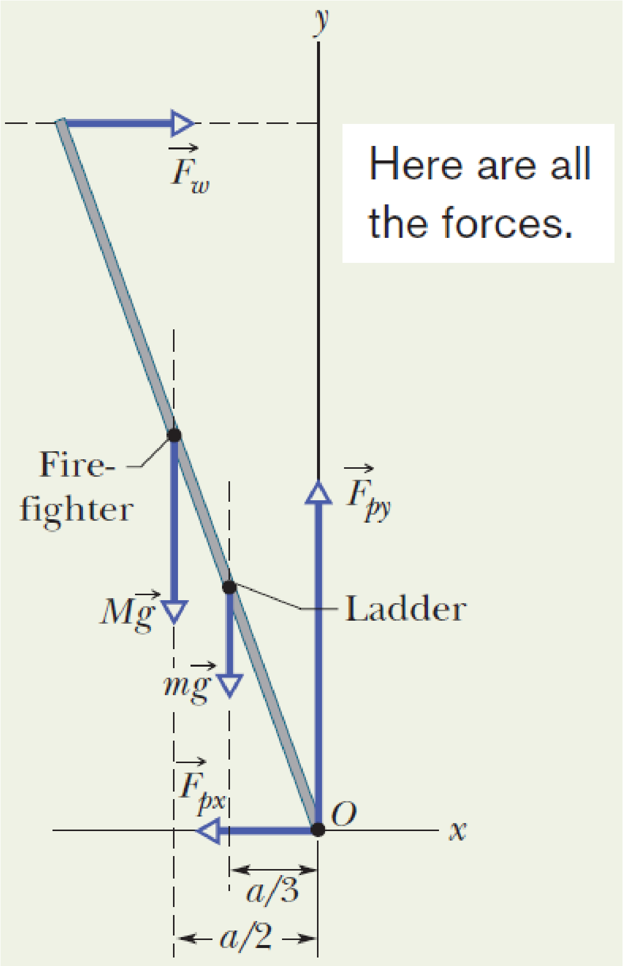
Using $\tau = r_{\perp}F$, the torques balancing equation $\tau_{\text{net},z} = 0$ about the origin O becomes

$$-hF_w + \left(\frac{a}{2}\right)Mg + \left(\frac{a}{3}\right)mg + (0)F_{py} + (0)F_{px} = 0.$$

We get then

$$\begin{aligned} F_w &= \frac{a}{6h}(3M + 2m)g \\ &= \frac{7.6\ \text{m}}{6(9.3\ \text{m})}[3(72\ \text{kg}) + 2(45\ \text{kg})](9.8\ \text{m/s}^2) \\ &\approx 410\ \text{N}. \end{aligned}$$

Here $a = \sqrt{(12\ \text{m})^2 - (9.3\ \text{m})^2} = 7.6\ \text{m}.$



4. Some Examples of Static Equilibrium

The equation $F_{\text{net},x} = 0$ reads

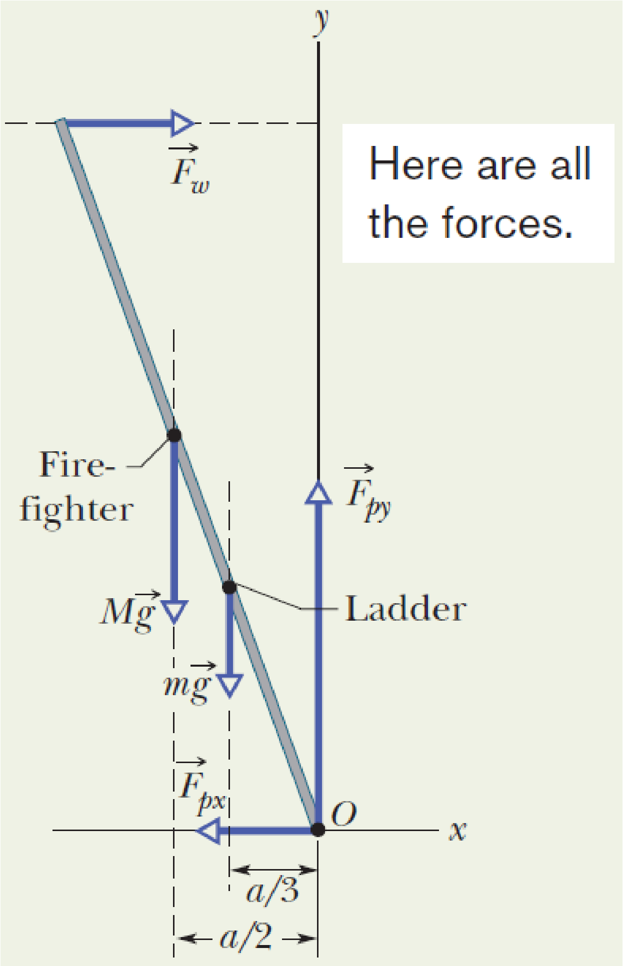
$$F_w - F_{px} = 0,$$

which yields $F_{px} = F_w = 410\ \text{N}$. The equation $F_{\text{net},y} = 0$ gives

