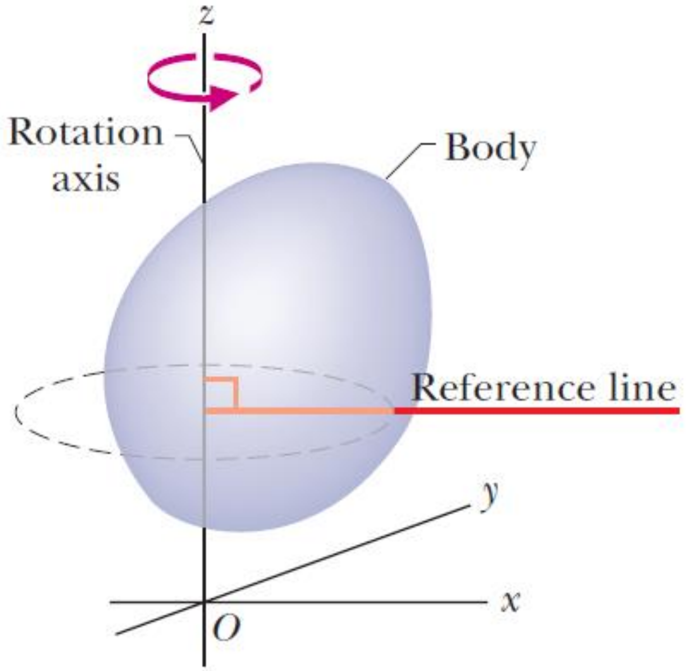
Chapter 10

Rotation

1. The Rotational Variables

- In the previous chapters we discussed only translational motion. We now discuss another type of motion: rotation.
- We examine the rotation of a rigid body about a fixed axis. A **rigid body** is a body that can rotate with all its parts locked together and without change in its shape. A **fixed axis** means that the rotation occurs about an axis that does not move.
- The figure shows a rigid body rotating about a fixed axis, called the **axis of rotation** or **rotational axis**. Every point of the body moves in a circle whose center lies on the axis or rotation, and every point moves through the same angle during a particular time interval.



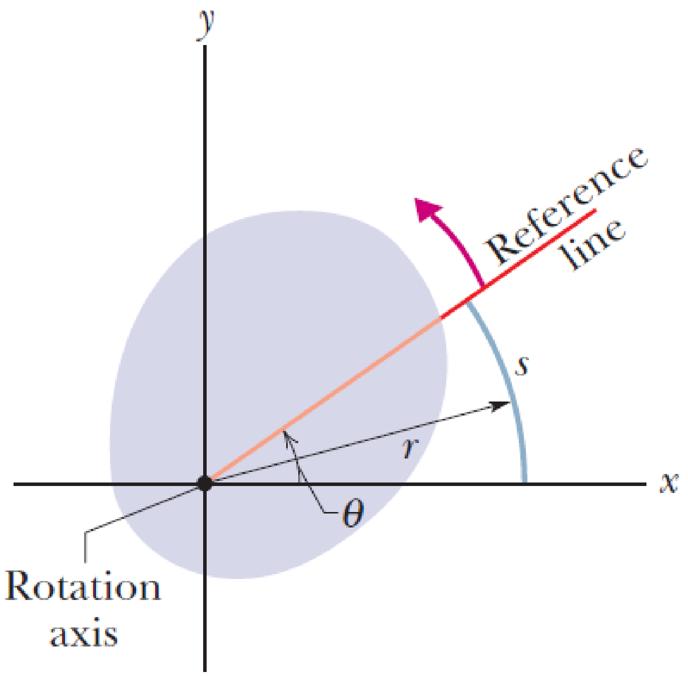
1. The Rotational Variables

- We deal now with the angular equivalence of the linear position, displacement, velocity and acceleration.
- <u>Angular Position</u>:

  The figure shows a *reference line*, fixed in the body, perpendicular to the rotation axis and rotating with the body. The **angular position** of this line is the angle it makes with a fixed direction, which we take as the **zero angular position**.

  In the figure, the angular position θ is measured relative to the positive direction of the x axis. From geometry,

$$\theta = \frac{s}{r}.$$



1. The Rotational Variables

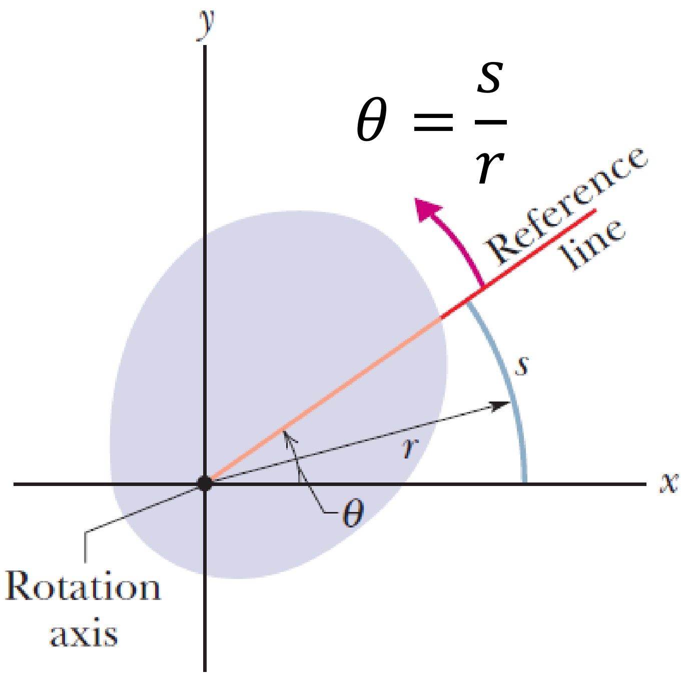
- <u>Angular Position</u>:

  Here s is the length of a circular arc that extends from the x axis to the reference line, and r is the radius of the circle.

  θ is in radians (dimensionless). If the reference line makes more that one revolution then $\theta > 2\pi$.

  We can know everything about the rotation of an object if we know $\theta(t)$, the angular position of the body's reference line as a function of time.

$$\theta = \frac{s}{r}$$



1. The Rotational Variables

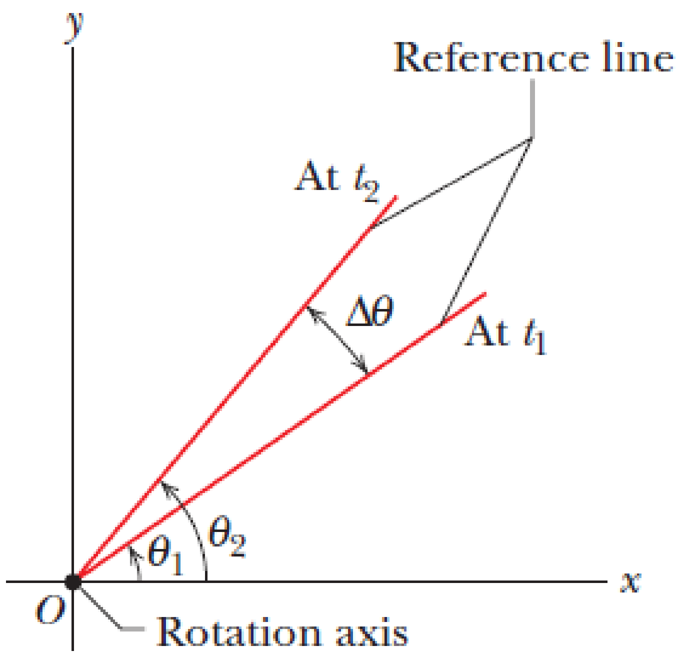
- <u>Angular Displacement</u>:

  If a body rotates about the rotation axis, changing the angular position of the reference line from θ_1 to θ_2, the body undergoes an **angular displacement** $\Delta\theta$ given by

  $$\Delta\theta = \theta_2 - \theta_1.$$

  In addition to the rigid body, this definition holds for every particle within it.

  Just like translational displacement, $\Delta\theta$ is either positive or negative, according to the following rule: Angular displacement in the counterclockwise direction is positive, and in the clockwise direction is negative.



1. The Rotational Variables

CHECKPOINT 1

A disk can rotate about its central axis like a merry-go-round. Which of the following pairs of values for its initial and final angular positions, respectively, give a negative angular displacement: (a) −3 rad, +5 rad, (b) −3 rad, −7 rad, (c) 7 rad, −3 rad?

(a) $\Delta\theta = 5\text{ rad} - (-3\text{ rad}) = 8\text{ rad}.$

(b) $\Delta\theta = -7\text{ rad} - (-3\text{ rad}) = -4\text{ rad}.$

(c) $\Delta\theta = -3\text{ rad} - 7\text{ rad} = -10\text{ rad}.$

(b) and (c) correspond to negative angular displacements.

1. The Rotational Variables

- Angular Velocity:

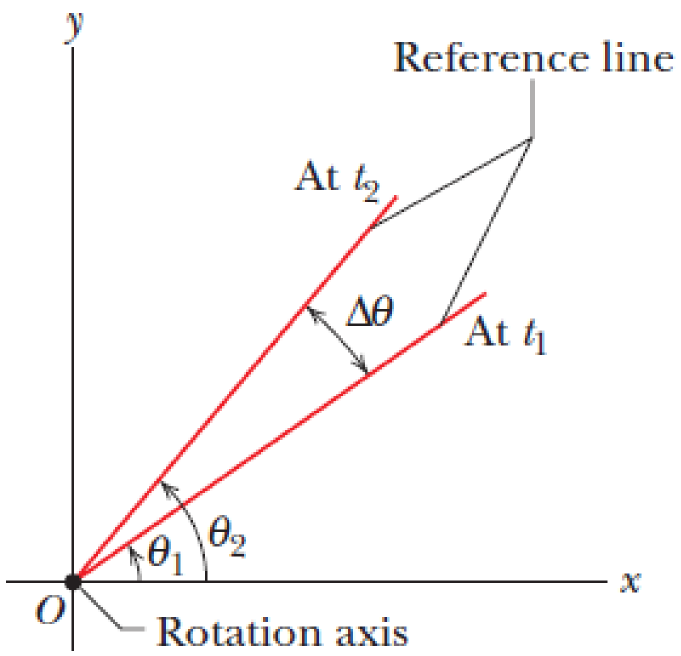
  Suppose the rotating body is at angular position θ_1 at time t_1 and at angular position θ_2 at time t_2. The **average angular velocity** of the of the body is defined as

$$\omega_{avg} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}.$$

  The **instantaneous angular velocity** ω is defined as

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}.$$

  The commonly used unit of angular velocity is radian per second (rad/s) or the revolution per second (rev/s) .



