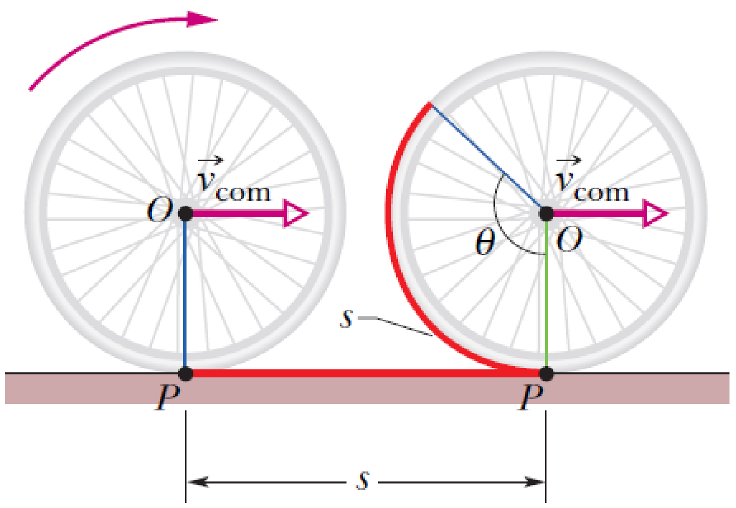
Chapter 11

Rolling, Torque and Angular Momentum

1. Rolling as Translation and Rotation Combined

- We consider here smoothly rolling objects along a straight surface; that is objects roll without slipping or bouncing on the surface.
- Consider a bicycle wheel as it rolls along a street. Both the center of mass O of the wheel and the point P of contact with the street move forward at speed v_{com}. During a time interval t, both O and P move forward by a distance s.



1. Rolling as Translation and Rotation Combined

- The distance s is related to the angle θ through which rotate about the center of the wheel by

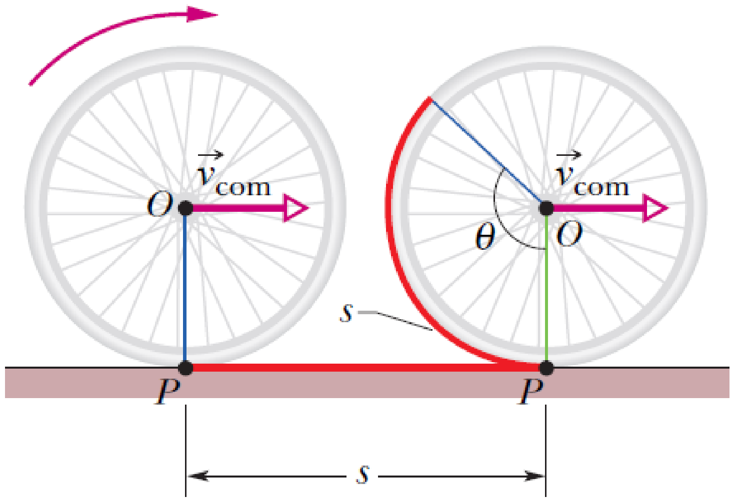
$$s = \theta R.$$

where R is the radius of the wheel.

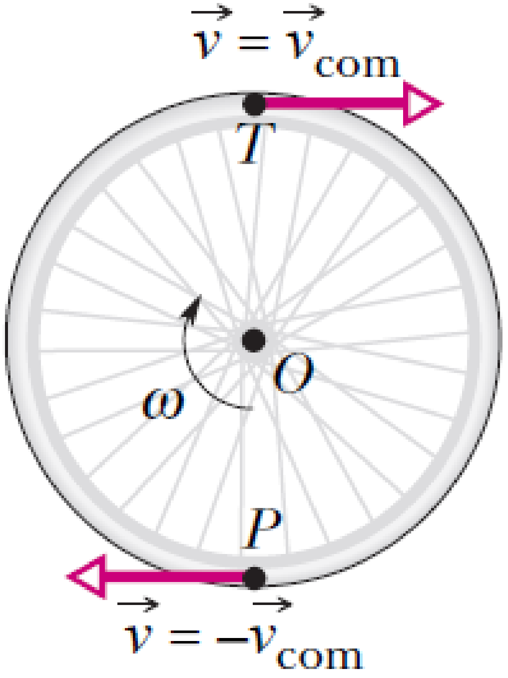
- We can relate v_{com} to the rotational speed ω of the wheel. Differentiating s with respect to time we get

$$v_{\text{com}} = \omega R.$$

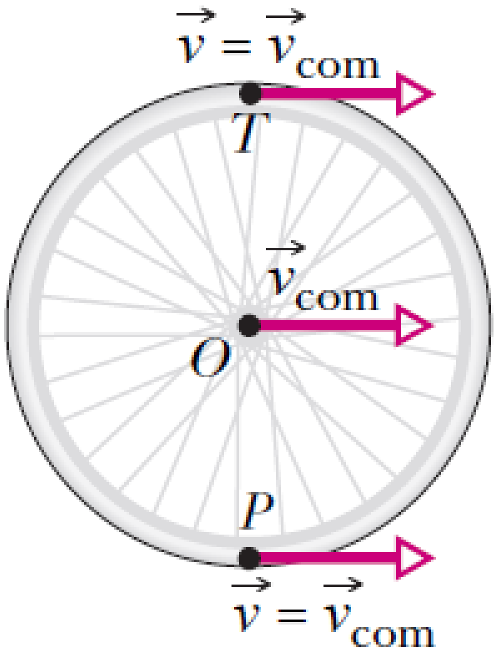
- The rolling motion of a wheel is a combination of (1) purely translational and (2) purely rotational motions.



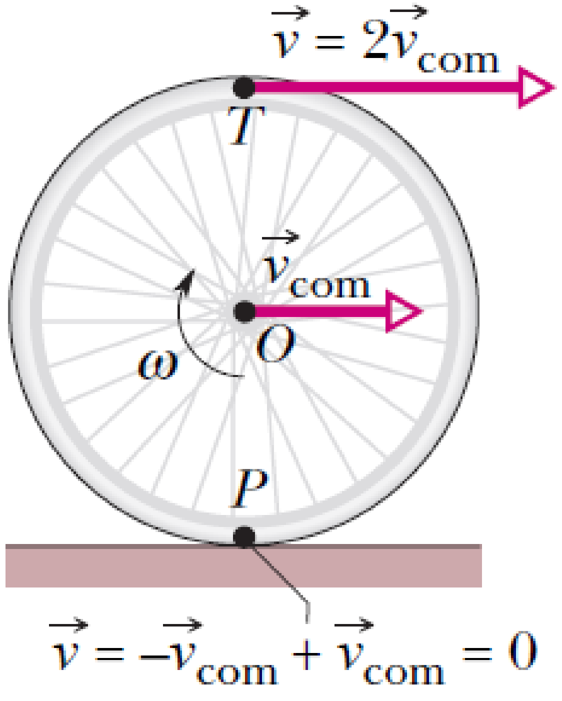
- In the purely rotational motion, every point on the wheel rotates about the center with angular speed ω. Every point on the outside edge of the wheel has linear speed v_{com}.



- In the purely translational motion every point on the wheel moves to the right with speed v_{com}.



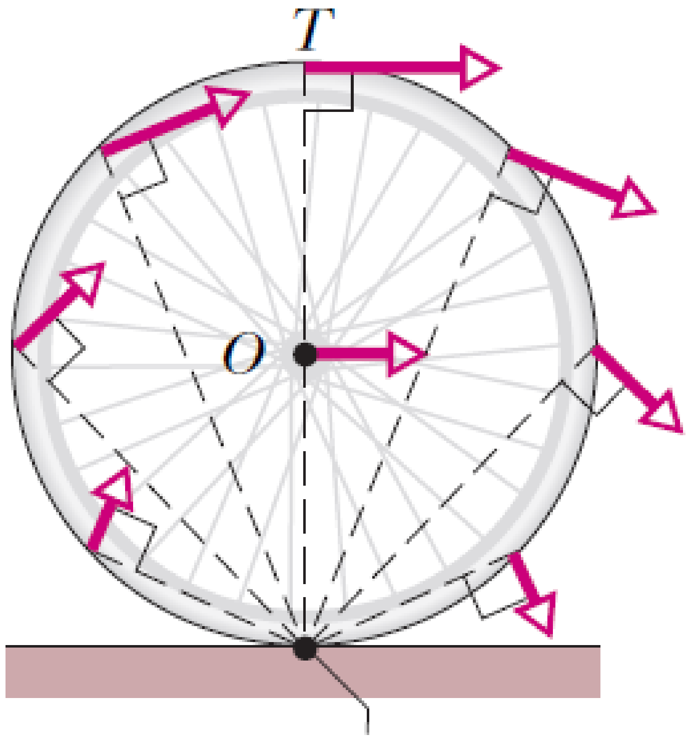
- The rolling motion of the wheel is the combination of the two motions. In this combination, point T is moving at speed $2v_{\mathrm{com}}$, where point P is stationary.



1. Rolling as Translation and Rotation Combined

- Rolling as Pure Rotation

Rolling can be viewed as pure rotation about an axis passing through point P, perpendicular to the plane of page, with angular speed ω.



https://www.youtube.com/watch?v=hnjszE3NS9E

1. Rolling as Translation and Rotation Combined

CHECKPOINT 1

The rear wheel on a clown's bicycle has twice the radius of the front wheel. (a) When the bicycle is moving, is the linear speed at the very top of the rear wheel greater than, less than, or the same as that of the very top of the front wheel? (b) Is the angular speed of the rear wheel greater than, less than, or the same as that of the front wheel?

