Chapter 8 _ Central-Force Motion

8.1 Introduction

When the force of interaction of two bodies is directed along the line joining the two bodies, we say that the force is central. Many forces in physics are of central nature: gravitational forces between planets and celestial bodies, electrical force between charged particles, etc... So the study of this problem is of fundamental importance.

8.2 Reduced Mass

Let us consider two masses m_1 with position vector $\vec{r_1}$ and m_2 with position vector $\vec{r_2}$. We define the center of mass and relative coordinates by

$$\vec{R} = \frac{m_1 \vec{r_1} + m_2 \vec{r_2}}{m_1 + m_2} \quad ; \quad \vec{r} = \vec{r_1} - \vec{r_2} \qquad —(1)$$

We restrict our attention to systems without frictional forces and for which U is a function of $\vec{r} = |\vec{r_1} - \vec{r_2}|$.

The Lagrangian in this case is:

$$L = \frac{1}{2} m_1 |\dot{\vec{r_1}}|^2 + \frac{1}{2} m_2 |\dot{\vec{r_2}}|^2 - U(r) \qquad —(2)$$

No external forces are applied to this system, therefore the motion of the C.M. is uniform $\Rightarrow$ $\vec{V}_{cm} = \text{Const.}$

This C.M. motion being trivial, we eliminate its consideration by choosing the origin of our coordinates at the c.m.

Recall that the coordinate system moving with constant velocity is an inertial frame so that ($\ddot{\vec{R}} = 0 \Rightarrow \dot{\vec{R}} = 0$ const.)

$$\vec{R} = 0 \quad \Rightarrow \quad \begin{cases} m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0 \\ \vec{r}_1 - \vec{r}_2 = \vec{r} \end{cases}$$

$$\Rightarrow \quad \vec{r}_1 = \frac{m_2}{m_1 + m_2} \vec{r} \quad \text{and} \quad \vec{r}_2 = -\frac{m_1}{m_1 + m_2} \vec{r} \qquad \text{— (3)}$$

Putting these equations (3) into (2) we obtain

$$L = \frac{1}{2} \mu \dot{r}^2 - U(r) \quad ; \quad \mu = \frac{m_1 m_2}{m_1 + m_2} \qquad \text{— (4)}$$

Therefore once the C.M. motion has been eliminated we reduce the problem to the motion of an effective particle of mass μ in a central potential $U(r)$.

μ is called the effective mass or reduced mass.

Once we obtain the solution $r(t)$, we can find the individual motions of the particles $r_1(t)$ and $r_2(t)$ by using eq. (3).

5.3 Conservation Theorems - First Integral of the Motion

We describe a particle of mass μ in a central-force field described by a potential $U(r)$.

Since U depends only on r $\Longrightarrow$ we have spherical symmetry. We expect $\vec{L} = \vec{r} \times \vec{p} = \text{Constant}$ — (5)

(space is isotropic)

The angular momentum of the particle is conserved.

Let $\hat{z}$ be the direction of $\vec{L}$ then eq. (5) implies that $\vec{r}$ and $\vec{p}$ are always $\perp$ to $\hat{z}$, thus the motion of the particle is constrained to the **xy** plane. e.g. let (r, θ) be the polar coord. of the particle. Then,

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - U(r) \qquad \text{— (6)}$$

θ is a cyclic coordinate $\Rightarrow \dot{p}_\theta = 0 \Rightarrow p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2 \dot{\theta} = \text{const}$ (7)

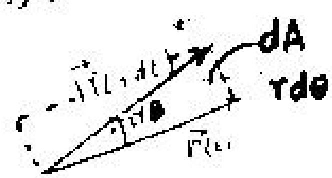
The quantity p_θ is <u>the first integral</u> of the motion.

Define $\ell = \mu r^2 \dot{\theta} = \text{constant} = I\dot{\theta} = I\omega$ is the angular momentum along the z-direction in **accordance with** eq (5).