1. The Rotational Variables

- Angular Velocity:

  Another common unit for angular velocity is the revolution per minute (rpm).

  The angular velocity ω is positive if the body is rotating counterclockwise and negative if the body is rotating clockwise. The magnitude of an angular velocity is called the **angular speed**.

- Angular Acceleration:

  Let ω_2 and ω_1 be the angular velocities of a body at times t_2 and t_1. The **average angular acceleration** α_{avg} of the body is defied as

$$\alpha_{avg} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}.$$

1. The Rotational Variables

- <u>Angular Acceleration</u>:

  The (**instantaneous**) **angular acceleration** α is defined as

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}.$$

  Angular acceleration is commonly measured in radians per second-squared (rad/s^2) or revolution per second squared (rev/s^2).

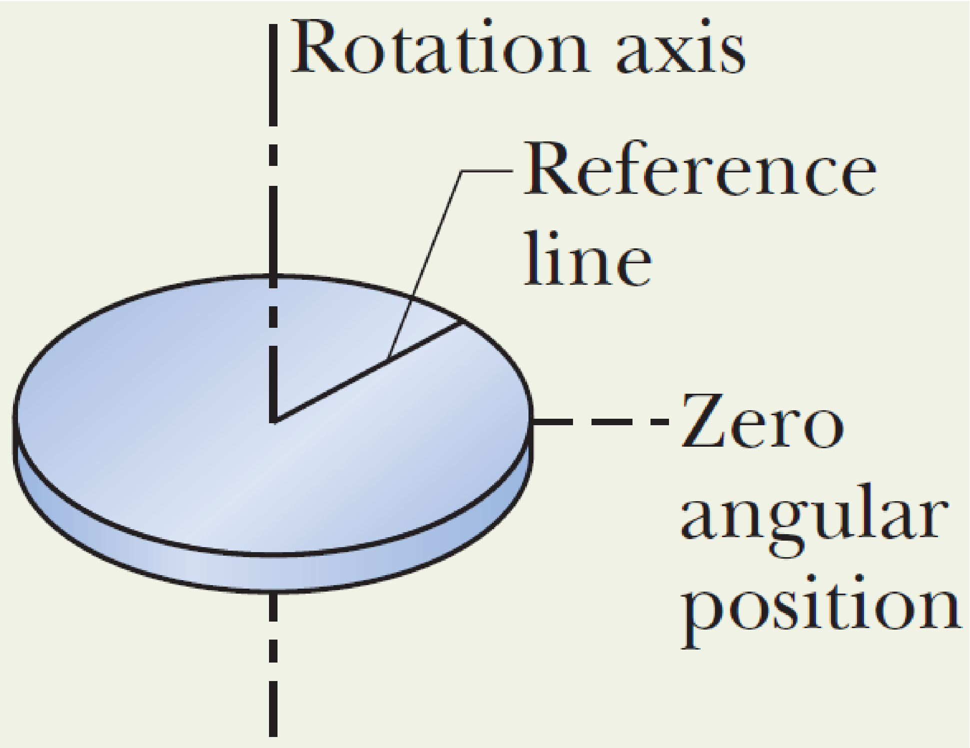
1. The Center of Mass

Example 1: The disk in the figure is rotating about its central axis like a merry-go-round. The angular position $\theta(t)$ of a reference line on the disk is given by

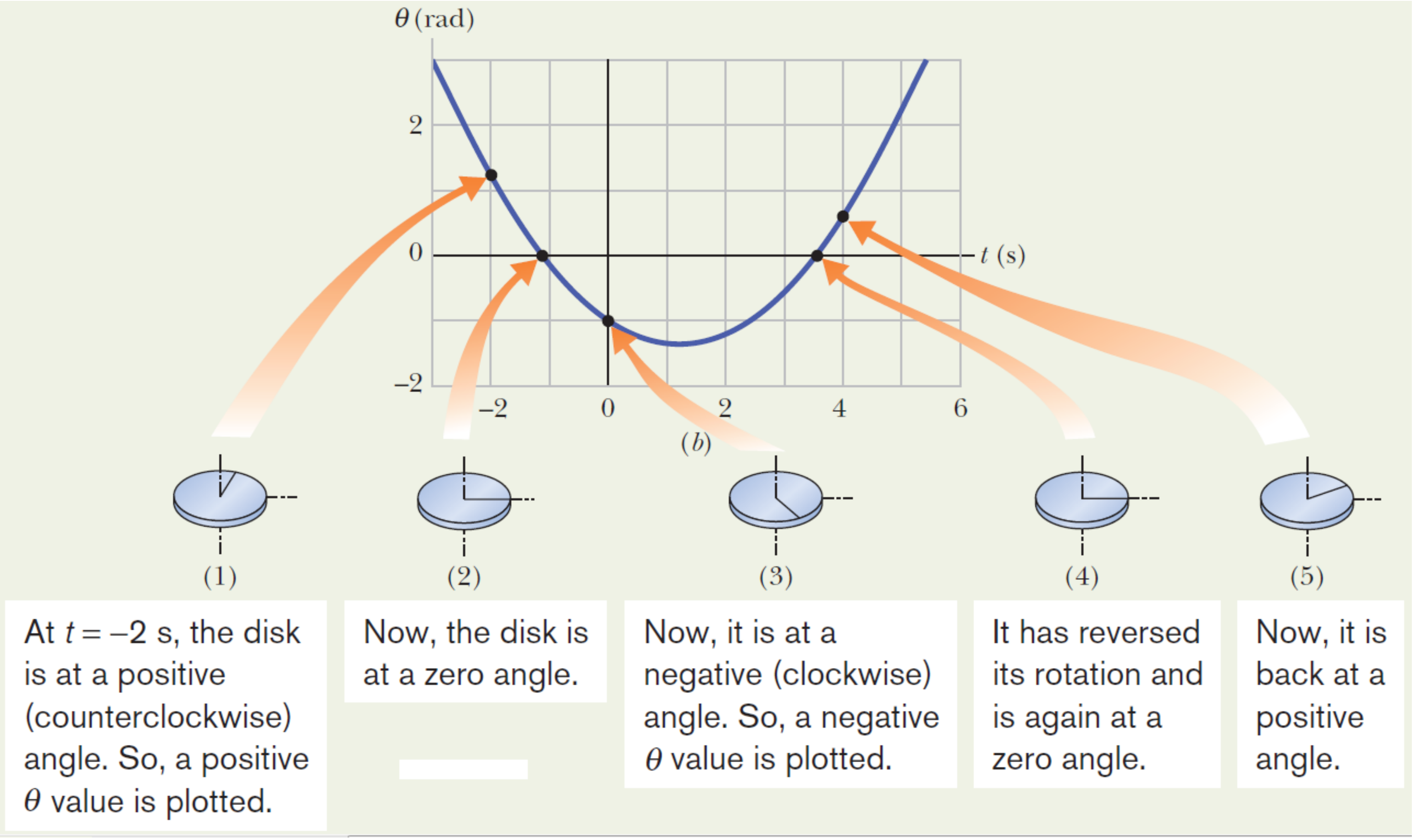
$$\theta(t) = -1.00 - 0.600t + 0.25t^2,$$

with t in seconds, θ in radians, and the zero angular positions indicated in the figure.

(a) Graph the angular position of the disk versus time from $t = -3.0$ s to $t = 5.4$ s. Sketch the disk and its angular position reference line at $t = -2.0$ s , 0 s, and 4.0 s, and when the curve crosses the t axis.