(a) Equal

(b) Less.

2. The Kinetic Energy of Rolling

- A rolling object has two types of kinetic energy: (1) a rotational kinetic energy ($\frac{1}{2}I_{\text{com}}\omega^2$) due to its rotation about its center of mass and (2) a translational kinetic energy ($\frac{1}{2}Mv_{\text{com}}^2$) due to translation of its center of mass. Therefore, the kinetic energy K of a rolling object is

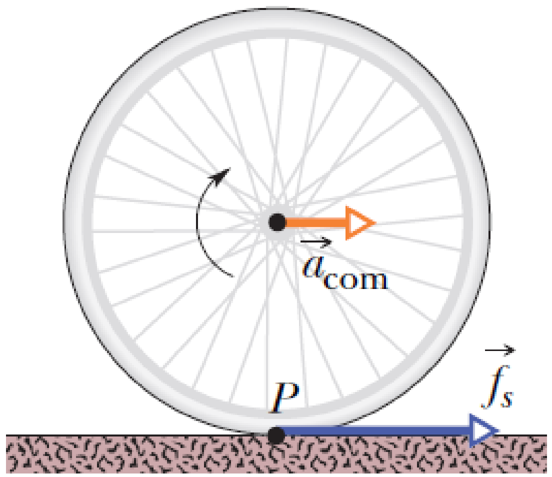
$$K = \frac{1}{2}I_{\text{com}}\omega^2 + \frac{1}{2}Mv_{\text{com}}^2.$$

3. The Forces of Rolling

<u>Friction and Rolling</u>

- A wheel that rolls at constant speed has no tendency to slide at the point of contact P, and thus no frictional force acts there. If a net force acts on the wheel, then it will have acceleration $\vec{a}_{\text{com}}$ of the center of mass, and angular acceleration α. These accelerations tend to make the wheel slide at P. Thus, a frictional force must act on the wheel at point P to oppose that tendency.
- If the wheel does not slide, the force is static frictional force $\vec{f}_s$ and the rolling is smooth. We can then relate $\vec{a}_{\text{com}}$ to α by differentiating $v_{\text{com}} = \omega R$ to get

$$a_{\text{com}} = \alpha R.$$



3. The Forces of Rolling

<u>Rolling Down a Ramp</u>

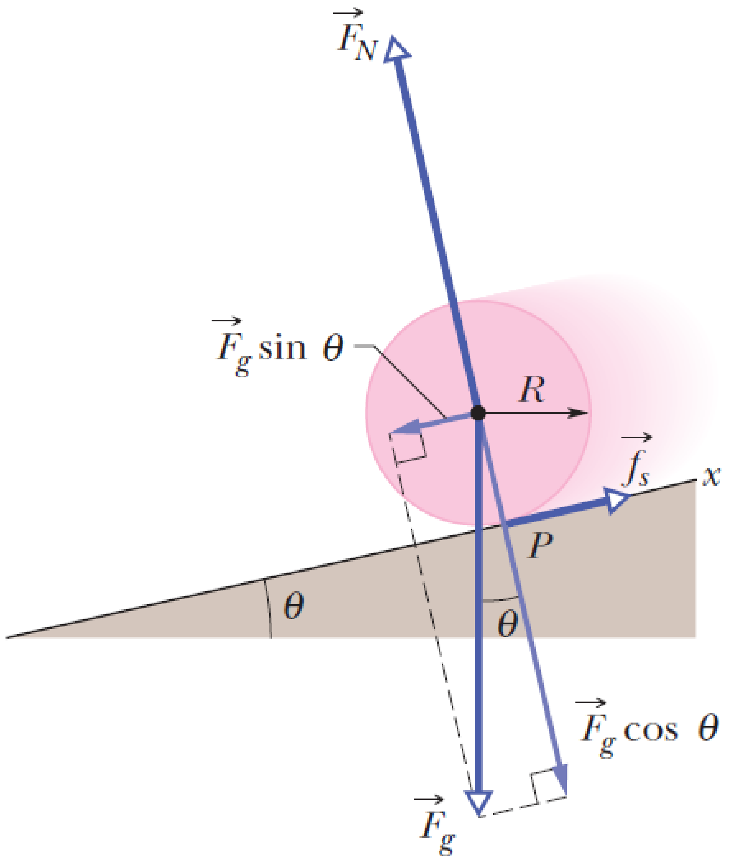
- Consider the rolling object in the figure. We want to find its acceleration $a_{\text{com},x}$ down the ramp. We use Newton's second law in both its linear version ($F_{\text{net}} = ma$) and its angular version ($\tau_{\text{net}} = I\alpha$):

  Along the x axis:

  $$f_s - Mg\sin\theta = Ma_{\text{com},x}.$$

  About an axis through the body's center of mass:

  $$Rf_s = I_{\text{com}}\alpha.$$



3. The Forces of Rolling

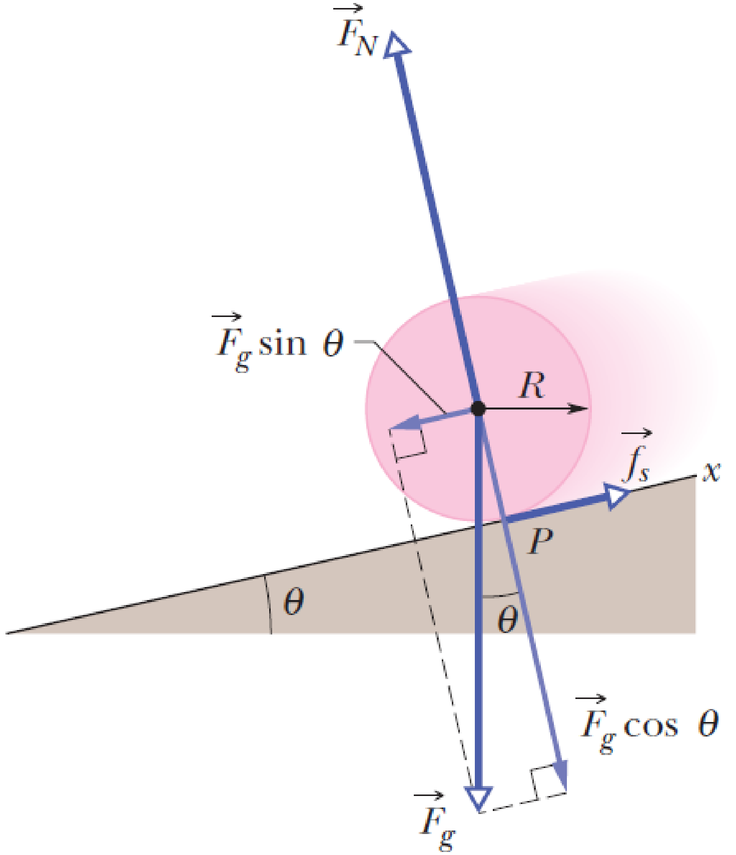
<u>Rolling Down a Ramp</u>

- Since the body is rolling smoothly, $a_{\text{com},x} = -\alpha R$. Solving for $a_{\text{com},x}$ we get

$$a_{\text{com},x} = -\frac{g \sin\theta}{1 + \dfrac{I_{\text{com}}}{MR^2}}.$$

The frictional force is given by

$$f_s = -I_{\text{com}} \frac{a_{\text{com},x}}{R^2}$$



3. The Forces of Rolling

CHECKPOINT 2

Disks A and B are identical and roll across a floor with equal speeds. Then disk A rolls up an incline, reaching a maximum height h, and disk B moves up an incline that is identical except that it is frictionless. Is the maximum height reached by disk B greater than, less than, or equal to h?

Less

3. The Forces of Rolling

Example 1: A uniform ball, of mass $M = 6.00\,\mathrm{kg}$ and radius R, rolls smoothly from rest down a ramp at angle $\theta = 30.0°$.

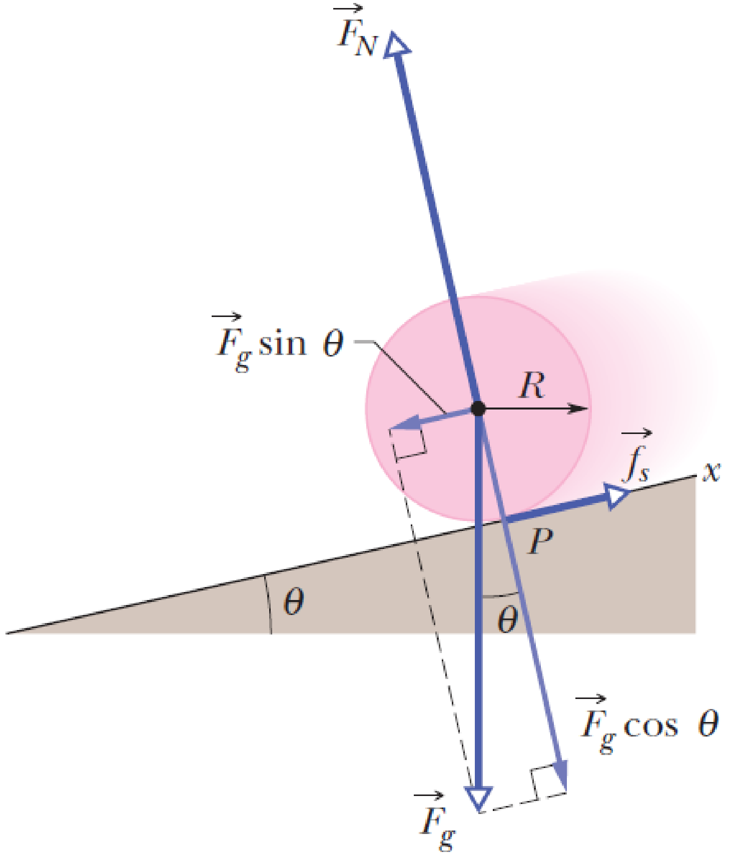
(a) The ball descends a vertical height $h = 1.20\ \mathrm{m}$ to reach the bottom of the ramp. What is its speed at the bottom?