$$F_{py} - Mg - mg = 0,$$

or

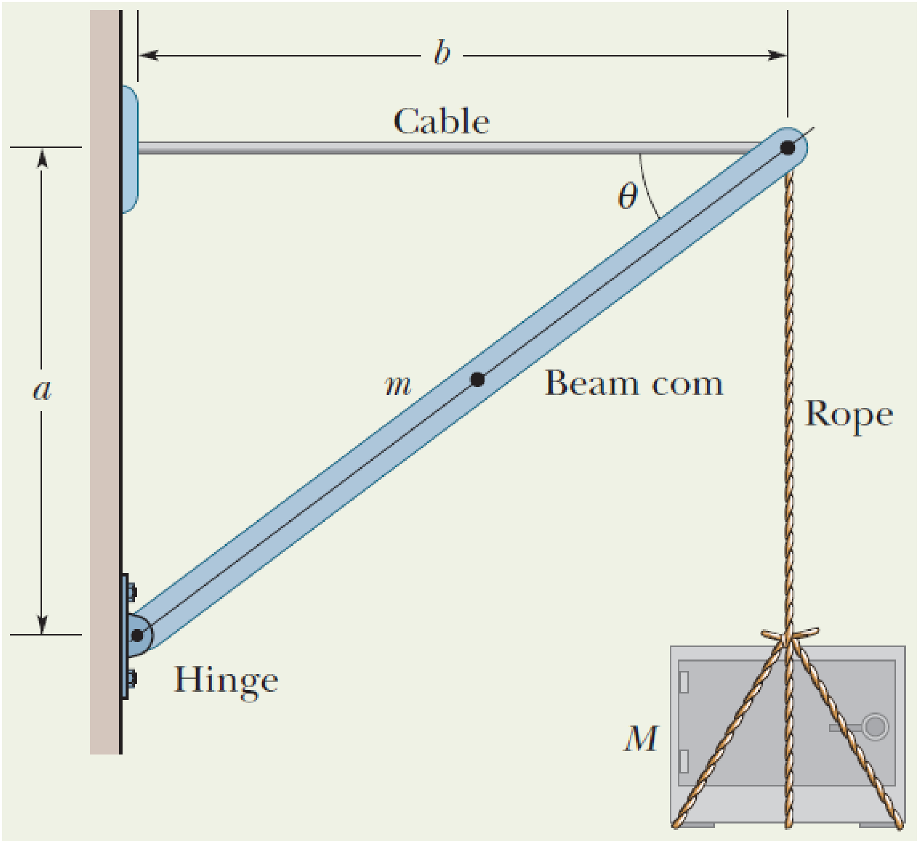
$$F_{py} = (M + m)g = (72\ \text{kg} + 45\ \text{kg})(9.8\ \text{m/s}^2) \approx 1100\ \text{N}.$$



4. Some Examples of Static Equilibrium

Example 2: The figure shows a safe (mass $M = 430\ \mathrm{kg}$) hanging by a rope (negligible mass) from a boom ($a = 1.9\ \mathrm{m}$ and $b = 2.5\ \mathrm{m}$) that consists of a uniform hinged beam ($m = 85\ \mathrm{kg}$) and horizontal cable (negligible mass).

(a) What is the tension T_c in the cable? In other words, what is the magnitude of the force T_c on the beam from the cable?



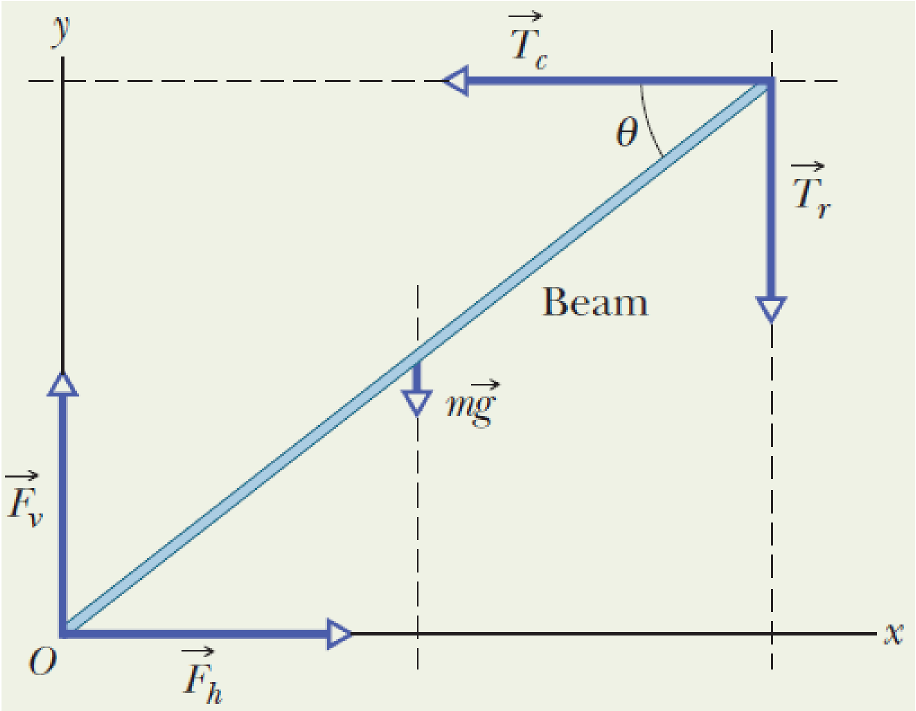
4. Some Examples of Static Equilibrium

Writing the torques in the form $\tau = r_\perp F$, the torques balancing equation $\tau_{\mathrm{net},z} = 0$ about O becomes