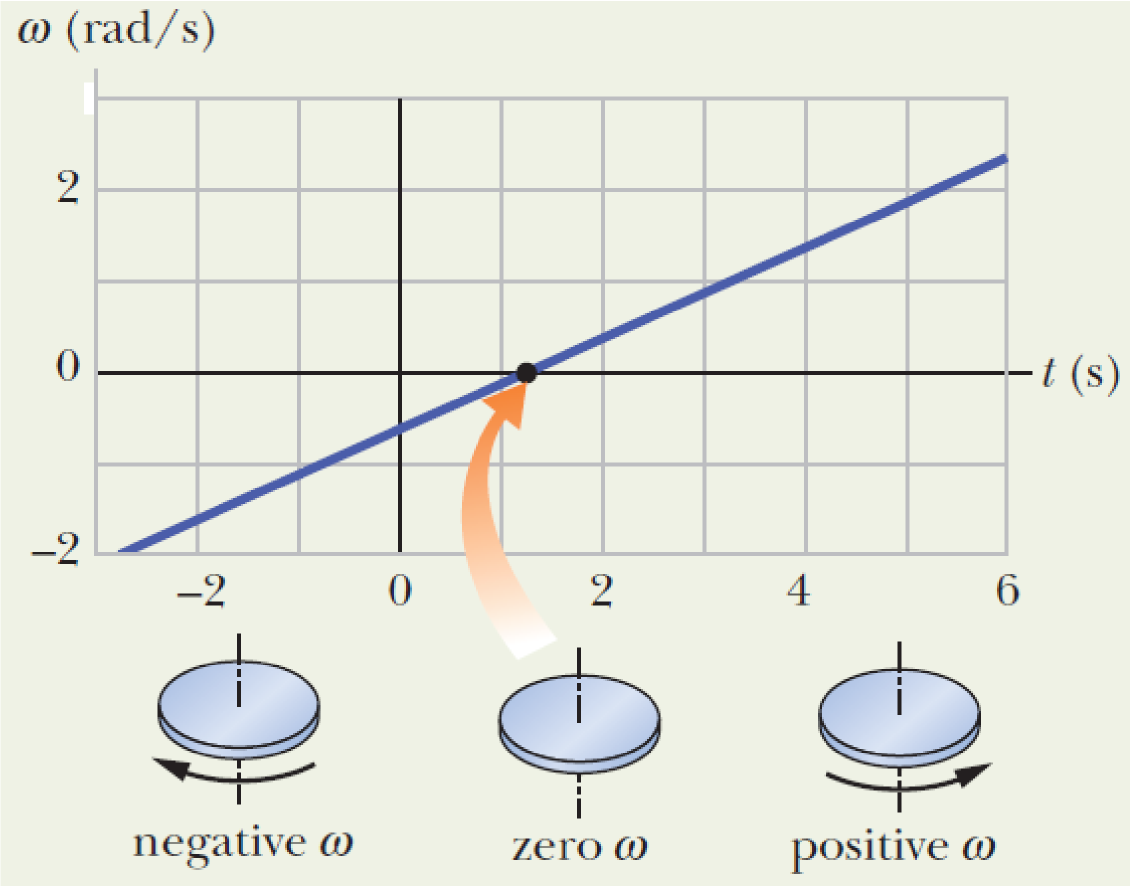
1. The Center of Mass



1. The Center of Mass

(b) Graph the angular velocity ω of the disk versus time from $t = -3.0$ s to $t = 6.0$ s. Sketch the disk and indicate the direction of turning and the sign of ω at $t = -2.0$ s, 4.0 s

$$\omega = \frac{d\theta}{dt} = -0.600 + 0.50t.$$



1. The Center of Mass

Example 2: The angular acceleration of an object is

$$\alpha = 5t^3 - 4t.$$

with t in seconds and a in radians per second-squared. At $t = 0$, the top has angular velocity 5 rad/s, and a reference line on it is at angular position $\theta = 2$ rad.

(a) Obtain an expression for the angular velocity $\omega(t)$ of the object.

$$d\omega = \alpha\, dt,$$

and therefore,

$$\int d\omega = \int \alpha\, dt\,,$$

or
$$\omega = \int (5t^3 - 4t)\, dt = \frac{5}{4}t^4 - 2t^2 + C.$$

1. The Center of Mass

To find C we use that $\omega(0) = 5 \text{ rad/s}$, or

or $$\omega(0) = \frac{5}{4}(0)^4 - 2(0)^2 + C = 5 \text{ rad/s}.$$

Therefore, $C = 5 \text{ rad/s}$ and

$$\omega = \frac{5}{4}t^4 - 2t^2 + 5.$$

(b) Obtain an expression for the angular position $\theta(t)$ of the object.

$$d\theta = \omega dt,$$

and therefore,

$$\int d\theta = \int \omega \, dt\,,$$

1. The Center of Mass

or

$$\theta = \int \left(\frac{5}{4}t^4 + 2t^2 + 5\right) dt = \frac{1}{4}t^5 - \frac{2}{3}t^3 + 5t + C'.$$

To find C' we use that $\theta(0) = 2 \text{ rad}$, or

or $$\theta(0) = \frac{1}{4}(0)^5 - \frac{2}{3}(0)^3 + 5(0) + C' = 2 \text{ rad/s}.$$

Therefore, $C' = 2 \text{ rad/s}$ and

$$\theta = \frac{1}{4}t^5 - \frac{2}{3}t^3 + 5t + 2.$$

2. Rotation with Constant Angular Acceleration

Table 10-1

Equations of Motion for Constant Linear Acceleration and for Constant Angular Acceleration

Linear Equation	Missing Variable		Angular Equation
$v = v_0 + at$	$x - x_0$	$\theta - \theta_0$	$\omega = \omega_0 + \alpha t$
$x - x_0 = v_0 t + \frac{1}{2}at^2$	v	ω	$\theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = v_0^2 + 2a(x - x_0)$	t	t	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$
$x - x_0 = \frac{1}{2}(v_0 + v)t$	a	α	$\theta - \theta_0 = \frac{1}{2}(\omega_0 + \omega)t$
$x - x_0 = vt - \frac{1}{2}at^2$	v_0	ω_0	$\theta - \theta_0 = \omega t - \frac{1}{2}\alpha t^2$

2. Rotation with Constant Angular Acceleration

Example 3: A disk rotates at constant angular acceleration $\alpha = 0.35\,\mathrm{rad}/s^2$. At time $t = 0$, it has an angular velocity of $\omega_0 = -0.46\,\mathrm{rad}/s$ and a reference line on it at the angular position $\theta_0 = 0$.

(a) At what time after $t = 0$ is the reference line at the angular position $\theta = 5.0\,\mathrm{rev}$?

We choose the relation

$$\theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2.$$

It then becomes

$$5.0\,(2\pi)\,\mathrm{rad} = (-0.46\,\mathrm{rad/s})t + \frac{1}{2}(0.35\,\mathrm{rad/s^2})t^2.$$

Note that $5.0\,\mathrm{rev} = 10\pi\,\mathrm{rad}$.

2. Rotation with Constant Angular Acceleration

Solving for t we get $t = 32\ \mathrm{s}$.

(b) At what time t does the disk momentarily stop?

We choose the equation

$$\omega = \omega_0 + \alpha t,$$

with $\omega = 0$. We then get

$$t = -\frac{\omega_0}{\alpha} = -\frac{-0.46\,\frac{\mathrm{rad}}{\mathrm{s}}}{0.35\,\frac{\mathrm{rad}}{\mathrm{s}^2}} = 13\ \mathrm{s}.$$

2. Rotation with Constant Angular Acceleration

Example 4: The angular velocity of a rotating cylinder is decreased from $3.40\ \mathrm{rad/s}$ to $2.00\ \mathrm{rad/s}$ in $20\ \mathrm{rev}$, at constant acceleration.

(a) What is the constant angular acceleration during this decrease in angular speed?

Using the relation

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0),$$

we write

$$\alpha = \frac{\omega^2 - \omega_0^2}{2(\theta - \theta_0)} = \frac{(2.00\ \mathrm{rad/s})^2 - (3.40\ \mathrm{rad/s})^2}{2(40\pi\ \mathrm{rad})} = -0.0301\,\frac{\mathrm{rad}}{\mathrm{s}^2}.$$

2. Rotation with Constant Angular Acceleration

(b) How much time did the speed decrease take?

$$t = \frac{\omega - \omega_0}{\alpha} = \frac{2.00\,\frac{\text{rad}}{\text{s}} - 3.40\,\frac{\text{rad}}{\text{s}}}{-0.0301\,\frac{\text{rad}}{\text{s}^2}} = 46.5\ \text{s}.$$

3. Relating the Linear and Angular Variables

- We wan to relate the linear variables s, v and a of a point in a rotating body to the angular variables θ, ω and α for that body.
- The two sets of variables are related by r, the perpendicular distance of the point from the rotating axis.
- The Position:

  If a reference line on a body rotates through an angle θ, a point within the body at a position r from the rotation axis moves a distance s along a circular arc, where

$$s = \theta r.$$

3. Relating the Linear and Angular Variables

- <u>The Speed</u>:

  Differentiating $s = \theta r$ with respect to time-with r held constant-gives

$$\frac{ds}{dt} = \frac{d\theta}{dt} r,$$

  or

$$v = \omega r.$$

  Since ω is the same for all points on the rotating body, points at a greater radius r have greater linear speed v. The velocity vector $\vec{v}$ is always tangent to the circular paths.

  For constant ω, the period of revolution T for the motion of each point and for the rigid body itself is given by

$$T = \frac{2\pi r}{v} = \frac{2\pi}{\omega}.$$

3. Relating the Linear and Angular Variables

- The Speed:

  The relation $T = 2\pi/\omega$ says that the time for one revolution is the angular distance 2π rad (1 rev) divided by the angular speed.

- The Acceleration:

  Differentiating $v = \omega r$ with respect to time-with r held constant-leads to

$$\frac{dv}{dt} = \frac{d\omega}{dt} r.$$

  dv/dt represents only the part of the linear acceleration that is responsible for changing the magnitude v of the linear velocity $\vec{v}$. That part of linear acceleration is tangent to the circular path.

3. Relating the Linear and Angular Variables

- The Acceleration:

  We call it the **tangential component** a_t of the linear acceleration and write it

$$a_t = \alpha r.$$

  The **radial component** a_r of the linear acceleration (from Ch. 4) can be written as

$$a_r = \frac{v^2}{r} = \omega^2 r.$$

  Note that a_r is not zero even when $\alpha = 0$.