Conservation of mechanical energy ($E_f = E_i$) can be written as

$$K_f + U_f = K_i + U_i,$$

or

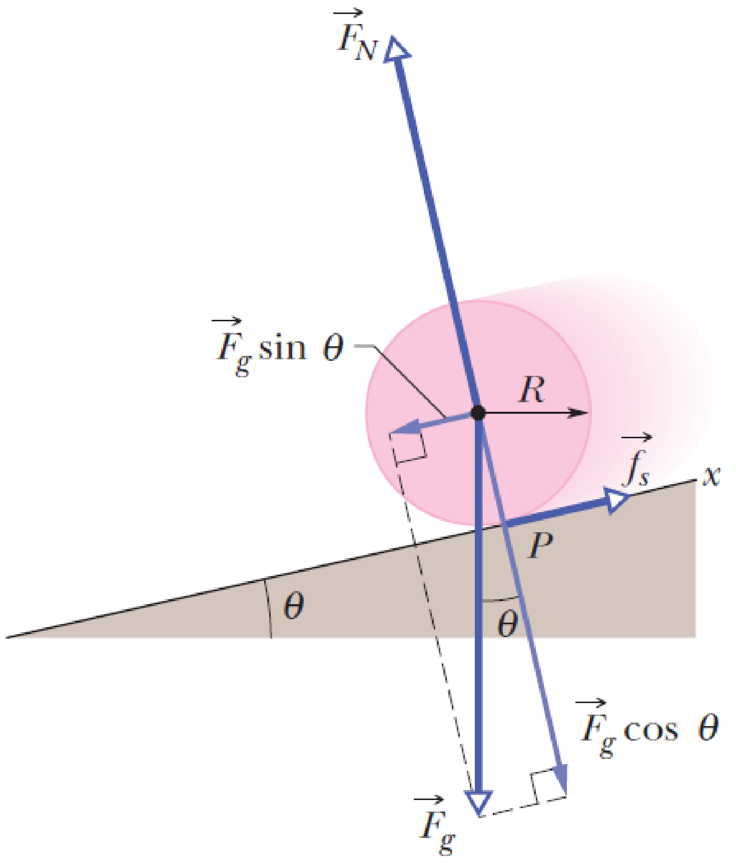
$$\left(\frac{1}{2} I_{\text{com}}\omega^2 + \frac{1}{2} M v_{\text{com}}^2\right) + 0 = 0 + Mgh$$



3. The Forces of Rolling

Because the rolling is smooth, $v_{\text{com}} = \omega R$. The moment of inertial I_{com} of a sold sphere is $\frac{2}{5}MR^2$. Substituting for ω and I_{com} and solving for v_{com} yield

$$v_{\text{com}} = \sqrt{(10/7)gh}$$
$$= \sqrt{(10/7)(9.8\ \text{m/s}^2)(1.2\ \text{m})} = 4.10\frac{\text{m}}{\text{s}}.$$



3. The Forces of Rolling

(b) What are the magnitude and direction of the frictional force on the ball as it rolls down the ramp?

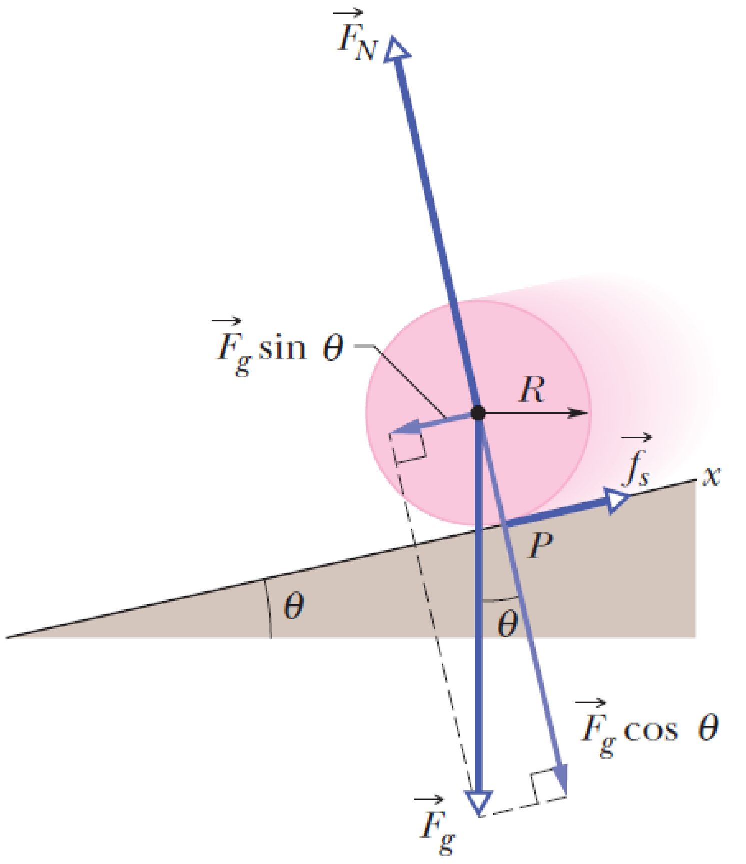
We need to calculate the acceleration $a_{\text{com,x}}$ first:

$$a_{\text{com},x} = -\frac{\left(9.8\,\frac{\text{m}}{\text{s}^2}\right)\sin 30.0°}{1 + \frac{2/5\,MR^2}{MR^2}}$$

$$= -\frac{\left(9.8\,\frac{\text{m}}{\text{s}^2}\right)\sin 30.0°}{1 + \frac{2}{5}} = -\frac{3.50\text{m}}{\text{s}^2}.$$

Then,

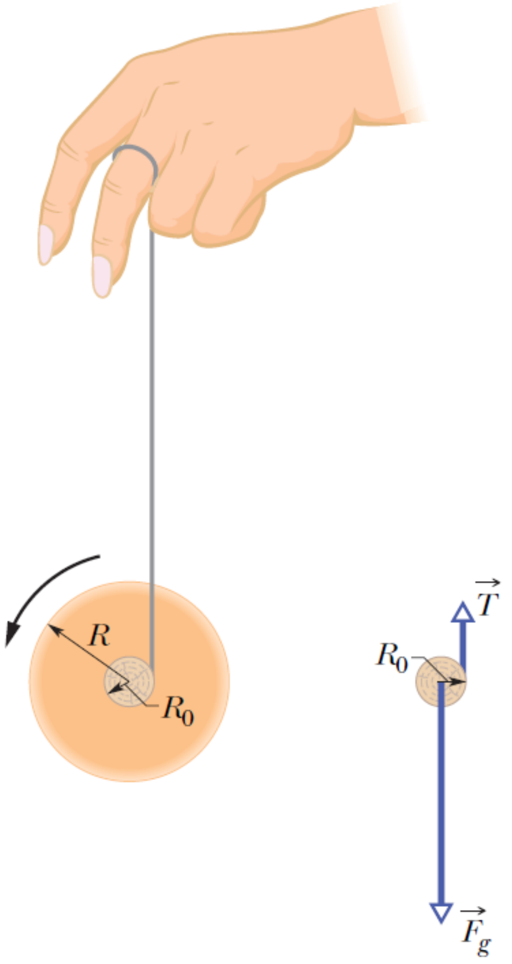
$$f_s = -\frac{I_{com}a_{\text{com},x}}{R^2} = -\frac{2}{5}Ma_{\text{com},x} = 8.40\text{ N}.$$



4. The Yo-Yo

- The Yo-Yo is similar to a rolling object down a ramp. The main differences between the two are:
  - (3) The yo-yo rolls down a string at an angle $\theta = 90°$.
  - (1) f_s is replaced by the tension $\vec{T}$.
  - (2) The yo-yo rolls on its axle of radius R_0.
- The downward acceleration of the yo-yo center of mass a_{com} is therefore

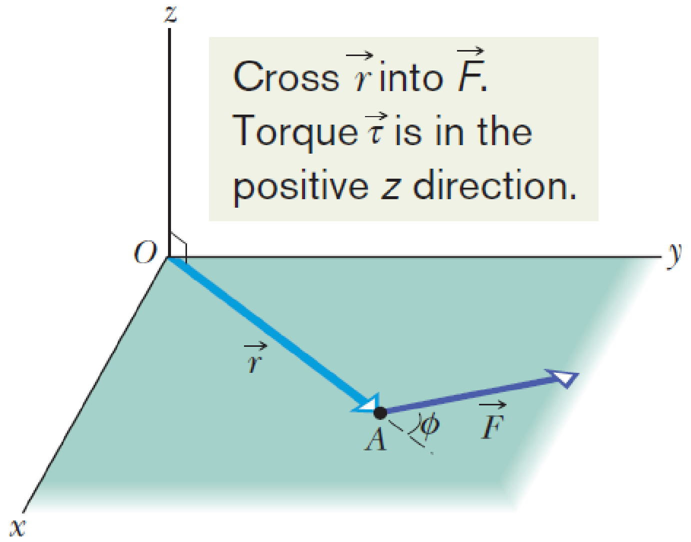
$$a_{\text{com}} = -\frac{g}{1 + \frac{I_{\text{com}}}{MR_0^2}}.$$