Geometrical interpretation of eq. (7):

$d\vec{A} \perp$ to the plane!

$$dA = \frac{1}{2}|\vec{r} \times d\vec{r}| = \frac{1}{2} r^2 d\theta$$



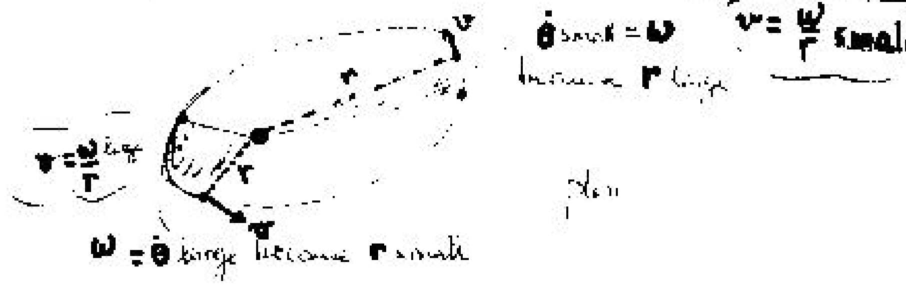
dA = area swept in time dt

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{1}{2} r^2 \frac{\ell}{\mu r^2} = \frac{\ell}{2\mu}$$

hence $\frac{dA}{dt}$ = Const. $\Rightarrow$ Kepler's second law.

known as the areal velocity.

Note: $\ell = \mu r^2 \dot\theta = \text{Const.} \not\Rightarrow \dot\theta = \text{Const}$ only, $r^2\dot\theta = \text{Const.}$



Conservation of Energy: $E = T + U$ (No friction involved)

$$E = \frac{1}{2}\mu\left(\dot r^2 + r^2\dot\theta^2\right) + U(r)$$

Now $\ell = \mu r^2 \dot\theta \Rightarrow \dot\theta^2 = \dfrac{\ell^2}{\mu^2 r^4}$

$$E = \frac{1}{2}\mu \dot r^2 + \frac{1}{2}\mu r^2 \frac{\ell^2}{\mu^2 r^4} + U(r)$$

(8) — $\boxed{E = \frac{1}{2}\mu \dot r^2 + \frac{1}{2}\frac{\ell^2}{\mu r^2} + U(r)}$ 1st integral of the motion

8.4 Equations of Motion

When $U(r)$ is specified, then we can solve equ (8) in terms of E and ℓ (both depend on initial conditions)

$$\dot r = \frac{dr}{dt} = \sqrt{\frac{2}{\mu}(E - U) - \frac{\ell^2}{\mu^2 r^2}} \quad \text{--- (9)}$$

can be Solved for dt and integrated to yield $t = t(r)$.
An inversion can give $r = r(t)$

At present we are interested in the eq. of the path in terms of r and θ. We can write

$$d\theta = \frac{d\theta}{dt}\,\frac{dt}{dr}\,dr = \frac{\dot{\theta}}{\dot{r}}\,dr$$

$$d\theta = \frac{\ell}{\mu r^2}\,\frac{1}{\sqrt{\frac{2}{\mu}(E-U) - \frac{\ell^2}{\mu^2 r^2}}}\,dr$$

or
$$\theta(r) = \int \frac{(\ell/r^2)\,dr}{\sqrt{2\mu(E-U) - \frac{\ell^2}{r^2}}}$$

or
$$\boxed{\theta(r) = \int \frac{(\ell/r^2)\,dr}{\sqrt{2\mu\left(E - U - \frac{\ell^2}{2\mu r^2}\right)}}} \qquad \text{— (10)}$$

The solution can be obtained for only for certain specific forms of the force law, i.e, only if $F \propto r^n$
where $n = 1, -2, -3$

harmonic oscillator ($n = 1$), inverse square law ($n = -2$), inverse cube law ($n = -3$)

The solutions are expressible in terms of circular functions.

The problem is formally solved once we get the equation of the orbit $\theta = \theta(r)$ by combinition of conservation of E and l in one equation!

