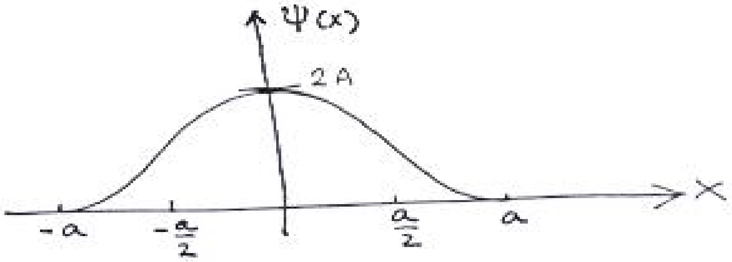
Problem 2: 2

$$\psi(x) = \begin{cases} 0 & x < -a, \; x > a \\ A\left(1 + \cos\frac{\pi x}{a}\right) & -a < x < a \end{cases}$$

a)



b) This wave function is indeed physically acceptable because it is square integrable, single-valued, continuous and has a continuous first derivative.

c)

$$1 = \int_{-a}^{a} |\psi(x)|^2 dx = |A|^2 \int_{-a}^{a} \left[1 + 2\cos\left(\frac{\pi x}{a}\right) + \cos^2\left(\frac{\pi x}{a}\right)\right] dx$$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$= |A|^2 \int_{-a}^{a} \left[\frac{3}{2} + 2\cos\left(\frac{\pi x}{a}\right) + \frac{1}{2}\cos\left(\frac{2\pi x}{a}\right)\right] dx$$

$$= |A|^2 \left[\frac{3}{2}x + \frac{2a}{\pi}\sin\left(\frac{\pi x}{a}\right) + \frac{a}{4\pi}\sin\left(\frac{2\pi x}{a}\right)\right]_{-a}^{a}$$

$$= 3aA^2 \Rightarrow \boxed{A = \frac{1}{\sqrt{3a}}}$$

d) since $\psi(x)$ is even $\Rightarrow$ $\langle x \rangle = \int_{-a}^{a} x\,|\psi(x)|^2 dx = 0$

since $\psi(x)$ is real $\Rightarrow$ $\langle p \rangle = 0$ $\qquad \langle p \rangle = -\int_{-a}^{a} \psi^* i\hbar \frac{\partial \psi}{\partial x} dx$

$$\langle x^2 \rangle = A^2 \int_{-a}^{a} x^2 \left(1 + \cos\frac{\pi x}{a}\right)^2 dx = A^2 \int_{-a}^{a} x^2 \left[\frac{3}{2} + 2\cos\left(\frac{\pi x}{a}\right) + \frac{1}{2}\cos\left(\frac{2\pi x}{a}\right)\right] dx$$

$$= \frac{a^2}{3}\left(1 - \frac{15}{4\pi^2}\right) \Rightarrow \boxed{\Delta x = \sqrt{\langle x^2 \rangle} = a\sqrt{\frac{1}{3} - \frac{5}{4\pi^2}}}$$

$$\langle p^2 \rangle = A^2 \int_{-a}^{a} \left(1 + \cos\frac{\pi x}{a}\right)\left(-\hbar^2 \frac{d^2}{dx^2}\right)\left(1 + \cos\frac{\pi x}{a}\right) dx$$

$$= \frac{\hbar^2 \pi^2}{a^2} A^2 \int_{-a}^{a} \left(\left[\cos\left(\frac{\pi x}{a}\right) + \cos^2\left(\frac{\pi x}{a}\right)\right] dx = \frac{\hbar^2 \pi^2}{a^2} \frac{1}{3a} a$$

$$\Rightarrow \boxed{\Delta p = \sqrt{\langle p^2 \rangle} = \frac{\hbar \pi}{\sqrt{3}\, a}}$$