$$aT_c - bT_r - (b/2)(mg) = 0.$$

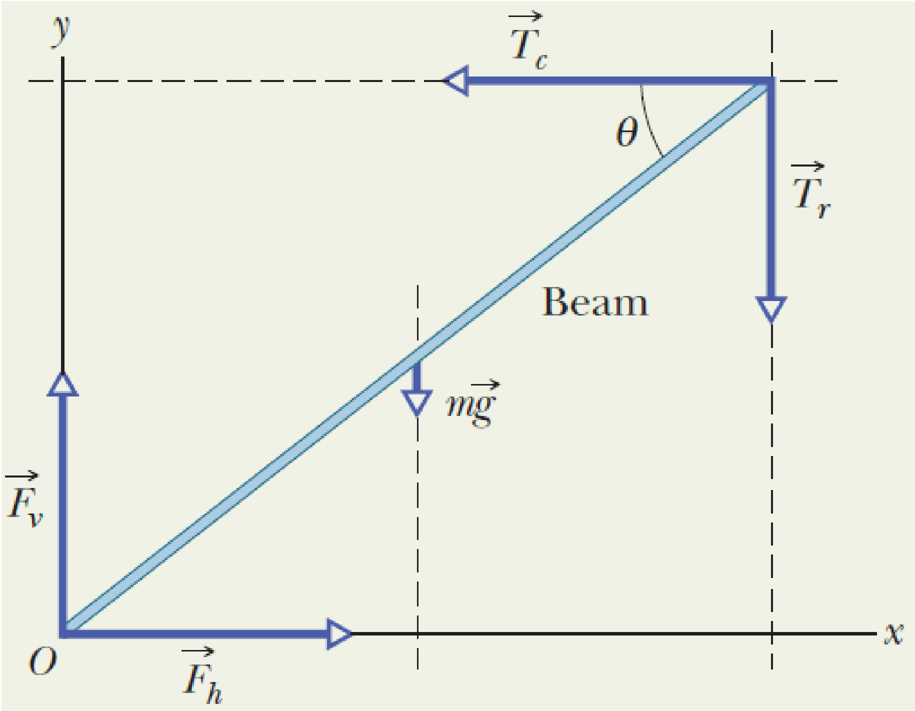
Solving for T_c, using $T_r = Mg$ and substituting yield

$$\begin{aligned} T_c &= \frac{b}{a}(M + m/2)g \\ &= \frac{2.5\ \mathrm{m}}{1.9\ \mathrm{m}}(430\ \mathrm{kg} + 45\ \mathrm{kg}/2)(9.8\ \mathrm{m/s^2}) \\ &= 6093\ \mathrm{N} \approx 6100\ \mathrm{N}. \end{aligned}$$



4. Some Examples of Static Equilibrium

(b) Find the magnitude F of the net force on the beam from the hinge.



The horizontal forces balancing equation $F_{\mathrm{net},x} = 0$ reads

$$F_h - T_c = 0,$$

which yields $F_h = T_c = 6093\ \mathrm{N}.$

The vertical forces balancing equation $F_{\mathrm{net},y} = 0$ reads

$$F_v - mg - T_r = 0,$$

which yields

4. Some Examples of Static Equilibrium

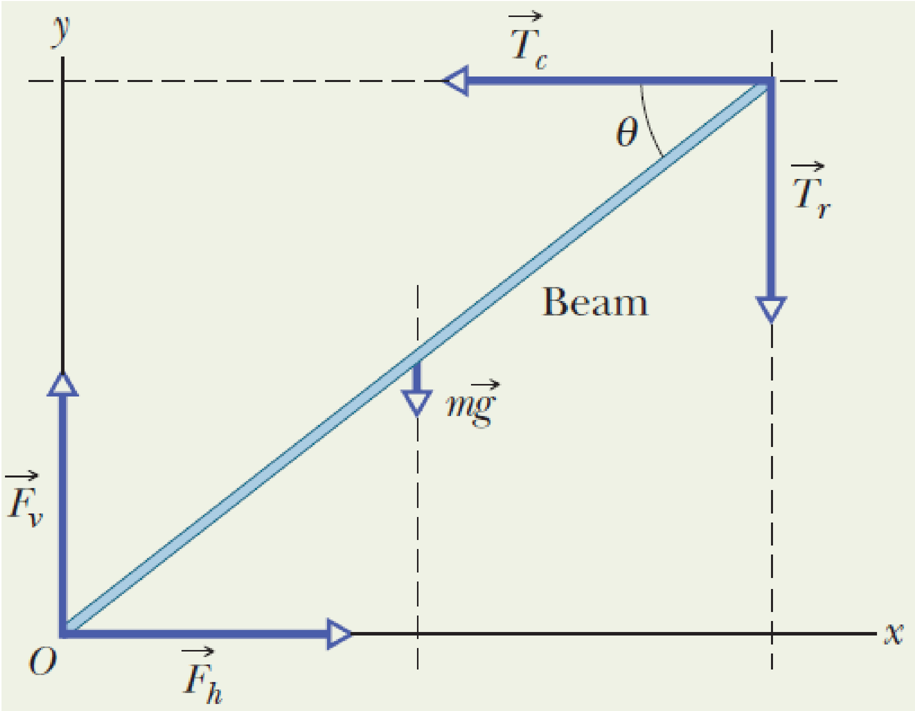
$$F_v = mg + T_r = (M + m)g.$$

Substituting gives

$$F_v = (430\ \mathrm{kg} + 45\ \mathrm{kg})(9.8\ \mathrm{m/s^2}) = 5047\ \mathrm{N}.$$

Therefore

$$F = \sqrt{F_h^2 + F_v^2}$$
$$= \sqrt{(6093\ \mathrm{N})^2 + (5047\ \mathrm{N})^2}$$
$$\approx 7900\ \mathrm{N}.$$