3. Relating the Linear and Angular Variables

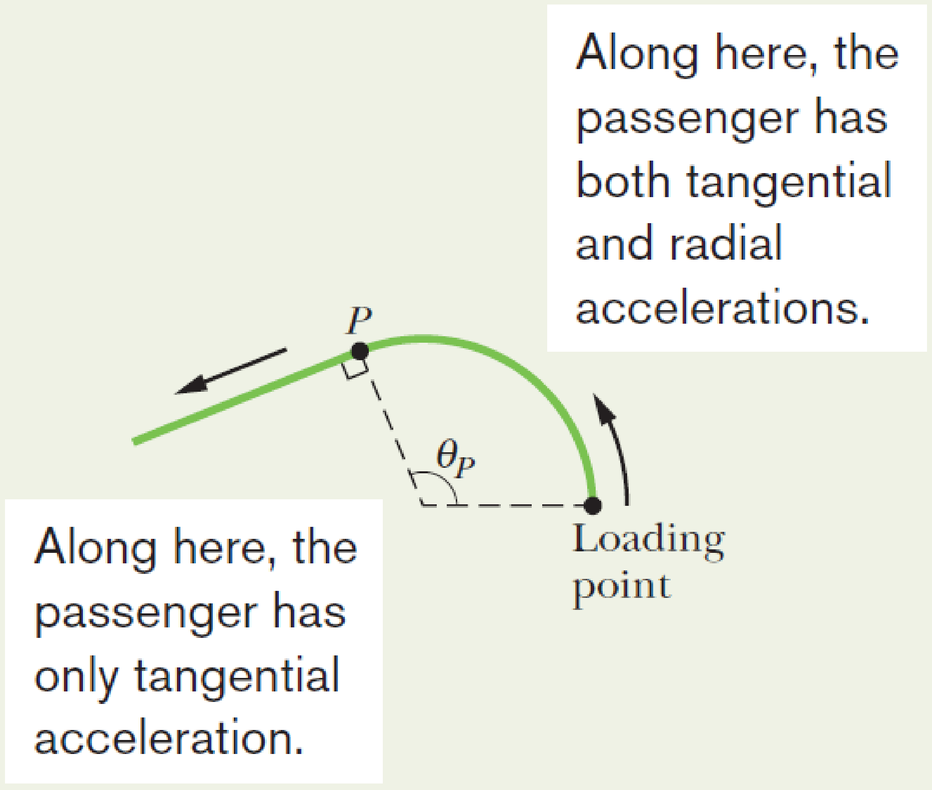
CHECKPOINT 3

A cat rides the rim of a rotating merry-go-round. If the angular speed of this system (*merry-go-round + cat*) is constant, does the cat have (a) radial acceleration and (b) tangential acceleration? If ω is decreasing, does the cat have (c) radial acceleration and (d) tangential acceleration?

(a) Yes.
(b) No.
(c) Yes.
(d) Yes.

2. Rotation with Constant Angular Acceleration

Example 5: A roller coaster, initially at rest, accelerates at g along a horizontal track that begins as a circular arc at the loading point and then, at point P, continues along a tangent to the arc. What angle θ_P should the arc subtend so that the coaster's acceleration a is $4g$ at point P?



The coaster acceleration is given by

$$a = \sqrt{a_t^2 + a_r^2},$$

since a_t and a_r are perpendicular. $a_t = g$ and we need to find express a_r in terms of θ_P.

2. Rotation with Constant Angular Acceleration

Using the relation

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0),$$

we find that

$$\omega^2 = 2\alpha\theta_P = \frac{2a_t\theta_P}{r}.$$

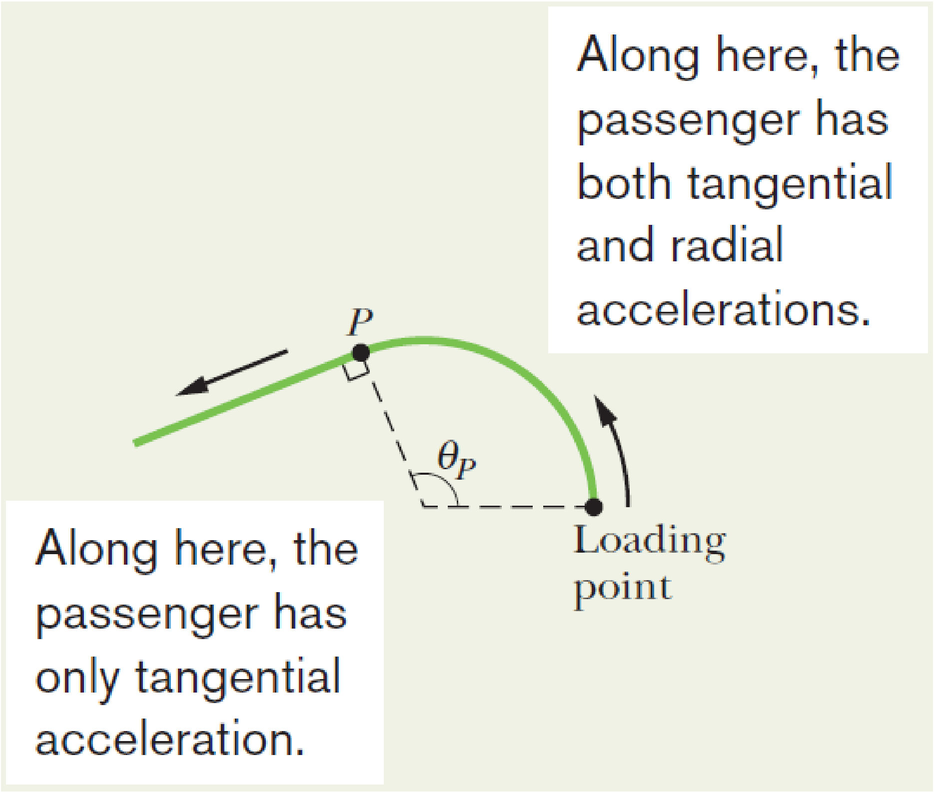
We know that $a_r = \omega^2 r$. Therefore,

$$a_r = 2a_t\theta_P.$$

Substituting in the expression $a = \sqrt{a_t^2 + a_r^2}$ leads to

$$4g = \sqrt{g^2 + (2g\theta_P)^2},$$

which yields $\theta_P = 1.94$ rad or 111°.



4. Kinetic Energy of Rotation

- A rotating body has kinetic energy equal to the kinetic energies of the particles constituting the body. We therefore write

$$K = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + \frac{1}{2}m_3 v_3^2 + \cdots$$
$$= \sum \frac{1}{2} m_i v_i^2.$$

Here m_i and v_i are the mass and speed of the ith particle, respectively.

We can replace v_i by ωr_i and write

$$K = \sum \frac{1}{2} m_i (\omega r_i)^2 = \frac{1}{2}\left(\sum m_i r_i^2\right)\omega^2.$$

Recall that ω is the same for all particle.

4. Kinetic Energy of Rotation

- The quantity in the parentheses is a measure of how the mass of the body is distributed about its rotation axis. That quantity is called the **rotational inertial** (or **moment of inertia**) I of the body with respect to the axis of rotation.
- We may now write

$$I = \sum m_i r_i^2 ,$$

and

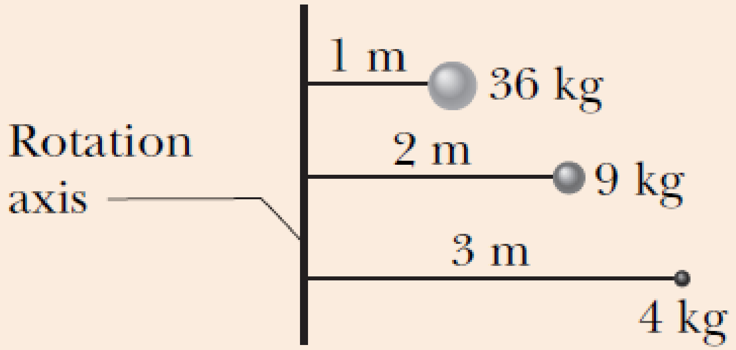
$$K = \frac{1}{2} I \omega^2 .$$

The SI unit for I is $(\mathrm{kg} \cdot \mathrm{m}^2)$.

4. Kinetic Energy of Rotation

CHECKPOINT 4

The figure shows three small spheres that rotate about a vertical axis. The perpendicular distance between the axis and the center of each sphere is given. Rank the three spheres according to their rotational inertia about that axis, greatest first.