Another approach

* Lagrangian **approach** $\Longrightarrow$ equation of the orbit $r = r(\theta)$

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = 0$$

$$L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r)$$

$$\frac{\partial L}{\partial r} = \mu r\dot{\theta}^2 - \frac{\partial U}{\partial r}$$

$$\frac{\partial L}{\partial \dot{r}} = \mu\dot{r} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \mu\ddot{r}$$

$$\Rightarrow \quad \mu\ddot{r} = \mu r\dot{\theta}^2 - \frac{\partial U}{\partial r}$$

or $$\mu\left(\ddot{r} - r\dot{\theta}^2\right) = -\frac{\partial U}{\partial r} = F(r)$$

To solve: set $u = \frac{1}{r}$ (**change of variables**)

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\theta} = -\frac{1}{r^2}\frac{\dot{r}}{\dot{\theta}}$$

Using $l = \mu r^2\dot{\theta}$ or $\dot{\theta} = \frac{l}{\mu r^2}$

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{\dot r}{\ell}\mu r^2 = -\frac{\mu}{\ell}\dot r$$

$$\frac{d^2u}{d\theta^2} = \frac{d}{dt}\left(-\frac{\mu}{\ell}\dot r\right)\frac{dt}{d\theta} = -\frac{\mu}{\ell}\ddot r\frac{1}{\dot\theta} = -\frac{\mu^2}{\ell^2}r^2\ddot r$$

$$\frac{d^2u}{d\theta^2} = \frac{d}{d\theta}\left(\frac{du}{d\theta}\right) = \frac{d}{dt}\left(\frac{du}{d\theta}\right)\frac{dt}{d\theta}$$

$$\Rightarrow \quad \ddot r = -\frac{\ell^2}{\mu^2}\frac{1}{r^2}\frac{d^2u}{d\theta^2} = -\frac{\ell^2}{\mu^2}u^2\frac{d^2u}{d\theta^2}$$

and $r\dot\theta^2 = \frac{\ell^2}{\mu^2}\frac{1}{r^3} = \frac{\ell^2}{\mu^2}u^3$

$$L.E \Rightarrow \quad \frac{d^2u}{d\theta^2} + u = -\frac{\mu}{\ell^2}\frac{1}{u^2}F\left(\frac{1}{u}\right)$$

or : $$\boxed{\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{\ell^2}F(r)} \quad \text{— (11)}$$

useful if we wish to find $F(r)$ given the orbit $r = r(\theta)$.

Example 8.1

logarithmic spiral $r = k e^{\alpha\theta}$ Find the force law.

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{d}{d\theta}\left(\frac{e^{-\alpha\theta}}{k}\right) = -\frac{\alpha e^{-\alpha\theta}}{k}$$

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{d}{d\theta}\left(-\frac{\alpha}{k} e^{-\alpha\theta}\right) = +\frac{\alpha^2}{k} e^{-\alpha\theta} = \frac{\alpha^2}{r}$$

$$\frac{\alpha^2}{r} + \frac{1}{r} = -\frac{\mu r^2}{l^2} F(r)$$

$$\Rightarrow F(r) = -\left(\frac{\alpha^2}{r} + \frac{1}{r}\right)\frac{l^2}{\mu r^2} = -\frac{l^2}{\mu r^3}\left(\alpha^2 + 1\right)$$

Force is attractive inverse cube.

Example 8.2

Write $\underline{\theta(t)}$ and $\underline{r(t)}$.

$$l = \mu r^2 \dot\theta \Rightarrow \dot\theta = \frac{l}{\mu r^2} = \frac{l}{\mu k^2 e^{2\alpha\theta}} = \frac{d\theta}{dt}$$

$$\int e^{2\alpha\theta}\, d\theta = \int \frac{l}{\mu k^2}\, dt$$

$$\frac{1}{2\alpha} e^{2\alpha\theta} = \frac{l t}{\mu k^2} + C$$

$$e^{2\alpha\theta} = \left(\frac{2\alpha l}{\mu k^2}\right) t + \underbrace{2\alpha C'}_{C}$$

$$\boxed{\theta(t) = \frac{1}{2\alpha} \ln\left(\frac{2\alpha l}{\mu k^2} t + C\right)}$$

To get $r(t)$,

$$r = k e^{\alpha\theta} \Rightarrow r^2 = k^2 e^{2\alpha\theta}$$

$$r^2 = k^2\left[\left(\frac{2\alpha l}{\mu k^2}\right) t + C\right]$$

$$\boxed{r(t) = \left[\frac{2\alpha l}{\mu} t + k^2 C\right]^{1/2}}$$

C and l are determined from the initial conditions.

