

Phys301 HW solution
Chapter 3

Pb#2. a) see equation (3.41) in the textbook.

$$|x_{en}| = A e^{-\beta t} = \frac{A}{2} \quad A: \text{initial amplitude}$$

$$\text{where } t = 10 \text{ sec} \Rightarrow e^{-10\beta} = \frac{1}{2} \Rightarrow -10\beta = \ln\left(\frac{1}{2}\right)$$

$$+ 10\beta = \ln(2) \Rightarrow \beta = \frac{\ln(2)}{10} = \underline{\underline{0.0693 \text{ sec}^{-1}}}$$

b) $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{10^4}{100}} = 10 \text{ rad/s}$$

$$\Rightarrow \omega_1 = \sqrt{100 - (0.0693)^2} \Rightarrow f_1 = \frac{\omega_1}{2\pi}$$

$$f_1 = \frac{10}{2\pi} (1 - 2.4 \times 10^{-5})$$

$$= \left(\frac{10}{2\pi}\right) \cdot [1 - 2.4 \times 10^{-5}] = f_0 [1 - 2.4 \times 10^{-5}]$$

$$f_1 = f_0 (1 - \delta) \quad \text{where } \delta = 2.4 \times 10^{-5}$$

$$\Rightarrow f_1 \approx f_0 \quad \sin \delta \text{ is small. } (f_1 \approx f_0 = 1.59 \text{ s}^{-1})$$

c) The decrement of the motion is defined to be

$$e^{\beta T_1} = e^{\frac{\beta}{f_1}} = e^{\frac{0.0693 \times 2\pi}{10}} \approx 1.0455.$$

Pb # 4. For simple harmonic oscillator:

The position $x(t) = A \sin \omega_0 t$

The velocity $\dot{x}(t) = \omega_0 A \cos \omega_0 t$ $\omega_0 = \sqrt{\frac{k}{m}}$

The time average of the kinetic energy is

$$\langle T \rangle = \frac{\omega_0}{2\pi} \int_0^{2\pi/\omega_0} \frac{1}{2} m \dot{x}^2 dt$$

$$\langle T \rangle = \frac{\omega_0}{2\pi} \int_0^{2\pi/\omega_0} \frac{1}{2} m \omega_0^2 A^2 \cos^2 \omega_0 t dt$$

$$= \frac{\omega_0^3 m}{4\pi} A^2 \int_0^{2\pi/\omega_0} \cos^2 \omega_0 t dt$$

$$\left(\frac{2\pi}{\omega_0} \right) \frac{1}{2}$$

T (period)

$$\boxed{\langle T \rangle = \frac{\omega_0^2 m A^2}{4}}$$

$$\langle U \rangle = \frac{1}{T} \int_0^T \frac{1}{2} k x^2 dt = \frac{1}{T} \frac{k A^2}{2} \int_0^T \sin^2 \omega_0 t dt$$

$$= \frac{k A^2}{4} = \frac{m A^2 \omega_0^2}{4}$$

$$\boxed{\langle U \rangle = \frac{\omega_0^2 m A^2}{4}}$$

$\Rightarrow \boxed{\langle T \rangle = \langle U \rangle}$ because the system is conservative and the total energy of the system is conserved.

$$E = T + U.$$

$$\Rightarrow \langle T \rangle = \langle U \rangle = \frac{E}{2}$$

b) Space average

kinetic energy: $\bar{T} = \frac{1}{A} \int_0^A \frac{1}{2} m \dot{x}^2 dx$

potential energy: $\bar{U} = \frac{1}{A} \int_0^A \frac{1}{2} k x^2 dx$

$$\begin{aligned} \dot{x}^2 &= \omega_0^2 A^2 \cos^2 \omega_0 t = \omega_0^2 A^2 (1 - \sin^2 \omega_0 t) \\ &= \omega_0^2 (A^2 - x^2) \end{aligned}$$

$$\bar{T} = \frac{m \omega_0^2}{2A} \int_0^A (A^2 - x^2) dx$$

$$= \frac{m \omega_0^2}{2A} \left(A^3 - \frac{A^3}{3} \right) = \frac{2m \omega_0^2 A^3}{6A} = \frac{2m \omega_0^2 A^2}{6}$$

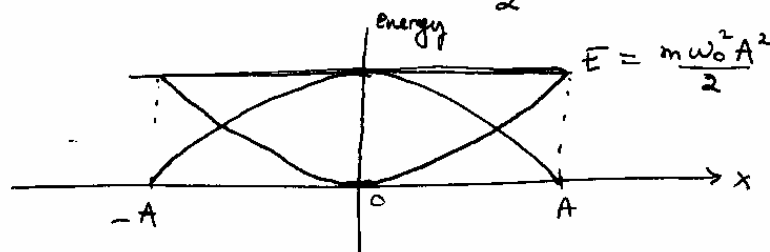
$$\boxed{\bar{T} = \frac{2m \omega_0^2 A^2}{6}}$$

$$\bar{U} = \frac{k}{2A} \int_0^A x^2 dx = \frac{m \omega_0^2}{2A} \frac{A^3}{3} = \frac{m \omega_0^2 A^2}{6A}$$

$$\boxed{\bar{U} = \frac{m \omega_0^2 A^2}{6A}}$$

$$\Rightarrow \boxed{\bar{T} = 2\bar{U}}$$

$$E = T + U = \frac{m \omega_0^2 A^2}{2}$$



Area under T vs. x is twice the

Pb#10.

