2. Consider the ground state wave function of a one-dimensional harmonic oscillator of frequency ω;

$$\psi_0(x,t) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}} e^{-i\frac{\omega t}{2}}$$

Calculate $\langle x \rangle, \langle x^2 \rangle$, and Δx for an electron in the ground state.

Given: $\int_0^\infty x^2 e^{-\alpha x^2} dx = \frac{1}{4\alpha}\sqrt{\frac{\pi}{\alpha}} \quad \alpha > 0$

$$\langle x \rangle = \int_{-\infty}^{+\infty} x\,|\psi|^2 dx = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \underbrace{\int_{-\infty}^{+\infty} x\, e^{-\frac{m\omega}{\hbar}x^2} dx}_{=0 \text{ odd function}}$$

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} x^2\,|\psi|^2 dx = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int_{-\infty}^{+\infty} x^2 e^{-\frac{m\omega}{\hbar}x^2} dx$$

$$= 2\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \underbrace{\int_0^{\infty} x^2 e^{-\frac{m\omega}{\hbar}x^2} dx}_{\frac{\hbar}{4m\omega}\left(\frac{\hbar\pi}{m\omega}\right)^{1/2}}$$

$$\Rightarrow \langle x^2 \rangle = \frac{\hbar}{2m\omega}$$

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\langle x^2\rangle} = \sqrt{\frac{\hbar}{2m\omega}}$$