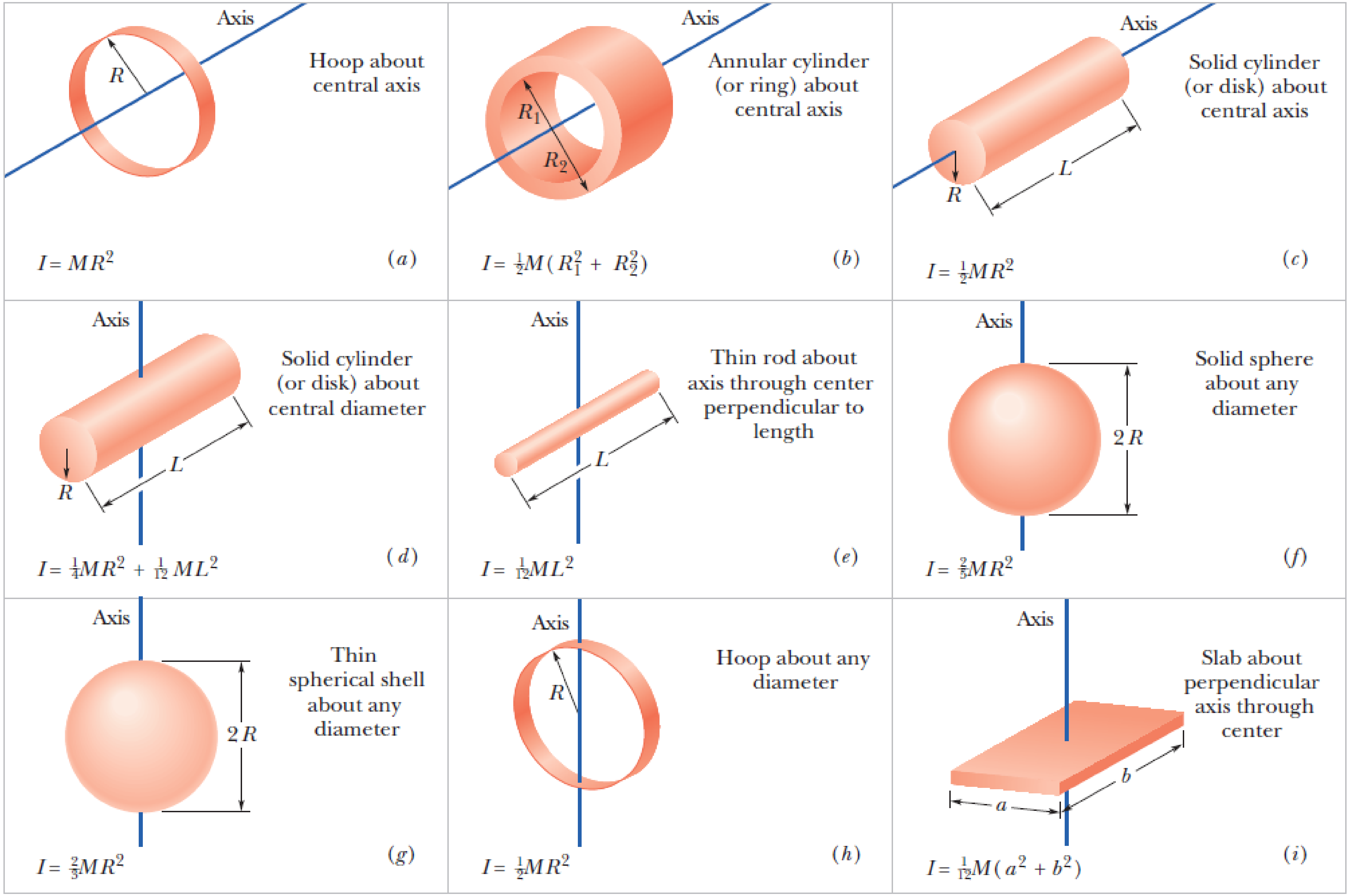
In all three cases $I = mr^2 = 36\ \mathrm{kg \cdot m^2}$.

5. Calculating the Rotational Inertia

- If a rigid body consists of a few particles, we can calculate its rotational inertia about a given rotation axis by $I = \sum m_i r_i^2$.
- If a rigid body consists of a great number of particles we replace the summation by the integral

$$I = \int r^2 dm \,.$$

- The table in the next slide gives the results of such integration for nine common body shapes and the indicated axes of rotation.


Axis
Hoop about central axis
R
$I = MR^2$
(a)
Axis
Annular cylinder (or ring) about central axis
R_1
R_2
$I = \frac{1}{2}M(R_1^2 + R_2^2)$
(b)
Axis
Solid cylinder (or disk) about central axis
R
L
$I = \frac{1}{2}MR^2$
(c)
Axis
Solid cylinder (or disk) about central diameter
R
L
$I = \frac{1}{4}MR^2 + \frac{1}{12}ML^2$
(d)
Axis
Thin rod about axis through center perpendicular to length
L
$I = \frac{1}{12}ML^2$
(e)
Axis
Solid sphere about any diameter
2R
$I = \frac{2}{5}MR^2$
(f)
Axis
Thin spherical shell about any diameter
2R
$I = \frac{2}{3}MR^2$
(g)
Axis
Hoop about any diameter
R
$I = \frac{1}{2}MR^2$
(h)
Axis
Slab about perpendicular axis through center
a
b
$I = \frac{1}{12}M(a^2 + b^2)$
(i)

5. Calculating the Rotational Inertia

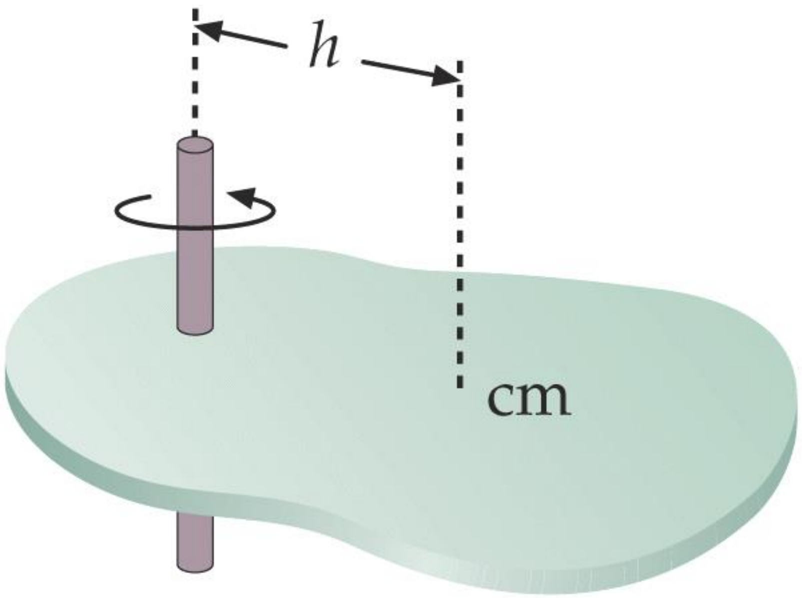
- <u>Parallel-Axis Theorem</u>:

  If we know the moment of inertia I_{com} of a body of mass M about an axis through its center of mass then we can find its moment of inertia I about any other parallel axis. If h is the perpendicular distance between the given axis and the axis though the center of mass then I and I_{com} are related by

$$I = I_{\text{com}} + Mh^2.$$

  This equation is know as the **parallel-axis theorem**.

  See you textbook for the proof.

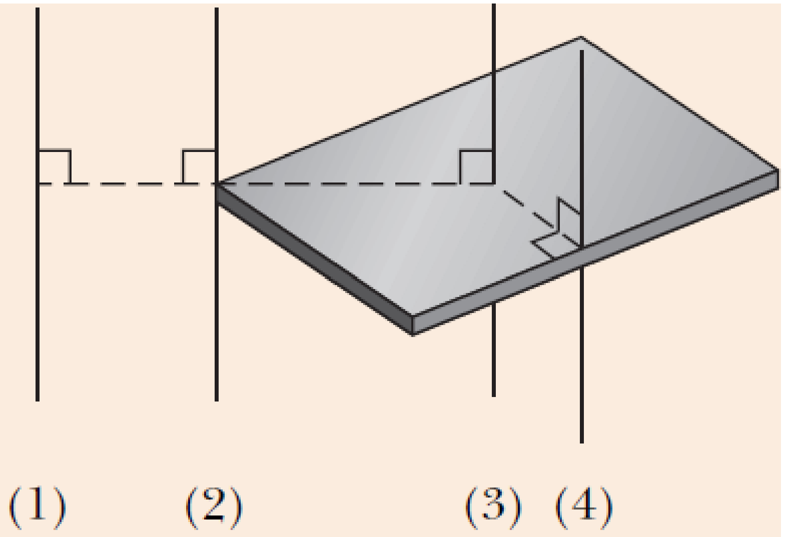


Source: http://www.phy.ohiou.edu/~hla/Ch9.htm

5. Calculating the Rotational Inertia

CHECKPOINT 5

The figure shows a book-like object (one side is longer than the other) and four choices of rotation axes, all perpendicular to the face of the object. Rank the choices according to the rotational inertia of the object about the axis, greatest first.



$$I = I_{com} + Mh^2.$$

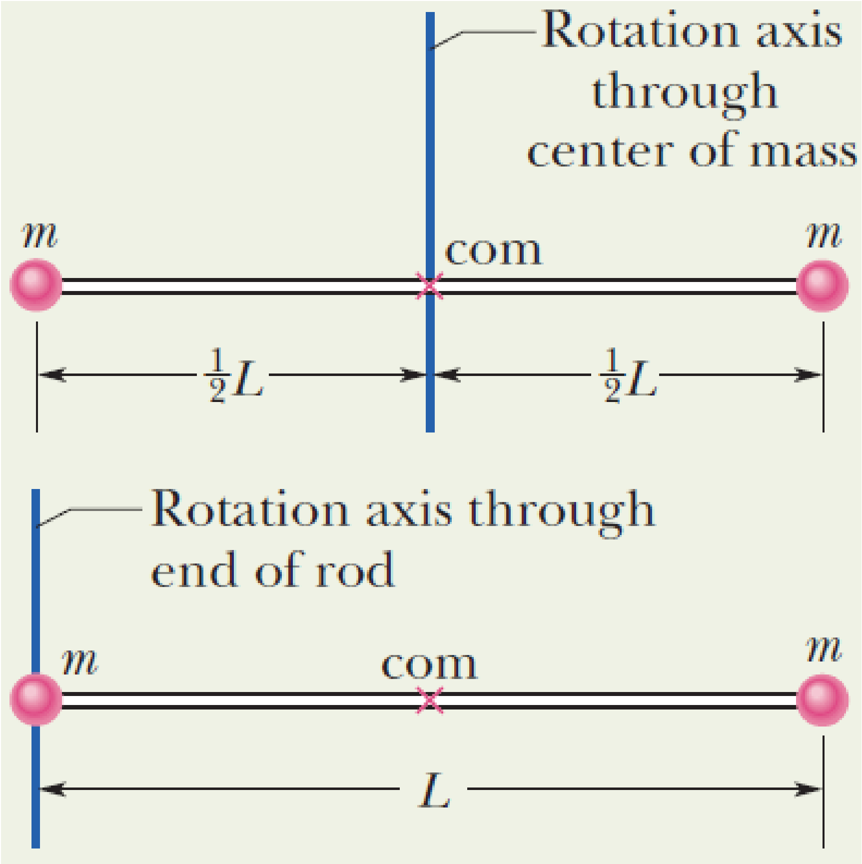
1, 2, 4, 3.