Take the underdamped situation.

$$X(t) = A e^{-\beta t} \cos(\omega_1 t - \delta)$$

The amplitude $A e^{-\beta t}$ decreases. $A e^{-1}$ of its initial value.

$$A e^{-1} = A e^{-\beta n T}$$

$$\Rightarrow \beta n T = 1 = \beta n \frac{2\pi}{\omega_1} \Rightarrow \omega_1 = 2\pi \beta n$$

● but: $\omega_1^2 = \omega_0^2 - \beta^2$

$$\Rightarrow \omega_0^2 = \beta^2 \left[1 + \frac{1}{4\pi^2 n^2} \right] =$$

$$\omega_1^2 = \omega_0^2 - \frac{\omega_1^2}{4\pi^2 n^2} \Rightarrow \omega_1^2 \left(1 + \frac{1}{4\pi^2 n^2} \right) = \omega_0^2$$

$$\frac{\omega_1}{\omega_0} = \sqrt{1 + \frac{1}{4\pi^2 n^2}} \cong 1 - \frac{1}{8\pi^2 n^2}$$

●

$$\boxed{\frac{\omega_1}{\omega_0} = 1 - (8\pi^2 n^2)^{-1}}$$

Pb# 15.

We need to plot

$$x(t) = A e^{-\beta t} \cos(\omega_0 t - \delta)$$

$$\text{and } v(t) = -A e^{-\beta t} [\beta \cos(\omega_0 t - \delta) + \omega_0 \sin(\omega_0 t - \delta)]$$

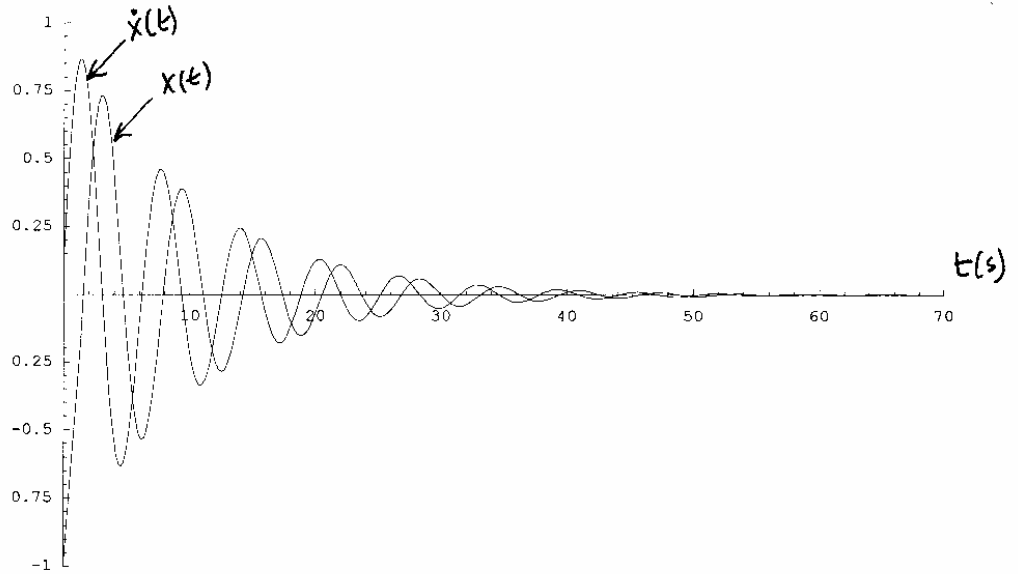
with $A = 1 \text{ cm}$, $\omega_0 = 1 \text{ rad/s}$, $\beta = 0.1 \text{ s}^{-1}$ and $\delta = \pi \text{ rad}$.

The plots (i) are $x(t)$ and $v(t)$ vs. t on the same

graph (ii) $\dot{x}(t)$ vs. $x(t)$ (phase diagram).