4. Some Examples of Static Equilibrium

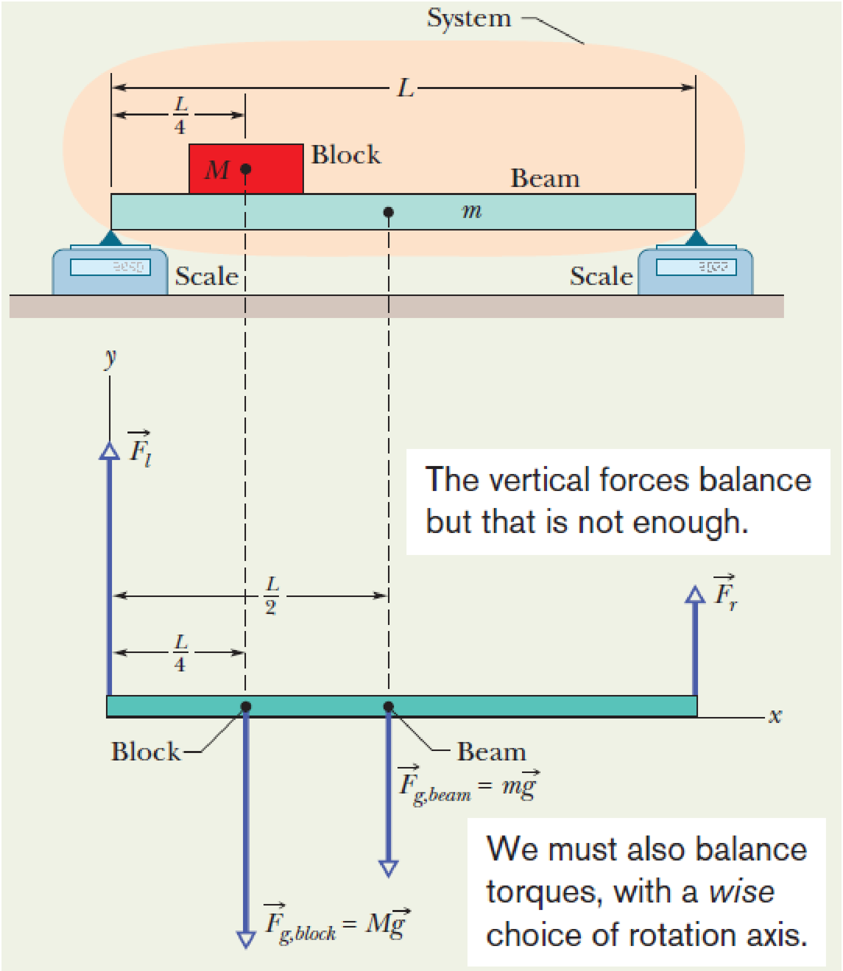
Example 3: In the figure, a uniform beam, of length L and mass $m = 1.8\,\text{kg}$, is at rest on two scales. A uniform block, with mass $M = 2.7\,\text{kg}$, is at rest on the beam, with its center a distance $L/4$ from the beam's left end. What do the scales read?

The torques balancing equation $\tau_{\text{net},z} = 0$ about the left end of the beam reads

$$LF_r - (L/2)(mg) - (L/4)(Mg) = 0.$$

F_r is then

$$F_r = (m/2 + M/4)g$$
$$= (1.8\,\text{kg}/2 + 2.7\,\text{kg}/4)(9.8\,\text{m/s}^2) \approx 15\,\text{N}.$$