5. Torque Revisited

- Now we expand the definition of torque to apply it to an individual particle that moves along any path relative to a fixed point, rather than a fixed axis. We therefore write the torque as a vector $\vec{\tau}$ that can have any direction.
- Consider a particle at a point A in an xy plane, having the position vector $\vec{r}$ relative to the origin O. The torque $\vec{\tau}$ acting on the particle relative to the fixed point O is a vector quantity defined as

$$\vec{\tau} = \vec{r} \times \vec{F}.$$



5. Torque Revisited

- The direction of $\vec{\tau}$ is determined by the right had rule.
- The magnitude of $\vec{\tau}$ is

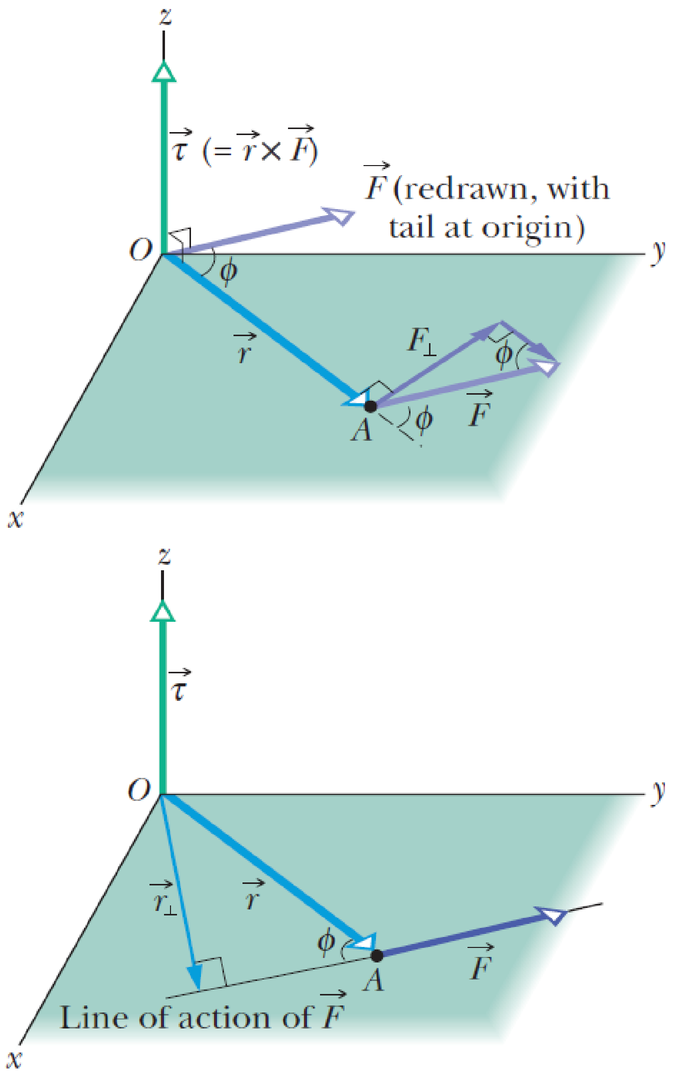
$$\tau = rF\sin\phi\,,$$

where ϕ is the smaller angle between $\vec{r}$ and $\vec{F}$.

We can also write

$$\tau = rF_{\perp} = r_{\perp}F,$$

where $F_{\perp}$ is the component of $\vec{F}$ perpendicular to $\vec{r}$ and $r_{\perp}$ is the moment arm of $\vec{F}$.



5. Torque Revisited

CHECKPOINT 3

The position vector $\vec{r}$ of a particle points along the positive direction of a z axis. If the torque on the particle is (a) zero, (b) in the negative direction of x, and (c) in the negative direction of y, in what direction is the force causing the torque?

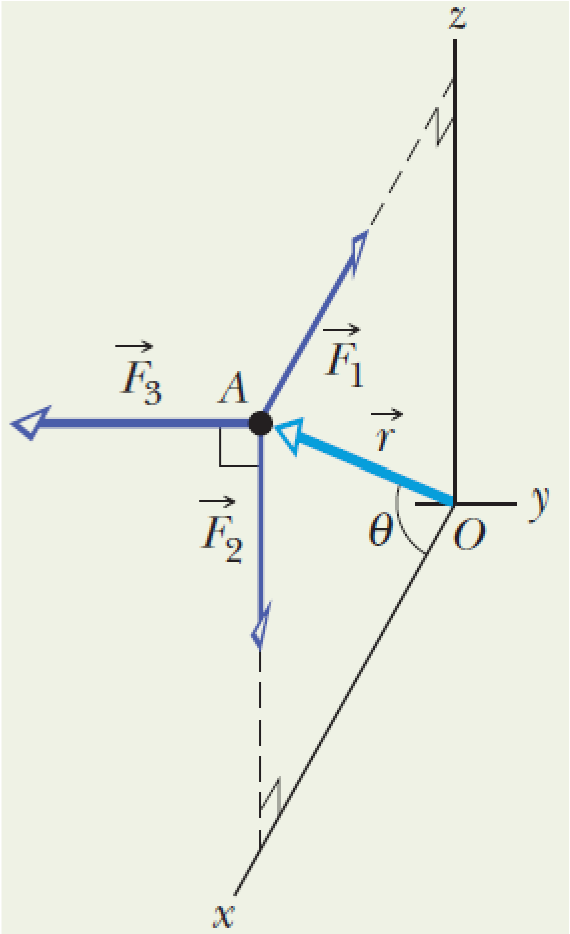
(a) $+\hat{k}$ or $-\hat{k}$.

(b) $+\hat{j}$.

(c) $-\hat{\imath}$.

5. Torque Revisited

Example 2: In the figure, three forces, each of magnitude 2.0 N, act on a particle. The particle is in the xz plane at point A given by position vector , where $r = 3.0$ m and $\theta = 30°$. Force $\vec{F}_1$ is parallel to the x axis, force $\vec{F}_2$ is parallel to the z axis, and force $\vec{F}_3$ is parallel to the y axis. What is the torque, about the origin O, due to each force?



The magnitudes of the torques are

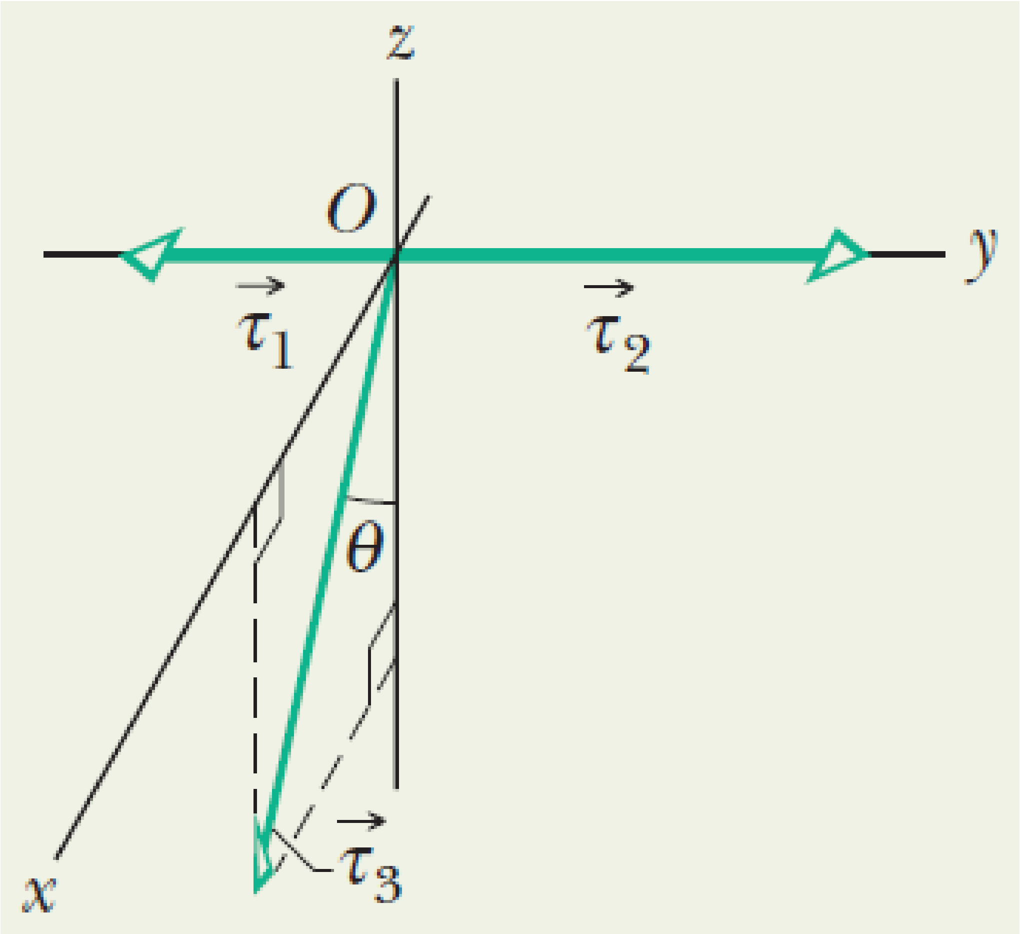
$$\tau_1 = rF_1 \sin\phi_1 = (3.0\text{ m})(2.0\text{ N})\sin 150° = 3.0\text{ N}\cdot\text{m}.$$