Example 8.3

what is the total energy E of the orbit?

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\frac{l^2}{\mu r^2} - U(r)$$

Need to find $U(r)$ and $\dot{r}$

$$U(r) = -\int \vec{F}\, dr = -\int -\frac{\ell^2}{\mu r^3}(\alpha^2+1)\, dr$$

$$= -\frac{(\alpha^2+1)\ell^2}{2\mu}\frac{1}{r^2}$$

$$\ell = \mu r^2\dot{\theta} \Rightarrow \dot{\theta} = \frac{\ell}{\mu r^2}$$

$$\dot{r} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{d}{d\theta}\left(k e^{\alpha\theta}\right)\dot{\theta} = k\alpha e^{\alpha\theta}\dot{\theta}$$

$$\dot{r} = k\alpha e^{\alpha\theta}\frac{\ell}{\mu r^2} = \frac{\alpha r\ell}{\mu r^2} = \frac{\alpha\ell}{\mu r}$$

$$\Rightarrow E = \frac{1}{2}\mu\frac{\alpha^2\ell^2}{\mu^2 r^2} + \frac{1}{2}\frac{\ell^2}{\mu r^2} - \frac{(\alpha^2+1)\ell^2}{2\mu}\frac{1}{r^2}$$

$$\frac{1}{2}\frac{(\alpha^2+1)\ell^2}{\mu^2 r^2} - \frac{(\alpha^2+1)}{2\mu^2 r^2} = 0$$

The total energy of the orbit is Zero if $U(r=\infty)=0$.

($\dot{\theta}\to 0$ when $r\to\infty$ and $\dot{r}\to 0$ when $r\to\infty$)
since E = constant $\Rightarrow$ it should be zero.

8.5 Orbits in a Central Field

We have found in previous section that

$$\dot{r} = \sqrt{\frac{2}{\mu}(E-U) - \frac{\ell^2}{\mu^2 r^2}}$$

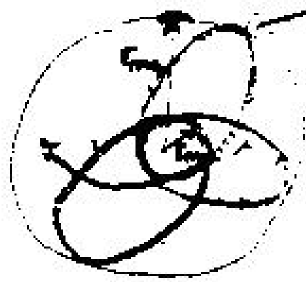
~~when~~ $\dot{r} = 0$ (turning points) then

$$\frac{2}{\mu}\,(E - U) - \frac{\ell^2}{\mu^2 r^2} = 0$$

or $$E - U = \frac{\ell^2}{2\mu r^2} \qquad \text{——} \qquad (12)$$

This equation in general has two roots, r_{max} and r_{min} and the region of possible motion is

$$r_{min} < r < r_{max}$$



For some conditions on ℓ and E, $r_{max} = r_{min}$ and hence the orbit will be __circular__. ($\dot{r} = 0$ and $r = $ const.)

If the motion is periodic, then the __orbit is closed__. that is after a finite excursions between r_{max} and r_{min} the motion exactly repeats itself.

The orbit is said to __be open__ when the motion never repeat itself.

The motion is symmetrical in time $r_{max} \xrightarrow{t_1} r_{min} \xrightarrow{t_2} r_{max}$

$t_1 = t_2$

The angular change $\Delta\theta = 2\Delta\theta(r_{min} \rightarrow r_{max})$

hence
$$\Delta\theta = 2\int_{r_{min}}^{r_{max}} \frac{(\ell^2/r^2)\ dr}{\sqrt{2\mu\left((E-U)-\frac{\ell^2}{2\mu r^2}\right)}} \qquad (13)$$

The path is closed if $\Delta\theta = \left(\frac{a}{b}\right) \cdot 2\pi$ rad where a and b are integers.

Then after $\underline{b}$ periods, the vector $\vec{r}$ will have completed $\underline{a}$ revolutions and have returned to its original position.