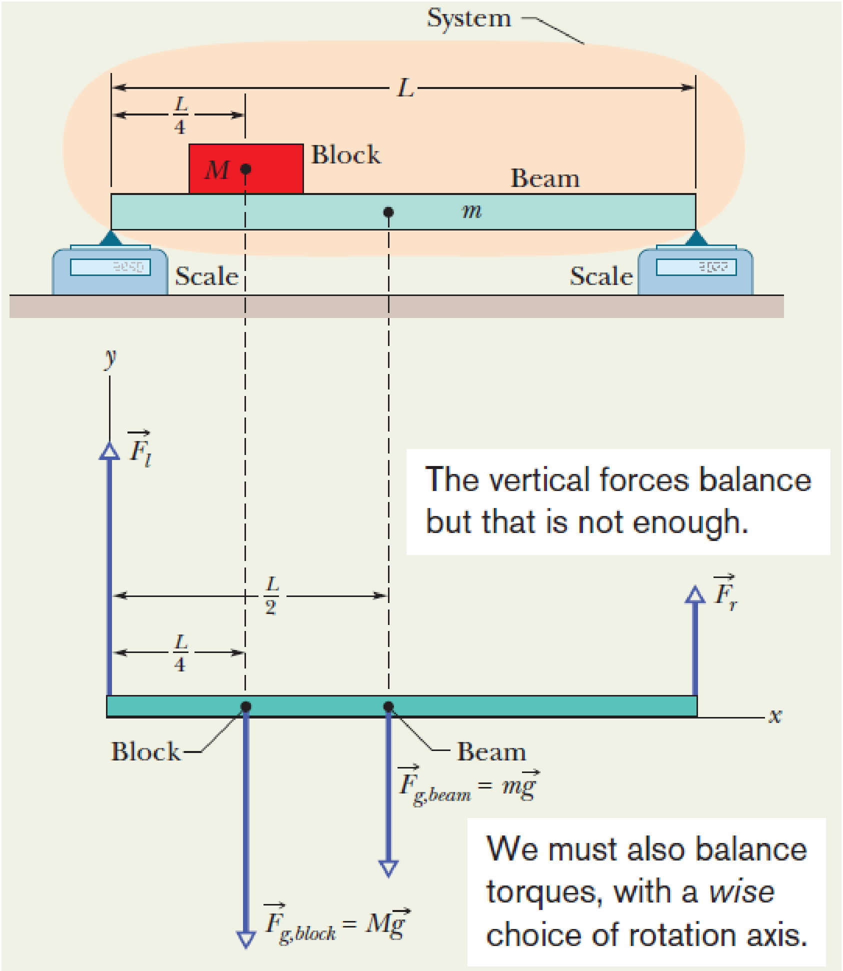
4. Some Examples of Static Equilibrium

The vertical balancing equation $F_{\text{net},y} = 0$ reads

$$F_l - Mg - mg + F_r = 0,$$

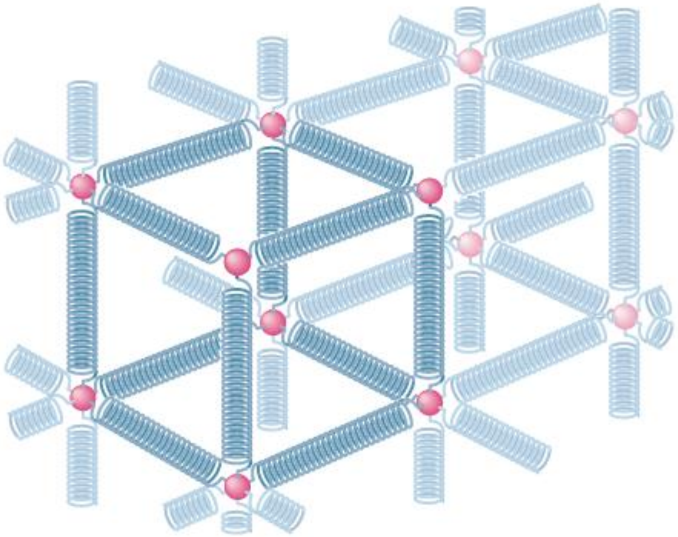
which gives

$$\begin{aligned} F_l &= (M + m)g - F_r \\ &= (2.7\text{ kg} + 1.8\text{ kg}/4)(9.8\text{ m/s}^2) - 15.4\text{ N} \\ &\approx 29\text{ N}. \end{aligned}$$



5. Elasticity

- The atoms in a solid settle into equilibrium positions in a three-dimensional *lattice*. A *lattice* is a repetitive arrangement in which each atom is at a well-defined equilibrium distance from its nearest neighbors.
- The atoms are held together by interatomic forces that are modeled as tiny springs. These springs are extremely stiff; the lattice is remarkably rigid. This is why we perceive ordinary objects as perfectly rigid.
- In soft material, such as rubber, the atoms do not form a rigid lattice but are aligned in a long, flexible molecular chains that are loosely bound to their neighboring chains.

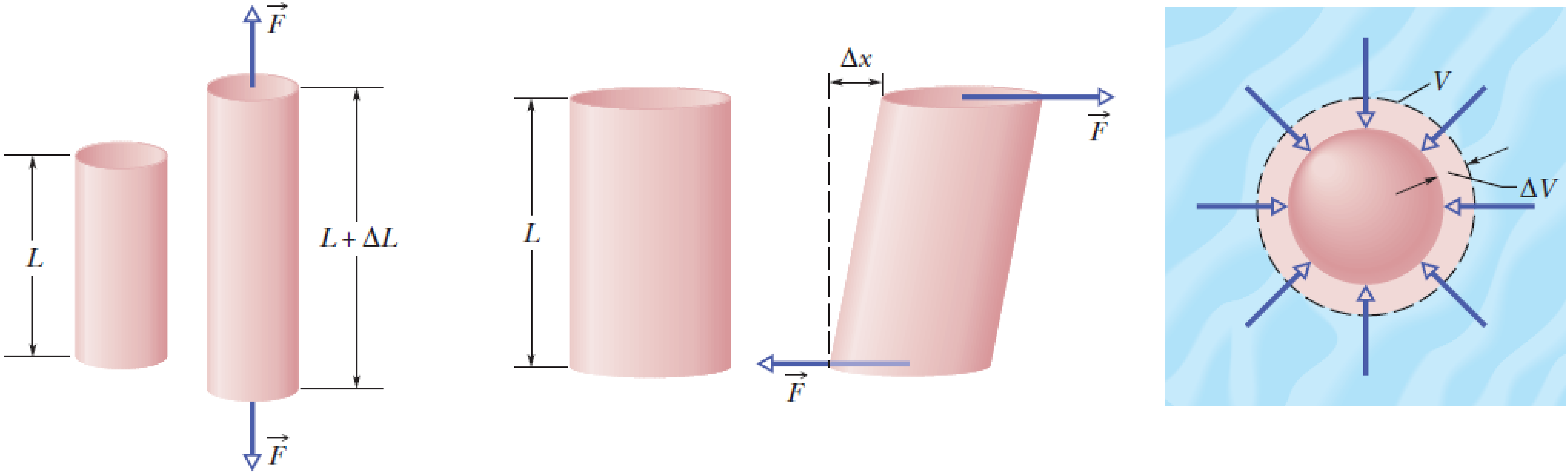


5. Elasticity

- All real rigid bodies are to some extent elastic; we can change their dimensions slightly by pulling, pushing, twisting, or compressing them. For example, if we hanged a small size car to a 1 m long and 1 cm in diameter steel rod, the rod will extend by only 0.5 mm. When the car is removed, the rod will return to its original length.
- If two cars are hanged to the rod, the rod will be permanently stretched. If three cars are attached to the rod, the rod will break, after it elongates by less than 2 mm.

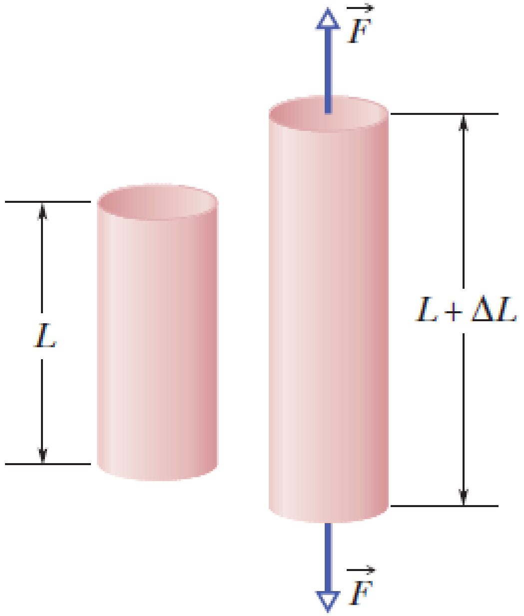
5. Elasticity

- The three figures show three ways in which a solid might change its dimension under the influence of an external force. In the three cases, there is a deforming force per unit area or **stress** that produces unit deformation or **strain**.