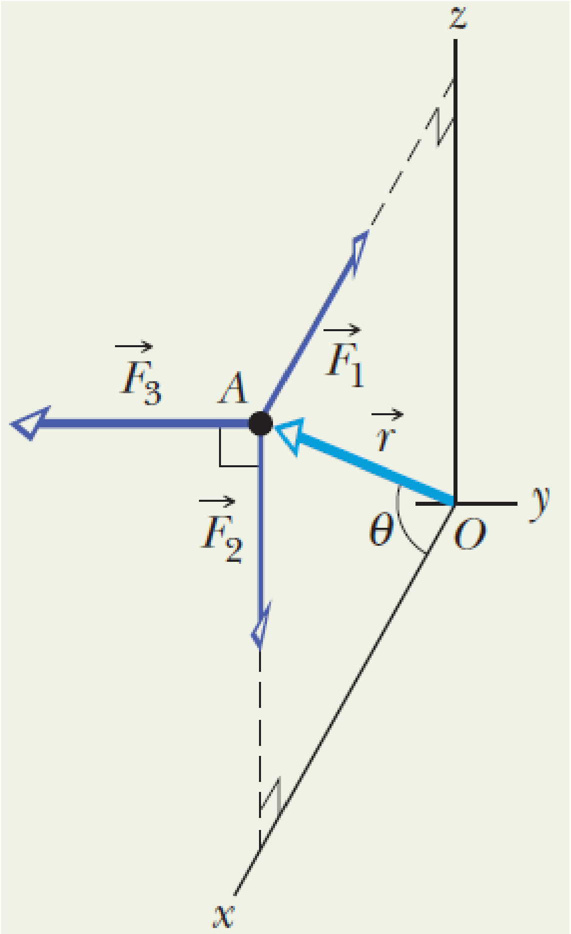
$$\tau_2 = rF_2 \sin\phi_2 = (3.0\text{ m})(2.0\text{ N})\sin 120° = 5.2\text{ N}\cdot\text{m}.$$

$$\tau_3 = rF_3 \sin\phi_3 = (3.0\text{ m})(2.0\text{ N})\sin 90° = 6.0\text{ N}\cdot\text{m}.$$

5. Torque Revisited

The directions of the torques are shown below:





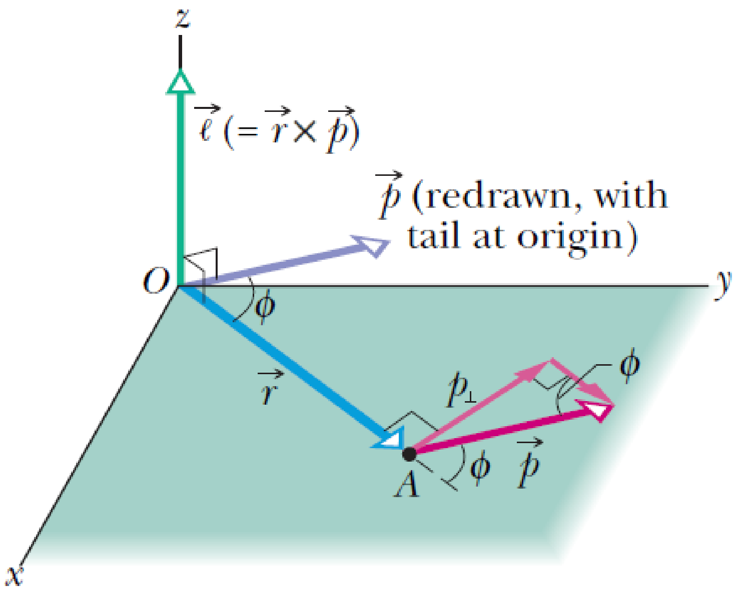
6. Angular Momentum

- We discuss now the angular counterpart of the linear momentum. The **angular momentum** $\vec{\ell}$ of a particle of mass m and velocity $\vec{v}$ (and $\vec{p} = m\vec{v}$) with respect to the origin O is a vector quantity defined as

$$\vec{\ell} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v}),$$

where $\vec{r}$ is the position vector of the particle with respect to O. As the particle moves, $\vec{r}$ rotates around O. Note that the particle does not have to rotate around O to have angular momentum about O.

The SI unit of angular momentum is $\mathrm{kg} \cdot \mathrm{m}^2/\mathrm{s}$, or $\mathrm{J} \cdot \mathrm{s}$.



6. Angular Momentum

- The direction of $\vec{\ell}$ is determined by the right hand rule.
- The magnitude of $\vec{\ell}$ is

$$\ell = rmv \sin\phi \,,$$

where ϕ is the smaller angle between $\vec{r}$ and $\vec{p}$.

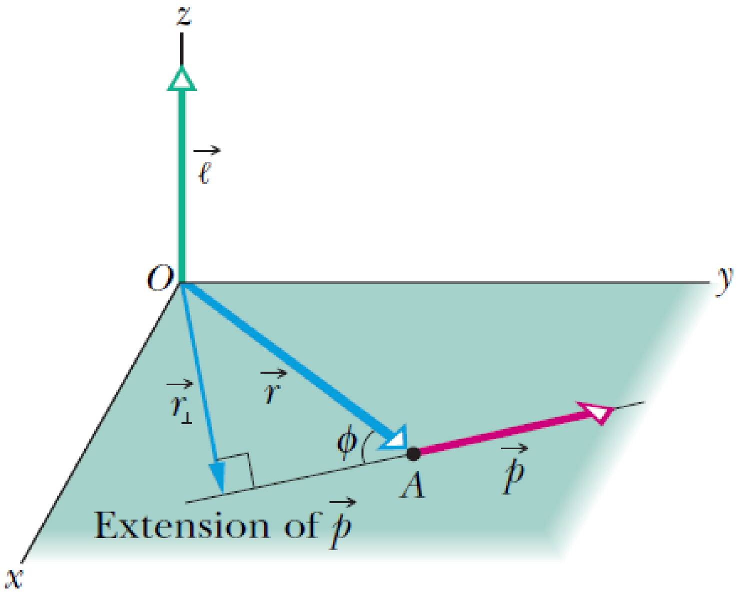
- We can rewrite ℓ as

$$\ell = rp_\perp = mrv_\perp,$$

where $p_\perp$ and $v_\perp$ are the components of $\vec{p}$ and $\vec{v}$ perpendicular to $\vec{r}$, respectively. We can also write

$$\ell = r_\perp p = r_\perp mv,$$

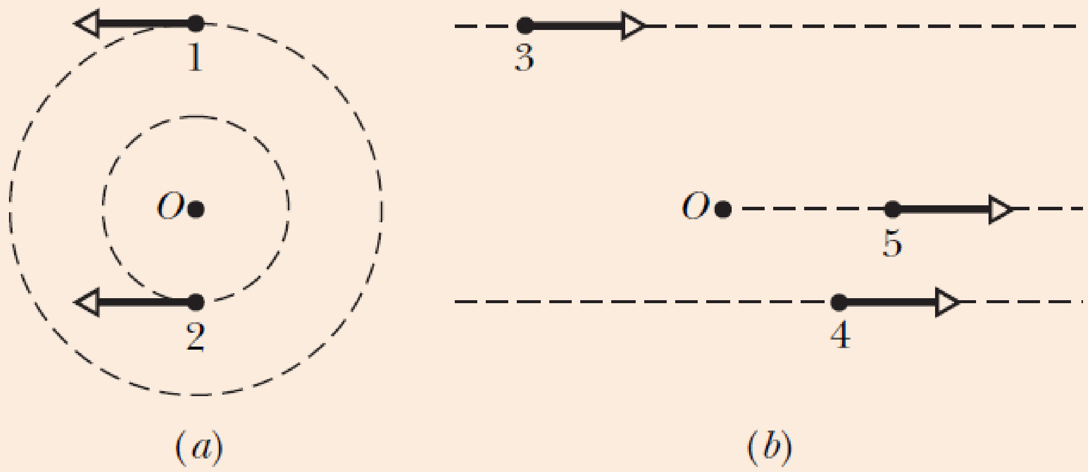
Where $r_\perp$ is the perpendicular distance between O and the extension of $\vec{p}$.



6. Angular Momentum

CHECKPOINT 4

In part *a* of the figure, particles 1 and 2 move around point *O* in circles with radii 2 m and 4 m. In part *b*, particles 3 and 4 travel along straight lines at perpendicular distances of 4 m and 2 m from point *O*. Particle 5 moves directly away from *O*.