5. Calculating the Rotational Inertia

Example 6: The figure shows a rigid body consisting of two particles of mass m connected by a rod of length L and negligible mass.

(a) What is the rotational inertia I_{com} about an axis through the center of mass, perpendicular to the rod as shown?

$$I = \sum m_i r_i^2 = m\left(\frac{1}{2}L\right)^2 + m\left(\frac{1}{2}L\right)^2 = \frac{1}{2}mL^2$$



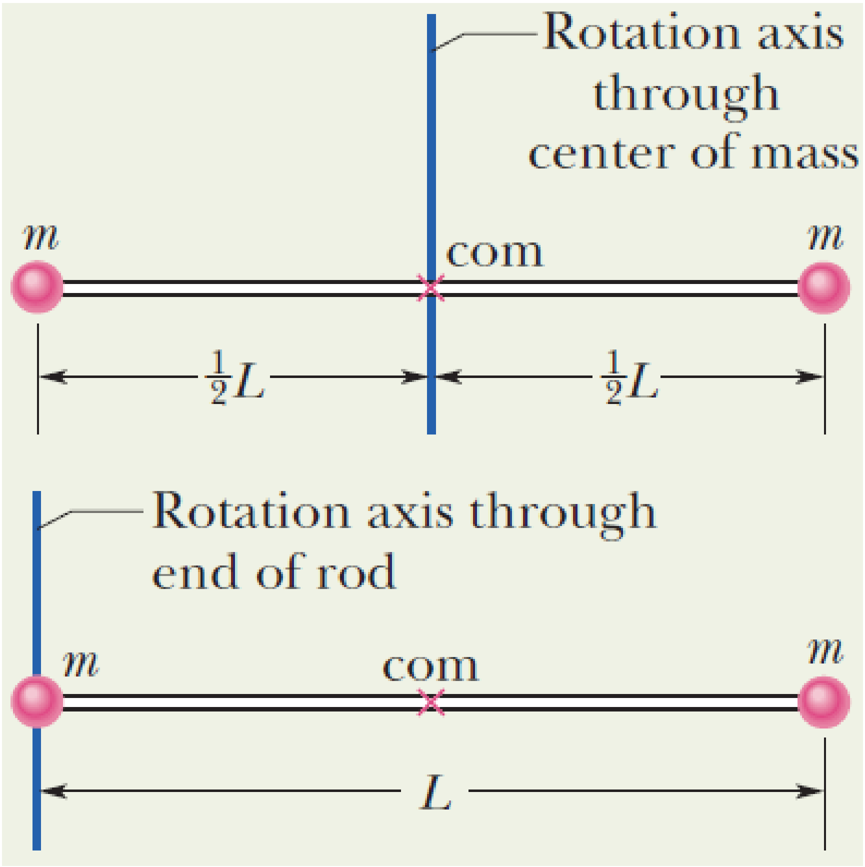
5. Calculating the Rotational Inertia

(b) What is the rotational inertia I of the body about an axis through the left end of the rod and parallel to the first axis?

$$I = \sum m_i r_i^2 = m(0)^2 + m(L)^2 = mL^2.$$

Alternatively, using the parallel-axis theorem

$$I = I_{com} + Mh^2 = \frac{1}{2}mL^2 + (2m)\left(\frac{1}{2}L\right)^2$$
$$= mL^2.$$



5. Calculating the Rotational Inertia

Example 7: Find the rotational kinetic energy of a solid steel disk of mass $M = 272\ \mathrm{kg}$ and radius $R = 38.0\ \mathrm{cm}$, rotating at $14000\ \mathrm{rev/min}$.

The moment of inertia of the disk is

$$I = \frac{1}{2}MR^2 = \frac{1}{2}(272\ \mathrm{kg})(0.38)^2 = 19.64\ \mathrm{kg}\cdot\mathrm{m}^2.$$

The rotational speed of the disk is

$$\omega = 14000\frac{\mathrm{rev}}{\mathrm{min}} \times \frac{1\mathrm{min}}{60\ \mathrm{s}} \times \frac{2\pi\ \mathrm{rad}}{1\ \mathrm{rev}} = 1.466 \times 10^3\frac{\mathrm{rad}}{\mathrm{s}}.$$

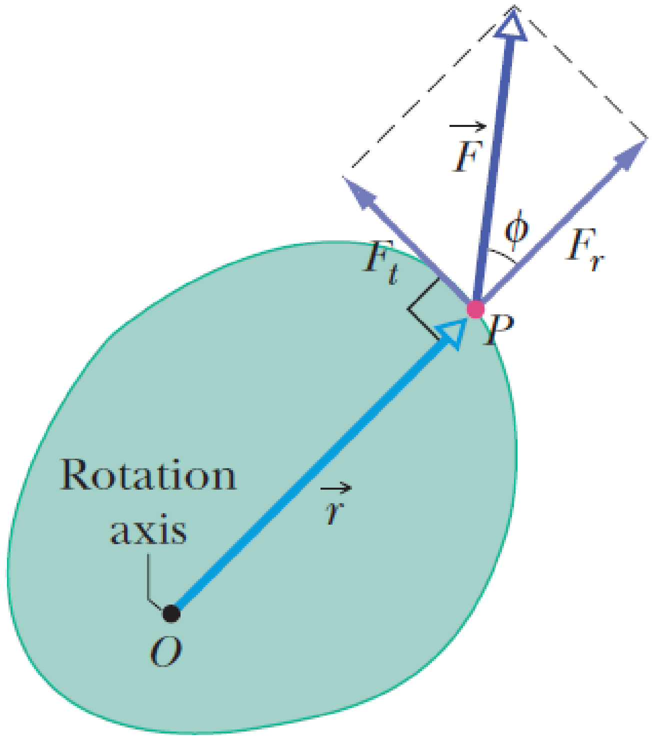
The kinetic energy is then

$$K = \frac{1}{2}I\omega^2 = \frac{1}{2}(19.64\ \mathrm{kg}\cdot\mathrm{m}^2)\left(1.466 \times 10^3\frac{\mathrm{rad}}{\mathrm{s}}\right)^2 = 2.1 \times 10^7\ \mathrm{J}.$$

6. Torque

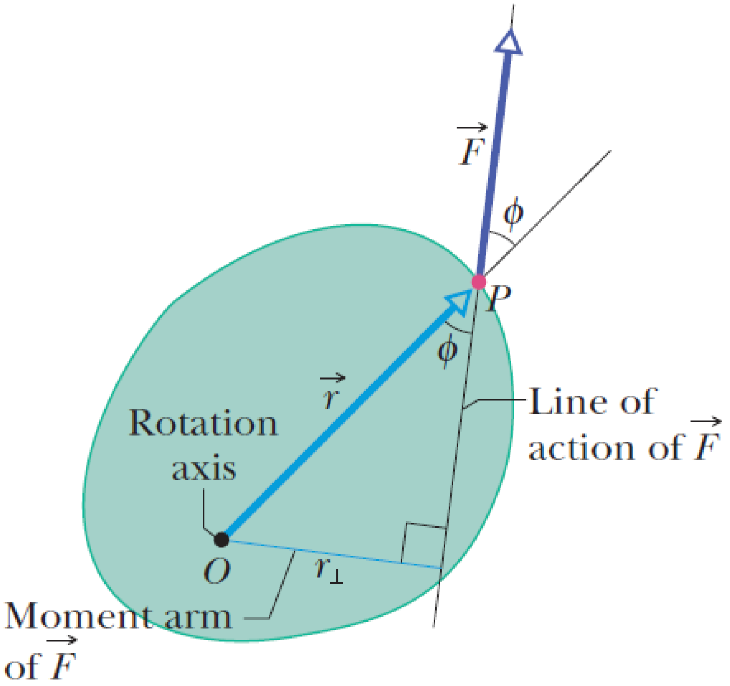
- Consider the situation shown in figure. The force $\vec{F}$ has a *radial component* F_r, pointing along $\vec{r}$ and a *tangential component* $F_t = F \sin \phi$, perpendicular to $\vec{F}$.
- The ability to rotate the object depends only on the tangential component F_t. It also depends on how far from the rotation axis the force is applied. We therefore define the torque

$$\tau = rF \sin \phi \, .$$



6. Torque

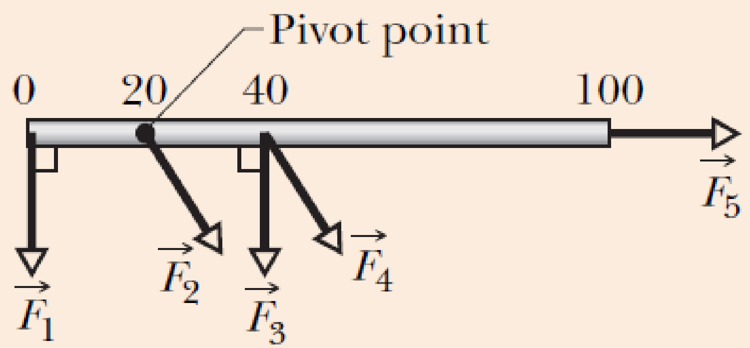
- We can also write that $\tau = r(F \sin\phi) = rF_t$ or $\tau = (r \sin\phi)F = r_\perp F$. $r_\perp$ is called the **moment arm** of $\vec{F}$, and the extended line between P and the moment of arm is called the **line of action** of $\vec{F}$.
- The SI unit for torque is $\mathrm{N \cdot m}$.
- Torques obey the superposition principle: When several torques act on a body the net torque τ_{net} (or resultant torque) is the sum of the individual torques.