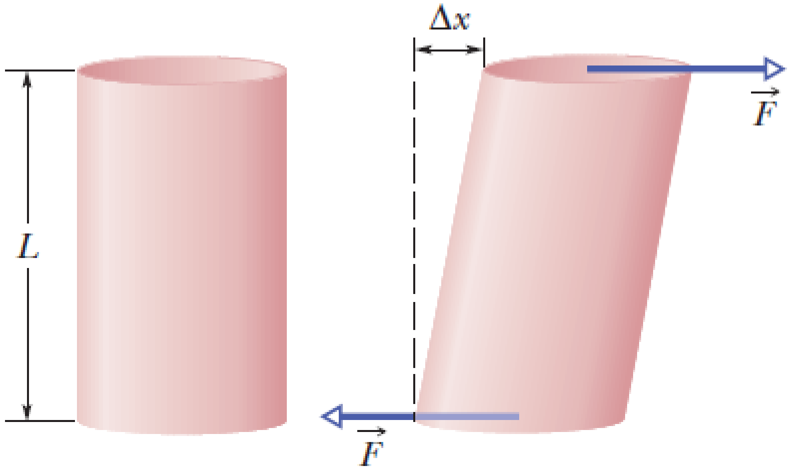


5. Elasticity

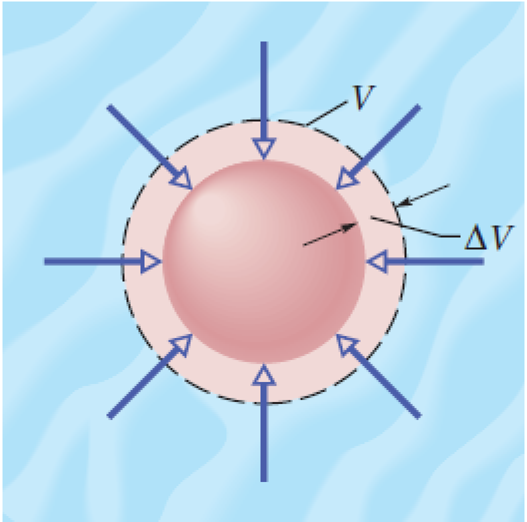
Tensile stress
↓
Stretching



Shearing stress
↓
Deformation



Hydraulic stress
↓
Compression

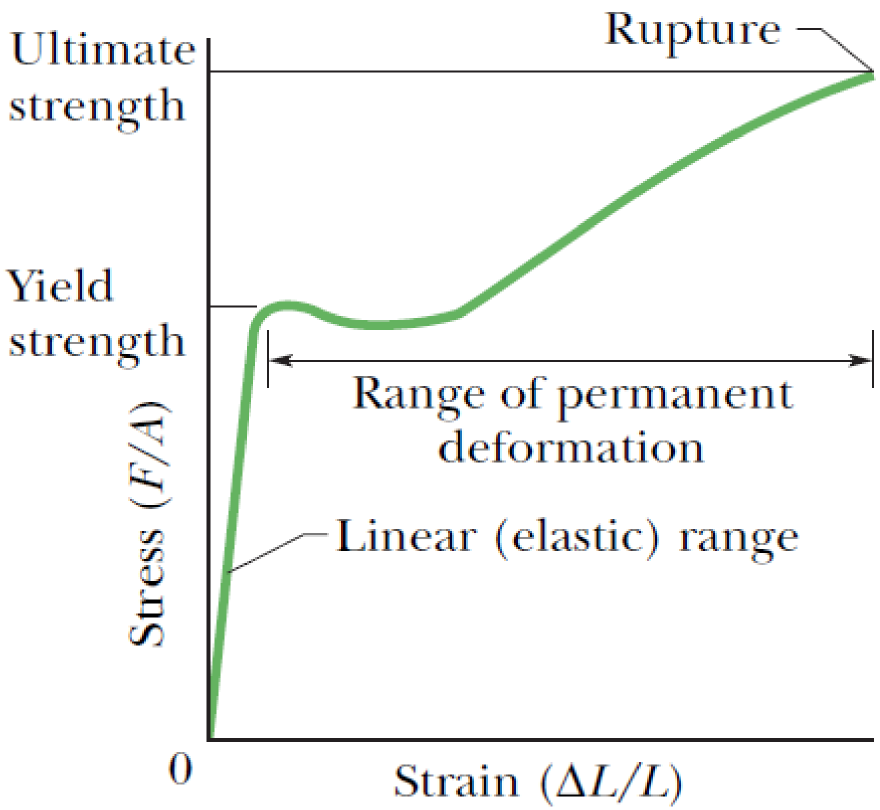


5. Elasticity

- Stresses and strains are proportional to each other. The constant of proportionality is called a **modulus of elasticity**. Therefore we write

$$\text{stress} = \text{modulus} \times \text{strain}.$$

- The figure shows a stress-strain curve for a steel test specimen. The stress-strain is linear before the **yield strength** S_y is reached. Beyond S_y, the specimen becomes permanently deformed. If the stress is increased further, the specimen eventually ruptures when the **ultimate strength** S_u is reached.