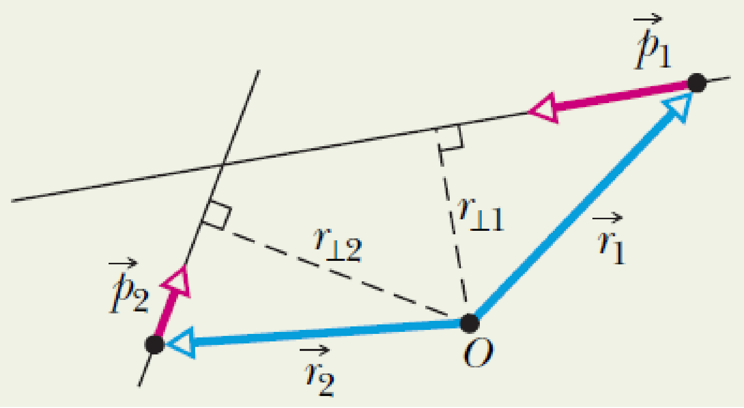
All five particles have the same mass and the same constant speed. (a) Rank the particles according to the magnitudes of their angular momentum about point *O*, greatest first. (b) Which particles have negative angular momentum about point *O*?

(a) 1 & 3 tie, 2 & 4 tie, 5.

(b) 2 and 3.

6. Angular Momentum

Example 3: The figure shows an overhead view of two particles moving at constant momentum along horizontal paths. Particle 1, with momentum magnitude $p_1 = 5.0 \text{ kg m/s}$, has position vector $\vec{r}_1$ and will pass 2.0 m from point O. Particle 2, with momentum magnitude $p_2 = 2.0 \text{ kg m/s}$, has position vector $\vec{r}_2$ and will pass 4.0 m from point O. What are the magnitude and direction of the net angular momentum about point O of the two particle system?



$$\ell_1 = r_{\perp 1} p_1 = (2.0 \text{ m})(5.0 \text{ kg} \cdot \text{m/s}) = 10 \text{ kg} \cdot \text{m}^2/\text{s}.$$

$$\ell_2 = -r_{\perp 2} p_2 = -(4.0 \text{ m})\left(2.0 \text{ kg} \cdot \frac{\text{m}}{\text{s}}\right) = -8.0 \text{ kg} \cdot \text{m}^2/\text{s}.$$

$$L = \ell_1 + \ell_2 = 2.0 \text{ kg} \cdot \text{m}^2/\text{s}.$$

7. Newton's Second Law in Angular Form

- The angular version of Newton's second law is

$$\vec{\tau}_{net} = \frac{d\vec{\ell}}{dt}.$$

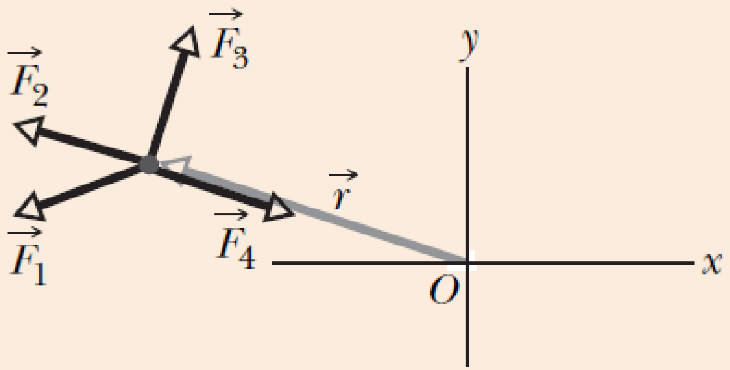
  See your text for the proof.

- In words, the vector sum of all torques acting on a particle is equal to the time rate of change of angular momentum of that particle.
- Both $\vec{\tau}$ and $\vec{\ell}$ here are defined with respect to the same point, usually the origin.

7. Newton's Second Law in Angular Form

CHECKPOINT 5

The figure shows the position vector $\vec{r}$ of a particle at a certain instant, and four choices for the direction of a force that is to accelerate the particle. All four choices lie in the xy plane. (a) Rank the choices according to the magnitude of the time rate of change ($d\vec{\ell}/dt$) they produce in the angular momentum of the particle about point O, greatest first. (b) Which choice results in a negative rate of change about O?



(a) 3, 1, 2 & 4 tie. (b) 3.

7. Newton's Second Law in Angular Form

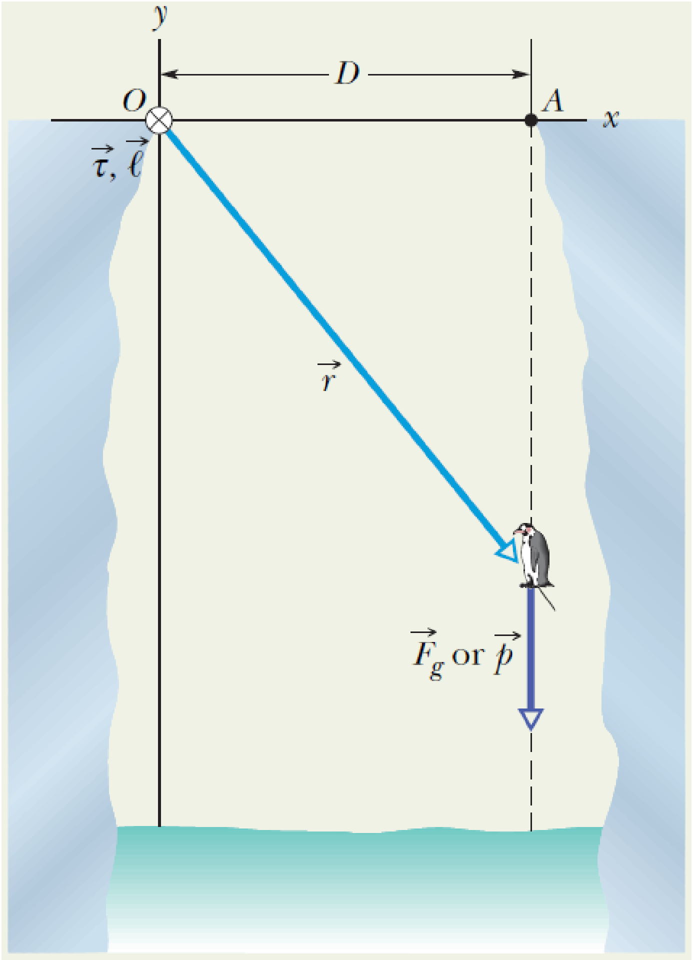
Example 4: In the figure, a penguin of mass m falls from rest at point A, a horizontal distance D from the origin O of an xyz coordinate system. (The positive direction of the z axis is directly outward from the plane of the figure.)

(a) What is the angular momentum $\vec{\ell}$ of the falling penguin about O?

ℓ is given by

$$\ell = r_{\perp}mv = Dmgt.$$

The direction of $\vec{\ell}$ is into the page , by the right hand rule.



7. Newton's Second Law in Angular Form

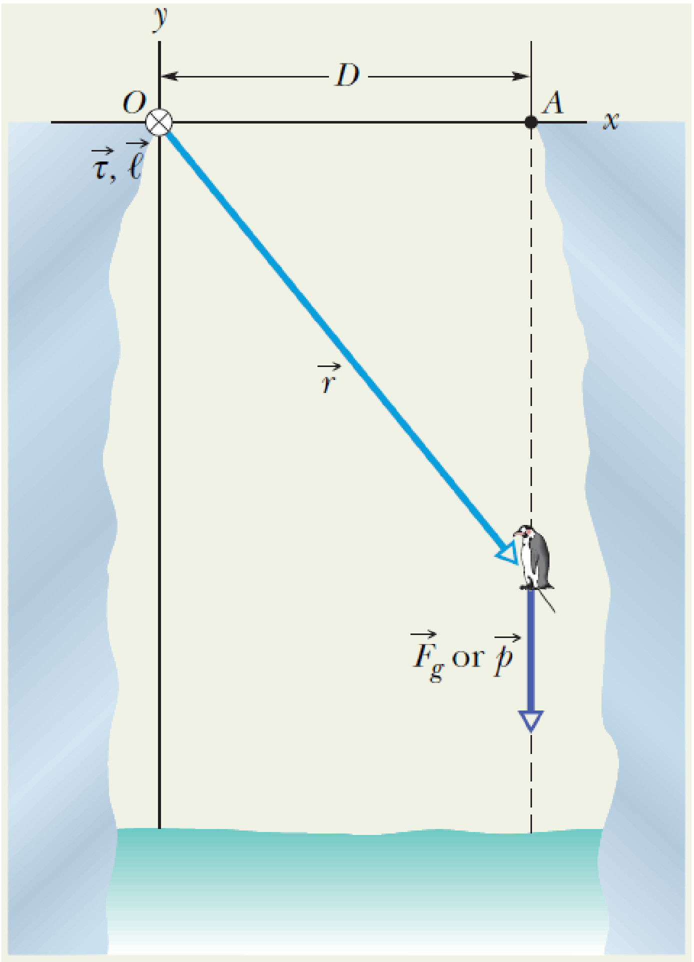
(b) About the origin O, what is the torque $\vec{\tau}$ on the penguin due to the gravitational force $\vec{F}_g$?

τ is given by

$$\tau = r_{\perp} F_g = Dmg.$$

The direction of $\vec{\tau}$ is into the page , by the right hand rule. Additionally, we can find τ by

$$\tau = \frac{d\ell}{dt} = \frac{d(Dmgt)}{dt} = Dmg.$$



8. The angular Momentum of a System of Particles

- The total angular momentum $\vec{L}$ of a system of particles is the vector sum of the angular momenta $\vec{\ell}$ of the individual particles:

$$\vec{L} = \vec{\ell}_1 + \vec{\ell}_1 + \cdots + \vec{\ell}_n = \sum_{i=1}^{n} \vec{\ell}_i \,.$$

- $\vec{L}$ can change with time due to the changes of the angular momenta $\vec{\ell}_i$ of individual particles:

$$\frac{d\vec{L}}{dt} = \sum_{i=1}^{n} \frac{d\vec{\ell}_i}{dt} = \sum_{i=1}^{n} \vec{\tau}_{\mathrm{net},i} = \vec{\tau}_{\mathrm{net}},$$

where $\vec{\tau}_{\mathrm{net},i}$ is the net torque on the ith particle and $\vec{\tau}_{\mathrm{net}}$ is the net external torque.