6. Torque

CHECKPOINT 6

The figure shows an overhead view of a meter stick that can pivot about the dot at the position marked 20 (for 20 cm). All five forces on the stick are horizontal and have the same magnitude. Rank the forces according to the magnitude of the torque they produce, greatest first.



$\vec{F}_1$ & $\vec{F}_3$, $\vec{F}_4$, $\vec{F}_2$ & $\vec{F}_5$.

7. Newton’s Second Law for Rotation

- Newton’s second law for the particle shown in the figure for components along the tangential direction reads

$$F_t = ma_t,$$

which can be rewritten as

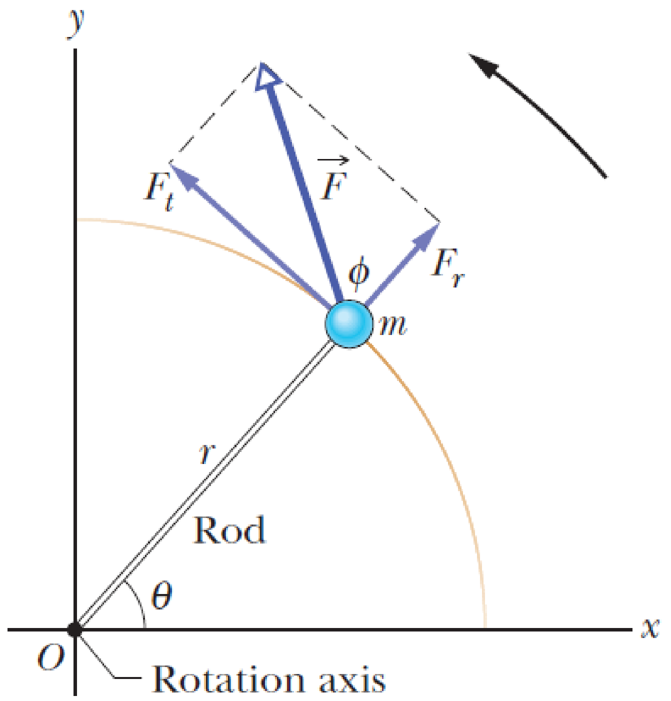
$$rF_t = mr^2 \frac{a_t}{r}.$$

Using $rF_t = \tau, mr^2 = I$ and $\frac{a_t}{r} = \alpha$ we get that

$$\tau = I\alpha.$$

This result is valid for any rigid body. When there are several forces applied to the particle we have

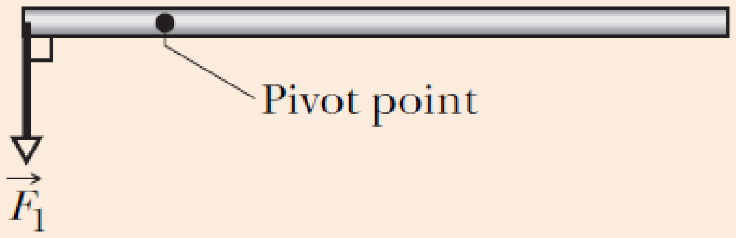
$$\tau_{\text{net}} = I\alpha.$$



7. Newton's Second Law for Rotation

CHECKPOINT 7

The figure shows an overhead view of a meter stick that can pivot about the point indicated, which is to the left of the stick's midpoint. Two horizontal forces, $\vec{F}_1$ and $\vec{F}_2$, are applied to the stick. Only $\vec{F}_1$ is shown. Force $\vec{F}_2$ is perpendicular to the stick and is applied at the right end. If the stick is not to turn, (a) what should be the direction of $\vec{F}_2$, and (b) should F_2 be greater than, less than, or equal to F_1?

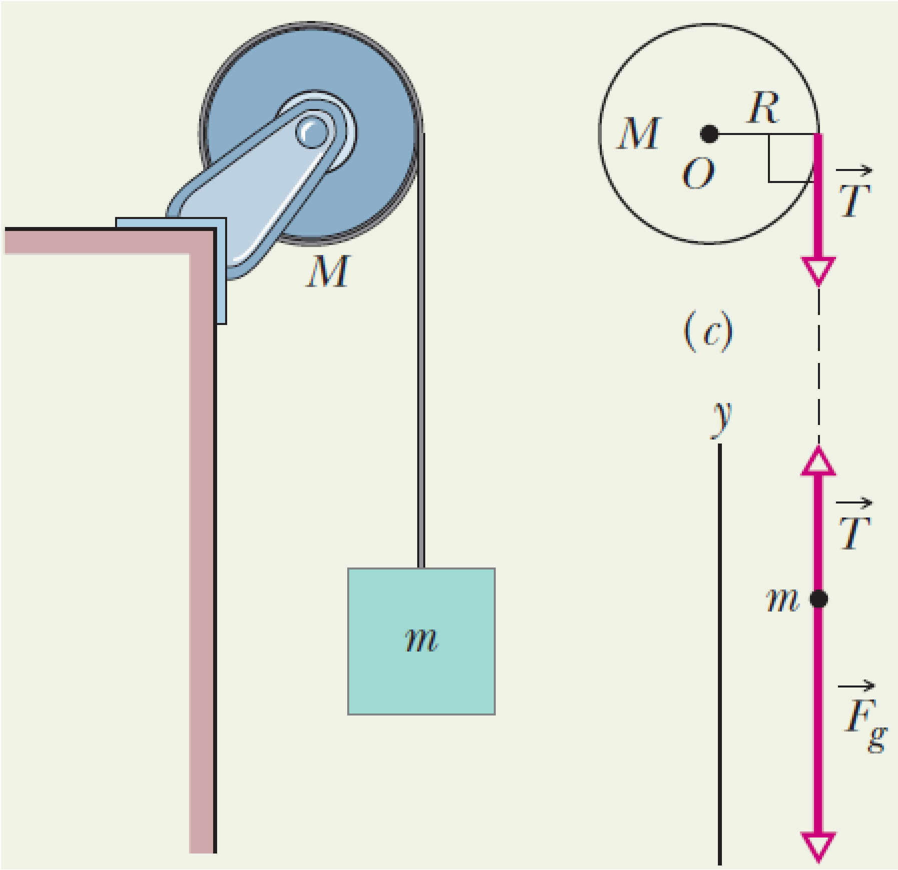


(a) Down.

(b) Less.

7. Newton's Second Law for Rotation

Example 8: The figure shows a uniform disk, with mass $M = 2.5$ kg and radius $\mathrm{R} = 20$ cm, mounted on a fixed horizontal axle. A block with mass $\mathrm{m} = 1.2$ kg hangs from a massless cord that is wrapped around the rim of the disk. Find the acceleration of the falling block, the angular acceleration of the disk, and the tension in the cord. The cord does not slip, and there is no friction at the axle.



Newton's second law along the vertical y axis ($F_{net,y} = ma_y$) gives

$$T - mg = ma_y$$

7. Newton's Second Law for Rotation

$$T - mg = ma_y. \quad (1)$$

Newton's second for the pulley ($\tau_{net} = I\alpha$) reads

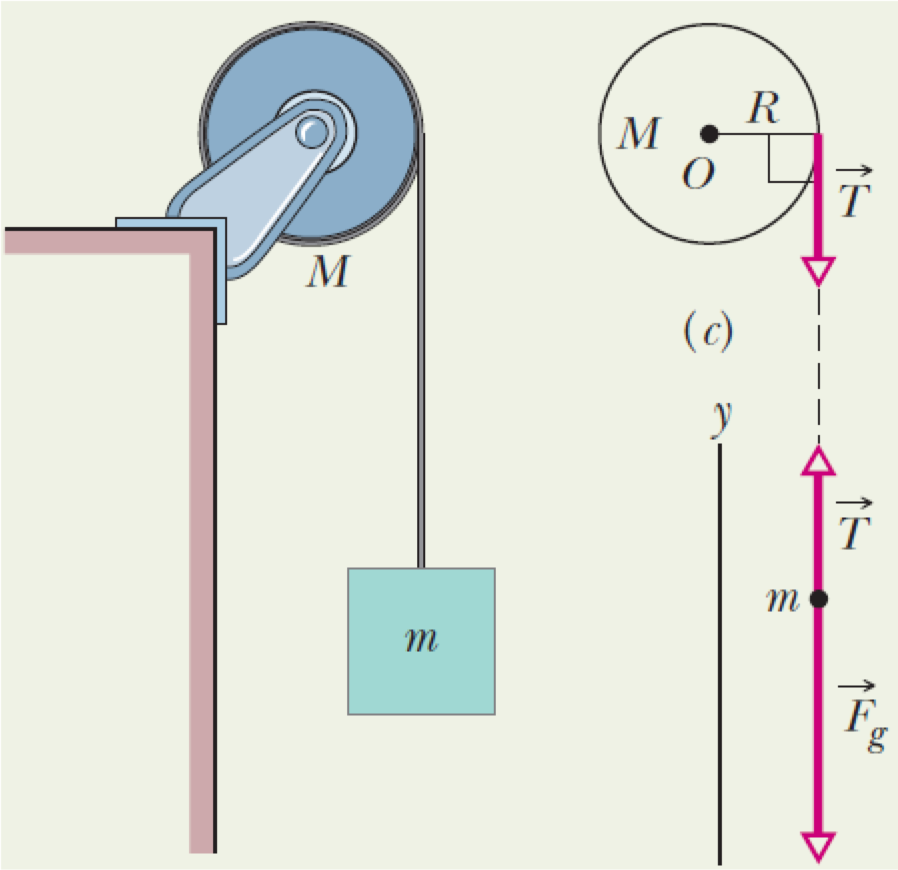
$$-RT = \left(\frac{1}{2}MR^2\right)(\alpha).$$