* For radial potentials such as $U(r) \propto r^{n+1}$, we can check that closed **orbits** are obtained for $n=1$ and $n=-2$

8.6 Centrifugal Energy and the Effective Potential

In our previous total energy $E = \frac{1}{2}\mu \dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r)$

we can think of $U_{eff} = U(r) + \frac{\ell^2}{2\mu r^2}$ as a new **effective** potential. The term $\frac{\ell^2}{2\mu r^2}$ is called the **centrifugal potential energy**, because we can define an associated

of the particle

Centrifugal force $F = -\frac{\partial}{\partial r}\left(\frac{l^2}{2\mu r^2}\right) = \frac{l^2}{\mu r^3} = \mu r \dot{\theta}^2 = \mu r \omega^2$

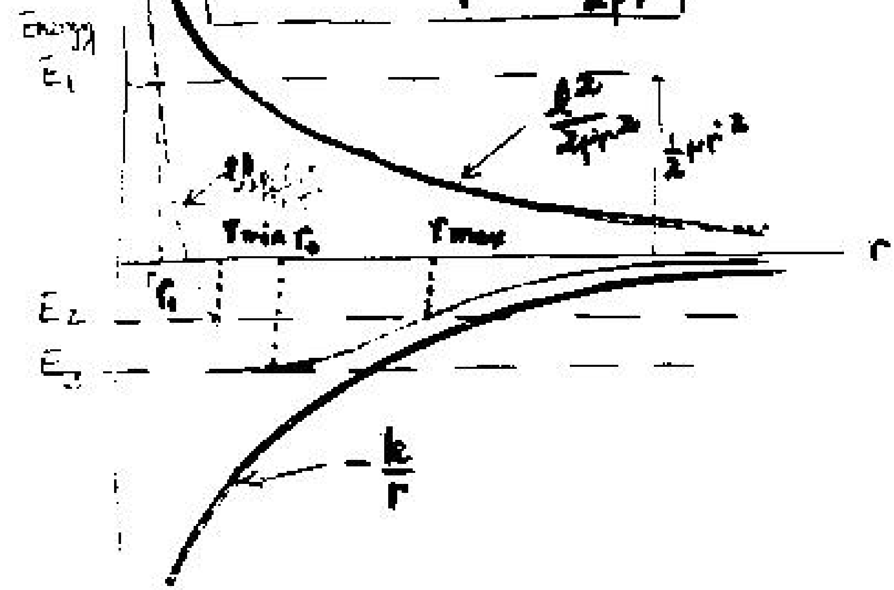
U_{eff} is a fictitious potential that combines the real potential function $U(r)$ with the energy associated with the angular motion about the center of force.

For the case of inverse square law (Coulomb and gravitational forces)

$$F = -\frac{k}{r^2} \Rightarrow U(r) = -\frac{k}{r} \quad (U(\infty) = 0)$$

Then

$$\boxed{U_{eff} = -\frac{k}{r} + \frac{l^2}{2\mu r^2}}$$



The possible types of motion allowed.

① $E_1 \geq 0$. motion is unbounded

Particle moves to force center, reaches r_1 then turns back and moves toward larger r.

$$E = U_{eff} + \frac{1}{2}\mu \dot{r}^2$$

$$\dot{r}^2 = \frac{2}{\mu}(E-U) - \frac{\ell^2}{\mu^2 r^2}$$

$\times \frac{1}{2}\mu$ $\Rightarrow \frac{1}{2}\mu\dot{r}^2 = E - \left(U + \frac{\ell^2}{2\mu r^2}\right) = E - U_{eff}$

Hence when $E = U_{eff} \Rightarrow \dot{r} = 0 \Rightarrow$ turning point!

② $E_2 < 0$ and lies between $U = 0$ and $U = U_{min}$

r_{min}, r_{max} are called apsidal distances of the orbit.

③ $E = E_3 = U_{min}$ Then $r_{max} = r_{min} = C$

orbit is a circle

Values for $E < U_{min}$ result in $\dot{r} < 0$ do not result in physically real motion.