8. The angular Momentum of a System of Particles

- Newton's second law in angular form for a system of particles:

$$\vec{\tau}_{\mathrm{net}} = \frac{d\vec{L}}{dt},$$

  In words, then net external torque acting on a system of particles is equal to the time rate of change of the system's total angular momentum $\vec{L}$.

- Remember that torques and angular momenta here are measured relative to the same origin.

9. The angular Momentum of a Rigid Rotating Body about a Fixed Axis

- Consider a rigid body that rotates about the z axis with constant angular speed ω. The angular momentum

$$L = I\omega,$$

where I is the moment of inertia of the body.

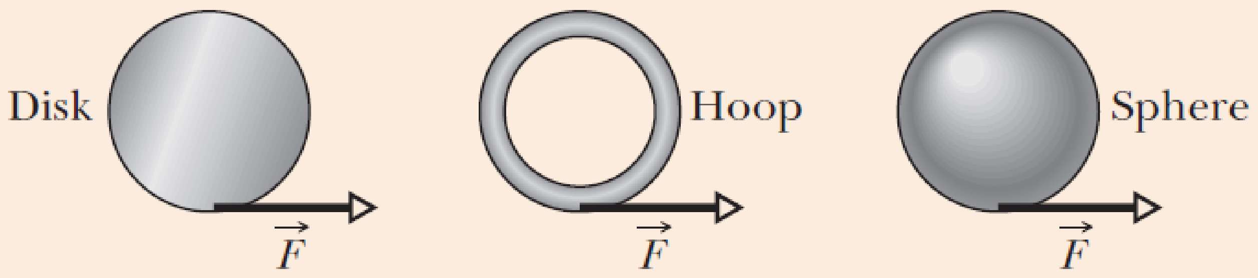
9. The angular Momentum of a Rigid Rotating Body about a Fixed Axis

More Corresponding Variables and Relations for Translational and Rotational Motion[a]

Translational		Rotational	
Force	$\vec{F}$	Torque	$\vec{\tau}\ (= \vec{r} \times \vec{F})$
Linear momentum	$\vec{p}$	Angular momentum	$\vec{\ell}\ (= \vec{r} \times \vec{p})$
Linear momentum[b]	$\vec{P}\ (= \Sigma \vec{p}_i)$	Angular momentum[b]	$\vec{L}\ (= \Sigma \vec{\ell}_i)$
Linear momentum[b]	$\vec{P} = M\vec{v}_{\text{com}}$	Angular momentum[c]	$L = I\omega$
Newton's second law[b]	$\vec{F}_{\text{net}} = \dfrac{d\vec{P}}{dt}$	Newton's second law[b]	$\vec{\tau}_{\text{net}} = \dfrac{d\vec{L}}{dt}$
Conservation law[d]	$\vec{P} =$ a constant	Conservation law[d]	$\vec{L} =$ a constant

9. The angular Momentum of a Rigid Rotating Body about a Fixed Axis

CHECKPOINT 6



In the figure, a disk, a hoop, and a solid sphere are made to spin about fixed central axes (like a top) by means of strings wrapped around them, with the strings producing the same constant tangential force $\vec{F}$ on all three objects. The three objects have the same mass and radius, and they are initially stationary. Rank the objects according to (a) their angular momentum about their central axes and (b) their angular speed, greatest first, when the strings have been pulled for a certain time t.

(a) All tie.

(b) Sphere, Disk, Hoop.

10. Conservation of Angular Momentum

- We now consider a third law of conservation, the conservation of annual momentum. If no external net torque acts on a system ($\vec{\tau}_{\mathrm{net}} = 0$), then $d\vec{L}/\mathrm{dt} = 0$, or

$$\vec{L} = \text{a constant.}$$

- This is the **law of conservation of angular momentum**. In words,

$$\begin{pmatrix}\text{net angular momentum} \\ \text{at some inital time } t_{\mathrm{i}}\end{pmatrix} = \begin{pmatrix}\text{net angular momentum} \\ \text{at some inital time } t_{\mathrm{f}}\end{pmatrix},$$

or

$$\vec{L}_{\mathrm{i}} = \vec{L}_{\mathrm{f}}.$$

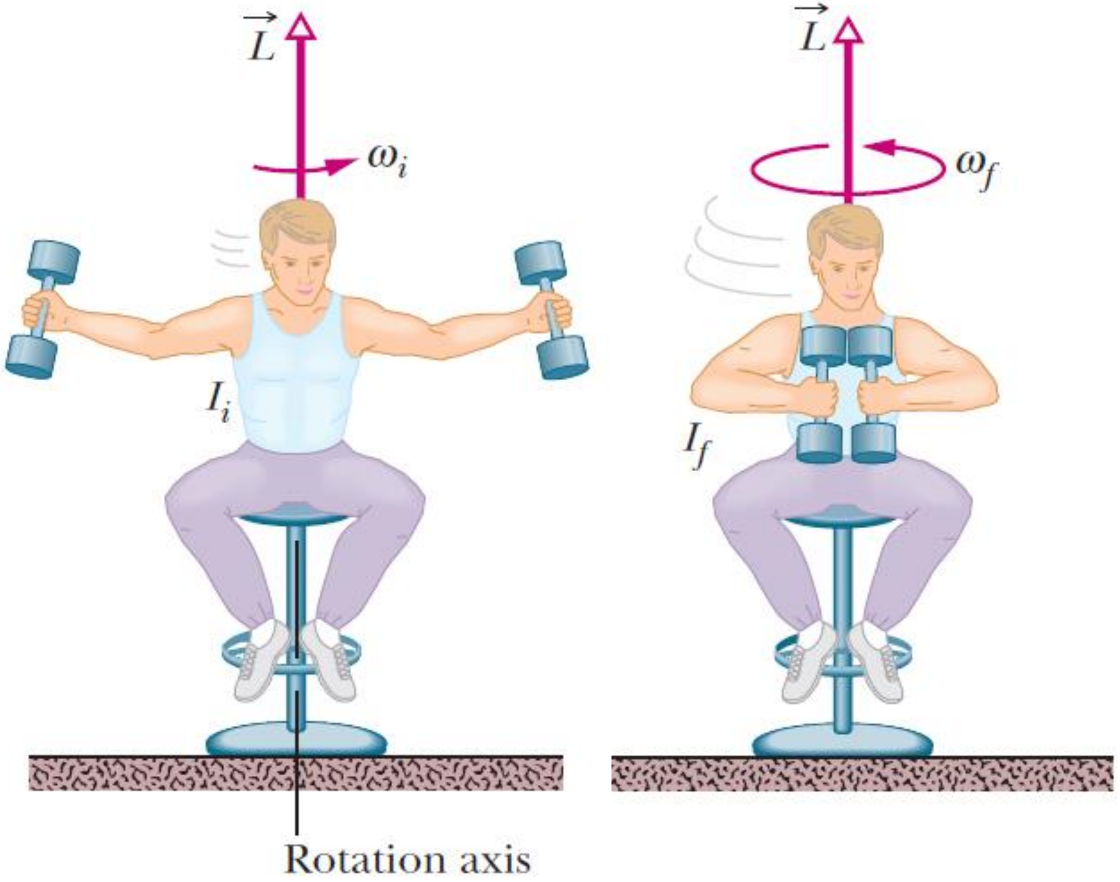
10. Conservation of Angular Momentum

- These two equations are vector equation; they have three components corresponding to the conservation of angular momentum in three mutually perpendicular directions. If the component of $\vec{\tau}_{\mathrm{net}}$ is zero along a certain axis, then the component of the angular momentum of the system along that axis cannot change.
- If an isolated body that rotates about the z axis changes its rotational inertia about tat axis by redistributing its mass then the angular speed of the body will change so that its angular momentum stays the same:

$$I_{\mathrm{i}}\omega_{\mathrm{i}} = I_{\mathrm{f}}\omega_{\mathrm{f}}.$$

10. Conservation of Angular Momentum

- Consider the demonstration shown in figure. The stool rotates at angular speed ω_i while the student's arms are outstretched. As he pulls his arms, decreasing the value of his moment of inertia from I_i to I_f, his rotational speed increases from ω_i to ω_f so that $I_\mathrm{i}\omega_\mathrm{i} = I_\mathrm{f}\omega_\mathrm{f}$.



10. Conservation of Angular Momentum

CHECKPOINT 7

A rhinoceros beetle rides the rim of a small disk that rotates like a merry-go-round. If the beetle crawls toward the center of the disk, do the following (each relative to the central axis) increase, decrease, or remain the same for the beetle–disk system: (a) rotational inertia, (b) angular momentum, and (c) angular speed?