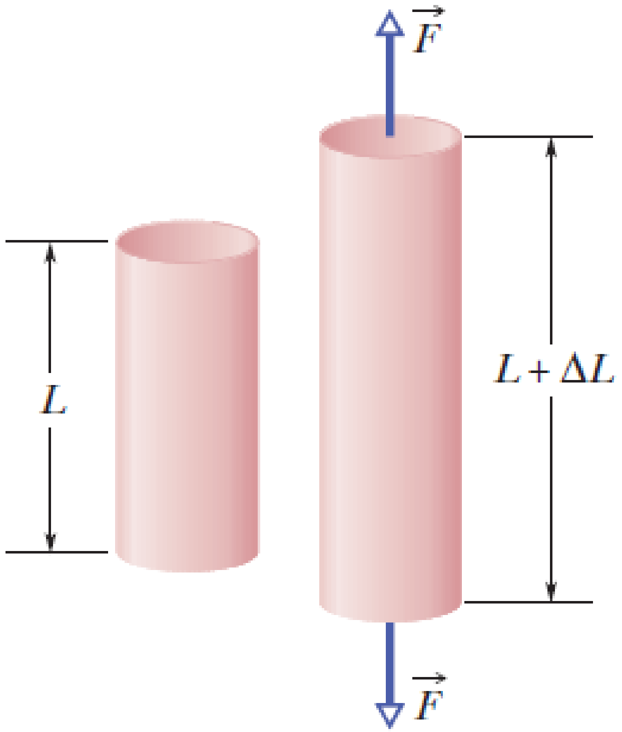
5. Elasticity

- **Tension and Compression:** For simple tension or compression, the stress on an object is F/A, where F is the magnitude of applied force perpendicular to an area A on the object.

  The strain or, unit deformation is the dimensionless quantity $\Delta L/L$. If the specimen is a long rod and the stress does not exceed the yield strength then every section of it experience the same strain when a given stress is applied.

  The modulus for tensile and compressive stresses is called the **Young's modulus** E. We can write

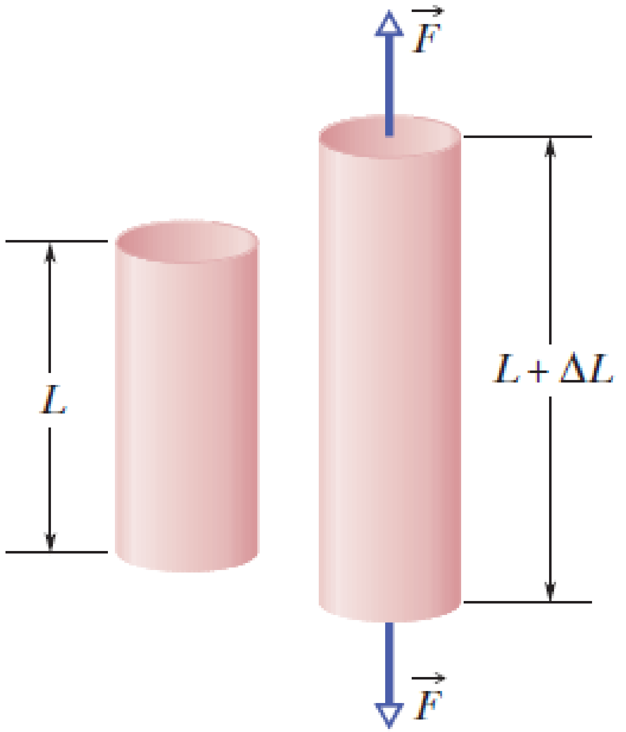
$$\frac{F}{A} = E\frac{\Delta L}{L}.$$



5. Elasticity

F/A has the SI unit of $\mathrm{N/m^2}$ and E has the same units as F/A since $\Delta L/L$ is dimensionless.

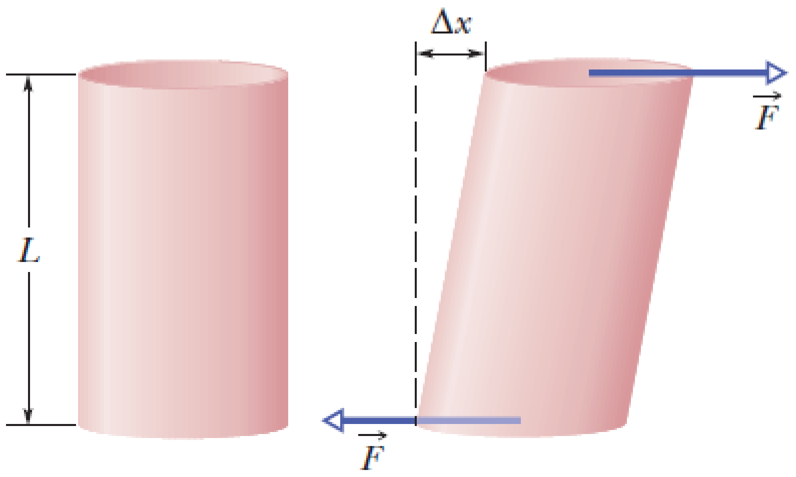
The Young's modulus may be nearly the same for tension and compression. However, the object's ultimate strength may be very different for the two types of stress.