Using $a_y = \alpha R$ we rewrite this expression as

$$T = -\frac{1}{2}Ma_y.$$

Combining this result with Eq. (1) we get that

$$a_y = -g\frac{2m}{M + 2m} = -g\frac{2(1.2\text{ kg})}{2.5\text{ kg} + 1.2\text{ kg}} = -4.8\ \frac{\text{m}}{\text{s}^2}.$$



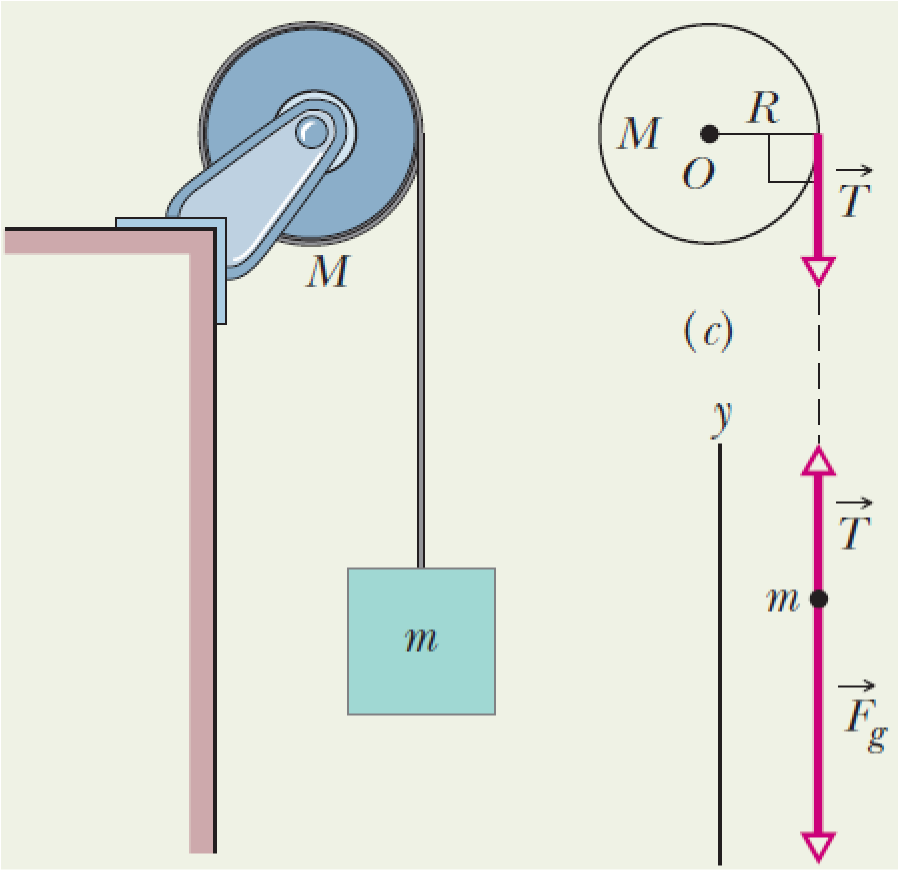
7. Newton's Second Law for Rotation

Substituting in the expression for T we get

$$T = -\frac{1}{2}Ma_y = -\frac{1}{2}(2.5\text{ kg})\left(-4.8\ \frac{\text{m}}{\text{s}^2}\right) = 6.0\text{ N}.$$

Finally, α is given by

$$\alpha = \frac{a_y}{R} = \frac{-4.8\,\frac{\text{m}}{\text{s}^2}}{0.20\text{ m}} = -24\ \frac{\text{rad}}{\text{s}}.$$



8. Work and Rotational Energy

- When a torque accelerates a rigid body in rotation about a fixed axis, the torque does work W on the body. The rotational kinetic energy $K = \frac{1}{2}I\omega^2$ of the body can change. If that is the only energy of the body that changes then we can relate the change ΔK in kinetic energy to the work W with the work-kinetic energy theorem:

$$\Delta K = K_f - K_i = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2.$$

- The work due to a torque τ is given by

$$W = \int_{\theta_i}^{\theta_f} \tau \, d\theta \,,$$

where θ_f and θ_i are the final and initial angular positions, respectively.

8. Work and Rotational Energy

- When the torque is constant the previous expression for work gives

$$W = \tau \int_{\theta_i}^{\theta_f} d\theta = \tau\left(\theta_f - \theta_i\right).$$

- The rate at which the work is done is the power, which we can find with the rotational equivalent

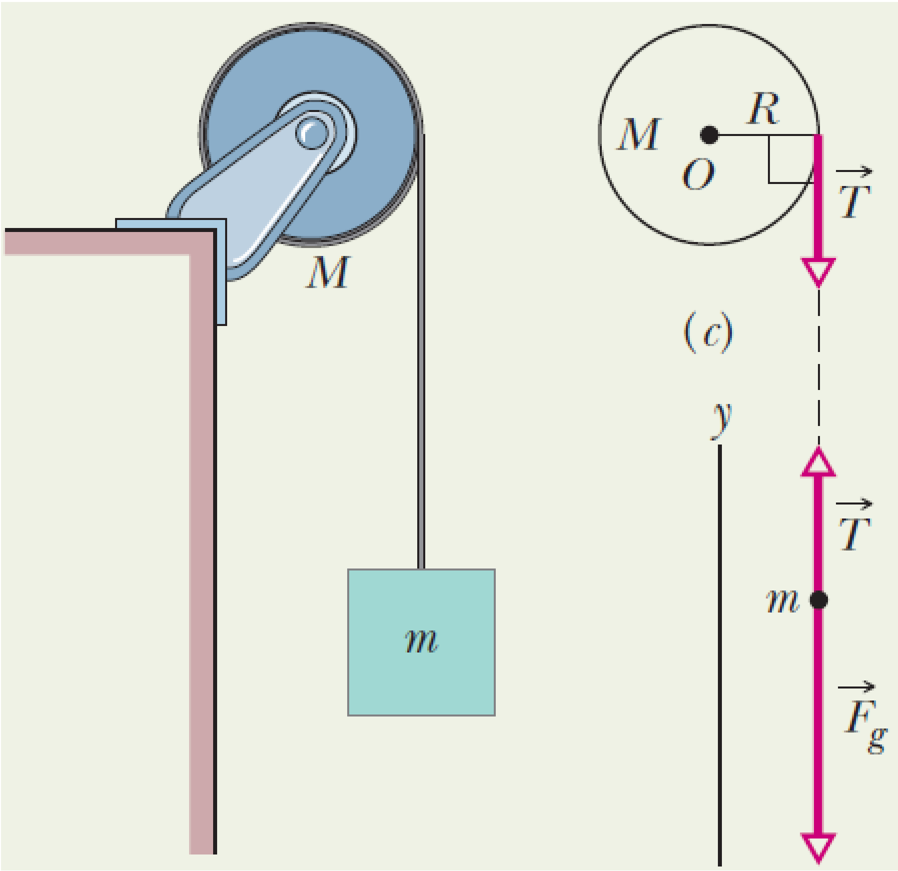
$$P = \frac{dW}{dt} = \tau\omega.$$

Some Corresponding Relations for Translational and Rotational Motion

Pure Translation (Fixed Direction)		Pure Rotation (Fixed Axis)	
Position	x	Angular position	θ
Velocity	$v = dx/dt$	Angular velocity	$\omega = d\theta/dt$
Acceleration	$a = dv/dt$	Angular acceleration	$\alpha = d\omega/dt$
Mass	m	Rotational inertia	I
Newton's second law	$F_{net} = ma$	Newton's second law	$\tau_{net} = I\alpha$
Work	$W = \int F\,dx$	Work	$W = \int \tau\,d\theta$
Kinetic energy	$K = \frac{1}{2}mv^2$	Kinetic energy	$K = \frac{1}{2}I\omega^2$
Power (constant force)	$P = Fv$	Power (constant torque)	$P = \tau\omega$
Work–kinetic energy theorem	$W = \Delta K$	Work–kinetic energy theorem	$W = \Delta K$

8. Work and Rotational Energy

Example 9: Let the disk in the figure start from rest at time $t = 0$ and also let the tension in the massless cord be 6.0 and the angular acceleration of the disk be $-24\ \text{rad/s}^2$. What is its rotational kinetic energy K at $t = 2.5$ s?



In order to find K we find ω at $t = 2.5$ s:

$$\omega = \omega_0 + \alpha t = \alpha t.$$

$$K = \frac{1}{2}\left(\frac{1}{2}MR^2\right)(\alpha t)^2 = \frac{1}{4}M(R\alpha t)^2$$

$$= \frac{1}{4}(2.0\ \text{kg})[(0.20\ \text{m})(-24\ \text{rad/s}^2)(2.5\ \text{s})]^2$$

$$= 90\ \text{J}.$$

8. Work and Rotational Energy

We can also get this answer by finding the disk's kinetic energy from the work done on the disk:

$$K_f = K_i + W = W = \tau(\theta_f - \theta_i).$$

We know that $\tau = -RT$, and

$$\theta_f - \theta_i = \omega_0 t + \frac{1}{2}\alpha t^2 = \frac{1}{2}\alpha t^2.$$

We now can write

$$\begin{aligned} K_f &= -\frac{1}{2}RT\alpha t^2 \\ &= -\frac{1}{2}(0.20\ \mathrm{m})(6.0\ \mathrm{N})\left(-24\frac{\mathrm{rad}}{\mathrm{s}^2}\right)(2.5\ \mathrm{s})^2 \\ &= 90\ \mathrm{J}. \end{aligned}$$