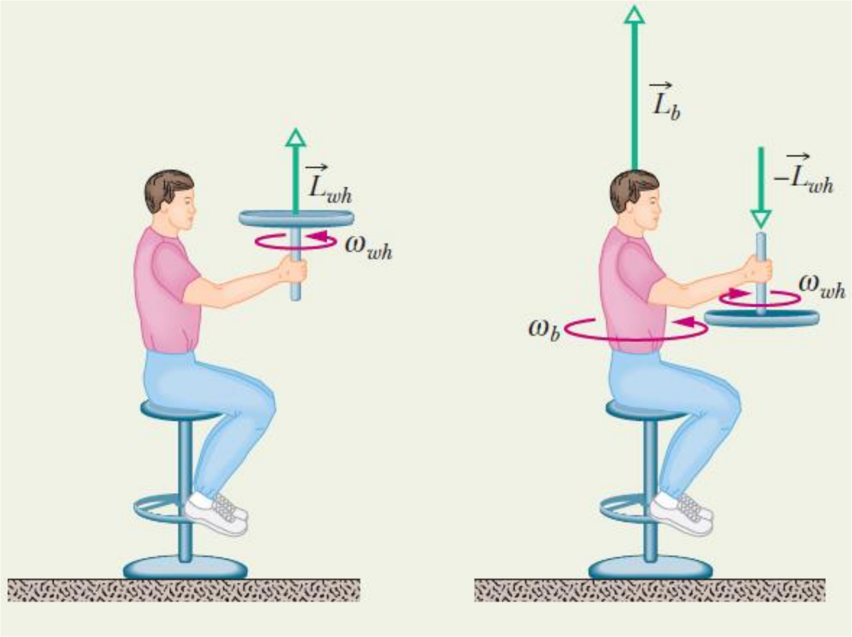
(a) Decreases.

(b) Remains the same.

(c) Increases.

10. Conservation of Angular Momentum

Example 5: The figure shows a student, sitting on a stool that can rotate freely about a vertical axis. The student, initially at rest, is holding a bicycle wheel whose rotational inertia I_{wh} about its central axis is $1.2\ \mathrm{kg}\cdot\mathrm{m}^2$. The wheel is rotating at an angular counterclockwise speed ω_{wh} of $3.9\ \mathrm{rev}/s$; as seen from overhead. The axis of the wheel is vertical, and the angular momentum of the wheel points vertically upward. The student now inverts the wheel so that it is rotating clockwise. Its angular momentum is now $-\vec{L}_{wh}$. The inversion results in the student, the stool, and the wheel's center rotating together as a composite rigid body about the stool's rotation axis, with rotational inertia $I_b = 6.8\ \mathrm{kg}\cdot\mathrm{m}^2$. With what angular speed ω_b and in what direction does the composite body rotate after the inversion of the wheel?



10. Conservation of Angular Momentum

The total conservation of the system is conserved:

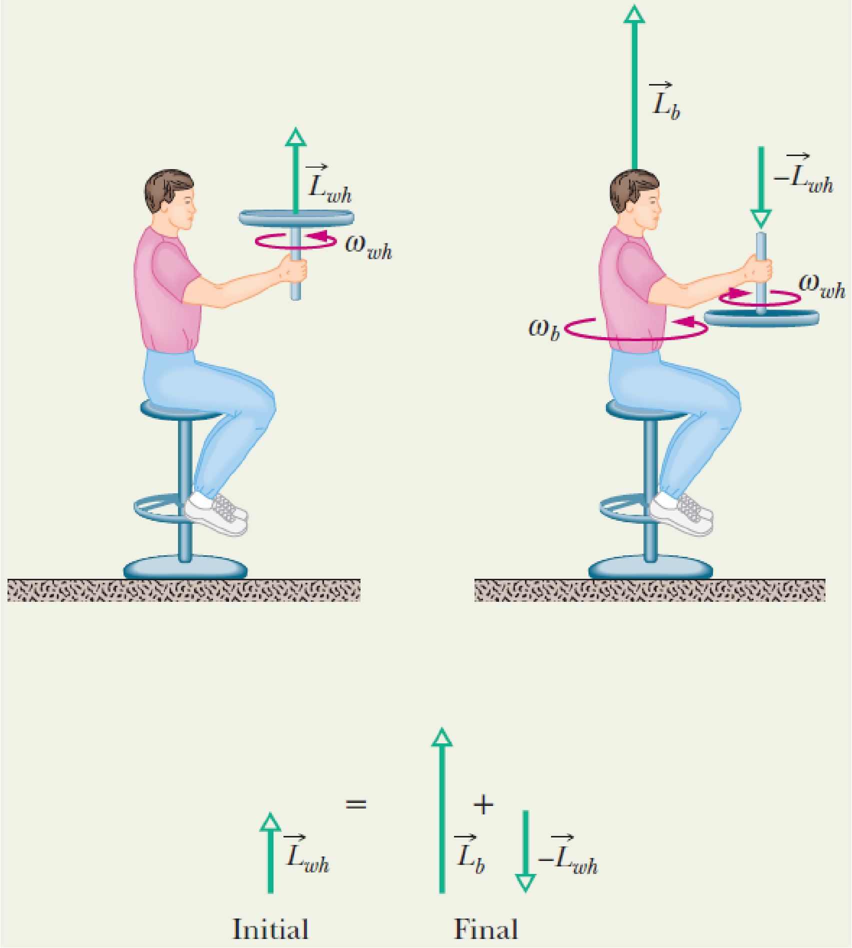
$$L_{b,i} + L_{wh,i} = L_{b,f} + L_{wh,f}.$$

or

$$\begin{aligned} L_{b,f} &= L_{b,i} + L_{wh,i} - L_{wh,f} \\ &= 0 + L_{wh,i} - \left(-L_{wh,i}\right) = 2L_{wh,i}. \end{aligned}$$

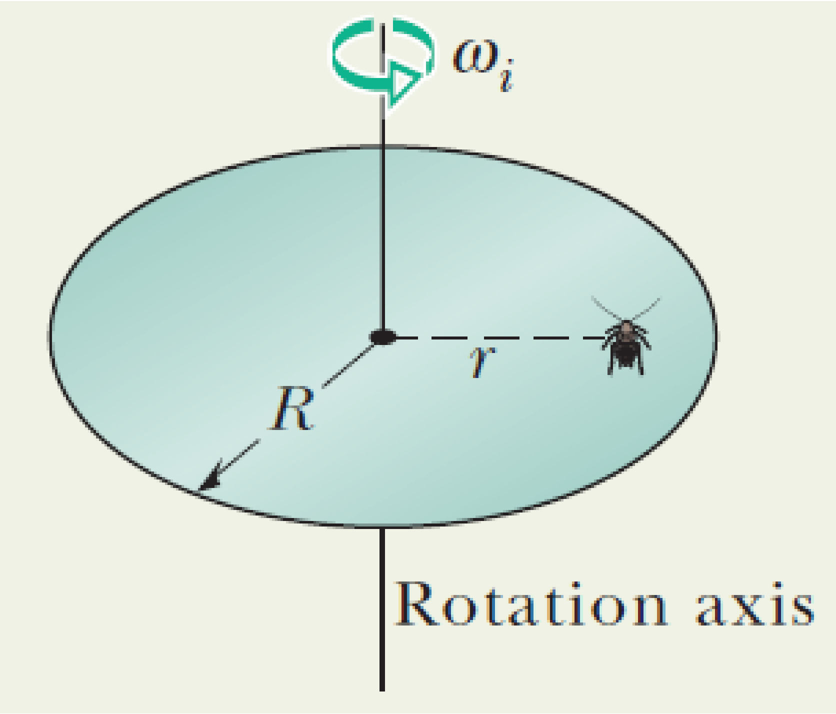
Using that $L = I\omega$ and solving for ω_b and substituting we find

$$\begin{aligned} \omega_b &= \frac{2I_{wh}}{I_b}\omega_{wh} \\ &= \frac{2(1.2\ \mathrm{kg}\cdot\mathrm{m}^2)}{6.8\ \mathrm{kg}\cdot\mathrm{m}^2}\left(3.9\ \frac{\mathrm{rev}}{\mathrm{s}}\right) = 1.4\ \frac{\mathrm{rev}}{\mathrm{s}}. \end{aligned}$$



10. Conservation of Angular Momentum

Example 6: In the figure, a bug with mass m rides on a disk of mass $M = 6.00m$ and radius R. The disk rotates like a merry-go-round around its central axis at angular speed $\omega_i = 1.50 \text{ rad/s}$. The bug is initially at radius $r = 0.80\,R$, but then it crawls out to the rim of the disk. Treat the bug as a particle. What then is the angular speed?



Because the angular momentum of the bug-disk system is conserved, we can write

$$I_\mathrm{i}\omega_\mathrm{i} = I_\mathrm{f}\omega_\mathrm{f}.$$

$$I_\mathrm{i} = I_\mathrm{d} + I_\mathrm{b,i} = \frac{1}{2}MR^2 + mr^2.$$

10. Conservation of Angular Momentum

$$I_{\mathrm{f}} = I_{\mathrm{d}} + I_{\mathrm{b},f} = \frac{1}{2}MR^2 + mR^2.$$

Solving for ω_{f} and substituting yield

$$\omega_{\mathrm{f}} = \frac{I_{\mathrm{i}}}{I_{\mathrm{f}}}\omega_{\mathrm{i}} = \frac{1/2\,MR^2 + mr^2}{1/2\,MR^2 + mR^2}\omega_{\mathrm{i}}$$

$$= \frac{1/2\,(6.00m)R^2 + m(0.80R)^2}{1/2\,(6.00m)R^2 + mR^2}\left(1.50\,\frac{\mathrm{rad}}{\mathrm{s}}\right)$$

$$= 1.36\,\frac{\mathrm{rad}}{\mathrm{s}}.$$