5. Elasticity

- **Shearing:** In the case of shearing the stress is also a force per unit area, but it lies in the plane of the area. The strain is $\Delta x/L$. The corresponding modulus is called the **shear modulus** G. For shearing we write

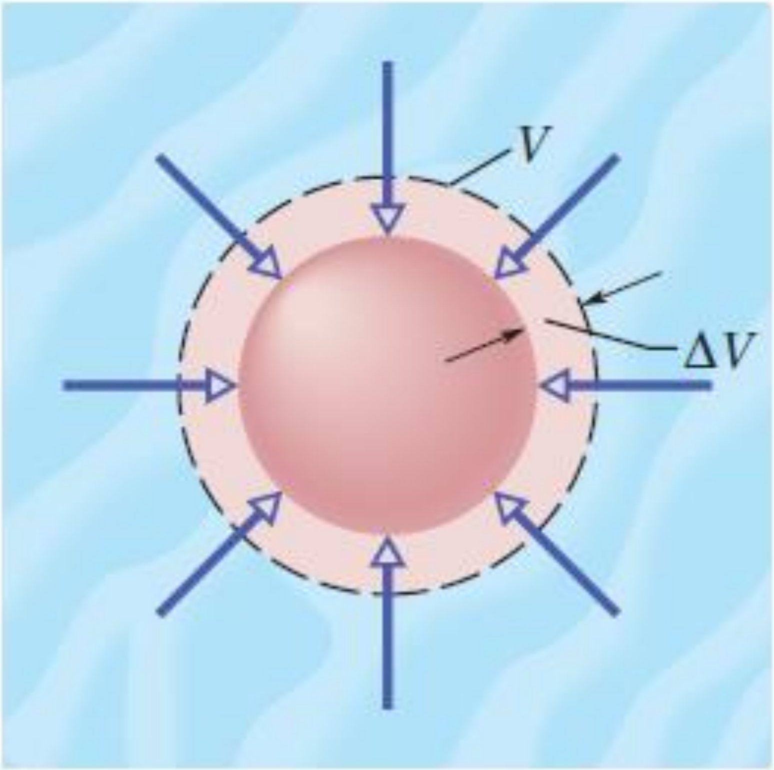
$$\frac{F}{A} = G\frac{\Delta x}{L}.$$



5. Elasticity

- **Hydraulic Stress:** The hydraulic stress is the pressure of a fluid (gas or liquid) on an object. The pressure p is a force per unit area. The strain is now $\Delta V/V$. The corresponding modulus is called the **bulk modulus** B. The object is said to be under hydraulic compression and the pressure can be called the hydraulic stress. We can write

$$p = B\frac{\Delta V}{V}.$$



5. Elasticity

Example 4: One end of a steel rod of radius $R = 9.5\ \mathrm{mm}$ and length $L = 81\ \mathrm{cm}$ is held in a vise. A force of magnitude $F = 62\ \mathrm{kN}$ is then applied perpendicularly to the end face (uniformly across the area) at the other end, pulling directly away from the vise. What are the stress on the rod and the elongation ΔL and strain of the rod? The Young's modulus for steel is $2.0 \times 10^{11}\ \mathrm{N/m^2}$.

The stress is

$$\frac{F}{A} = \frac{62 \times 10^3\ \mathrm{N}}{\pi(9.5 \times 10^{-3}\ \mathrm{m})^2} = 2.2 \times 10^8\ \mathrm{N/m^2}.$$

The elongation of the rod is

$$\Delta L = \frac{F}{A}\frac{L}{E} = (2.2 \times 10^8\ \mathrm{N/m^2})\frac{0.81\ \mathrm{m}}{2.0 \times 10^{11}\ \mathrm{N/m^2}} = 8.9 \times 10^{-4}\ \mathrm{m} = 0.89\ \mathrm{mm}.$$

5. Elasticity

The strain is then

$$\frac{\Delta L}{L} = \frac{8.9 \times 10^{-4}\ \mathrm{m}}{0.81\ \mathrm{m}} = 1.1 \times 10^{-3} = 0.11\%.$$

5. Elasticity

Example 5: A table has three legs that are $1.00\ \mathrm{m}$ in length and a fourth leg that is longer by $d = 0.50\ \mathrm{mm}$, so that the table wobbles slightly. A steel cylinder with mass $M = 290\ \mathrm{kg}$ is placed on the table (which has a mass much less than M) so that all four legs are compressed but unbuckled and the table is level but no longer wobbles. The legs are wooden cylinders with cross-sectional area $A = 1.0\ \mathrm{cm}^2$; Young's modulus is $E = 1.3 \times 10^{10}\ \mathrm{N/m^2}$. What are the magnitudes of the forces on the legs from the floor?

Each of the three short legs is compressed by ΔL_3 under the influence of force F_3 . The long leg is compressed by ΔL_4 under the influence of force F_4. Because the tabletop is level

$$\Delta L_4 = \Delta L_3 + d.$$

5. Elasticity

We can relate a change of length to its corresponding force by $\Delta L = \frac{FL}{AE}$. The last equation becomes

$$\frac{F_4 L}{AE} = \frac{F_3 L}{AE} + d.$$

Now we use the balance of vertical forces ($F_{\text{net},y} = 0$) and write

$$3F_3 + F_4 - Mg = 0.$$

Solving for F_3 we find

$$F_3 = \frac{1}{4}\left(Mg - \frac{dAE}{L}\right)$$

$$= \frac{1}{4}\left[(290\ \mathrm{kg})(9.8\ \mathrm{m/s^2}) - \frac{(5.0 \times 10^{-4}\mathrm{m})(10^{-4}\mathrm{m}^2)(1.3 \times 10^{10}\ \mathrm{N/m^2})}{1.00\ \mathrm{m}}\right] \approx 550\ \mathrm{N}$$

5. Elasticity

Then F_4 is given by

$$F_4 = Mg - 3F_3 = \left[(290\ \mathrm{kg})\left(9.8\ \frac{\mathrm{m}}{\mathrm{s}^2}\right) - 3(548\ \mathrm{N})\right] \approx 1200\ \mathrm{N}.$$

Exercise: Calculate ΔL_3 and ΔL_4.