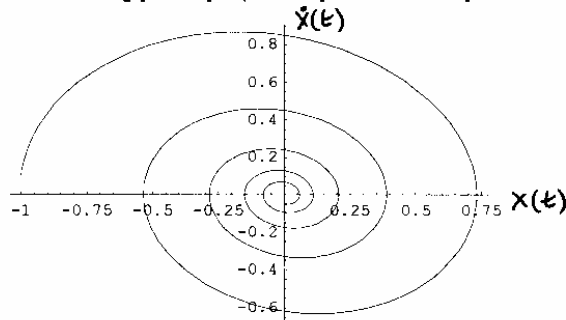
In[7]:=

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(i) a = 1; w0 = 1; b = 0.1; delta = Pi; w1 = Sqrt[w0^2 - b^2]
Plot[{a * Exp[-b * t] * Cos[w1 * t - delta],
      -a * Exp[-b * t] * (b * Cos[w1 * t - delta] + w1 * Sin[w1 * t - delta])},
      {t, 0, 70}, PlotRange -> {{0, 70}, {-1, 1}}]
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Out[8]= - Graphics -

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(ii) In[16]:- ParametricPlot[{a * Exp[-b * t] * Cos[w1 * t - delta],
                              -a * Exp[-b * t] * (b * Cos[w1 * t - delta] + w1 * Sin[w1 * t - delta])}, {t, 0, 30}]
```



Out[16]= - Graphics -

Pb # 21.

We want to plot $x(t) = (A + Bt) e^{-\beta t}$

and $v(t) = [B - \beta(A + Bt)] e^{-\beta t}$

$$x(0) = \boxed{A = x_0}$$

$$v(0) = B - \beta A = v_0 \Rightarrow \boxed{B = v_0 + \beta x_0}$$

We can easily see that as $t \rightarrow \infty$ $x(t) = Bt e^{-\beta t}$

and $v(t) = -\beta Bt e^{-\beta t}$

$$\Rightarrow \cancel{v(t) = -\beta x(t)} \quad \boxed{\dot{x}(t) = -\beta x(t)}$$

This is the equation of a line with slope $-\beta$ on the phase diagram.

If $\beta = 1 \text{ s}^{-1}$ we take as initial conditions:

$$x_0, v_0 = (-2, 4), (1, 4), (-4, -4) \quad (\text{above the line})$$

$$x_0, v_0 = (-1, -4), (1, -4), (-4, 0) \quad (\text{below the line})$$

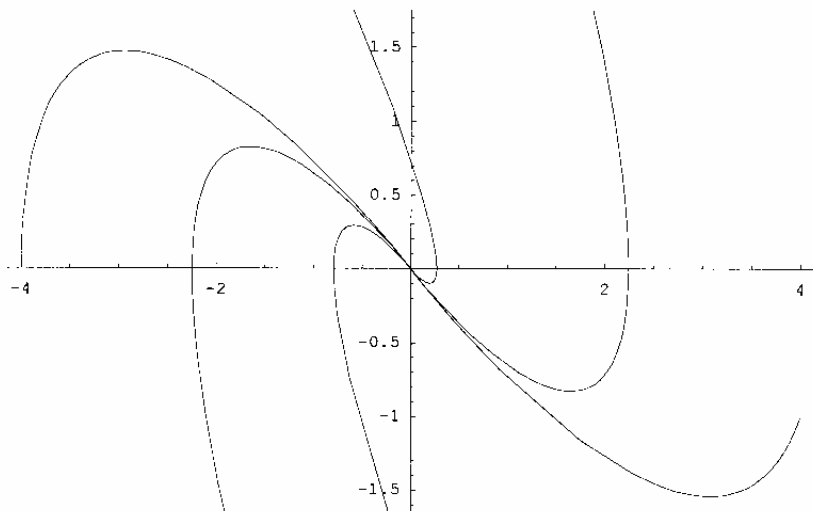
The graphs are shown below.

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In[187]:= x02 = -2; v02 = 4; x03 = 1; v03 = 4; x04 = 4; v04 = -1;
x05 = 1; v05 = -4; x06 = -1; v06 = -4; x07 = -4; v07 = 0; b = 1
x2[t_] := (x02 + (v02 + b*x02) * t) * Exp[-b * t]
v2[t_] := ((v02 + b*x02) - b * (x02 + (v02 + b*x02) * t)) * Exp[-b * t]
x3[t_] := (x03 + (v03 + b*x03) * t) * Exp[-b * t]
v3[t_] := ((v03 + b*x03) - b * (x03 + (v03 + b*x03) * t)) * Exp[-b * t]
x4[t_] := (x04 + (v04 + b*x04) * t) * Exp[-b * t]
v4[t_] := ((v04 + b*x04) - b * (x04 + (v04 + b*x04) * t)) * Exp[-b * t]
x5[t_] := (x05 + (v05 + b*x05) * t) * Exp[-b * t]
v5[t_] := ((v05 + b*x05) - b * (x05 + (v05 + b*x05) * t)) * Exp[-b * t]
x6[t_] := (x06 + (v06 + b*x06) * t) * Exp[-b * t]
v6[t_] := ((v06 + b*x06) - b * (x06 + (v06 + b*x06) * t)) * Exp[-b * t]
x7[t_] := (x07 + (v07 + b*x07) * t) * Exp[-b * t]
v7[t_] := ((v07 + b*x07) - b * (x07 + (v07 + b*x07) * t)) * Exp[-b * t]
ParametricPlot[{{x2[t], v2[t]}, {x3[t], v3[t]}, {x4[t], v4[t]},
{x5[t], v5[t]}, {x6[t], v6[t]}, {x7[t], v7[t]}], {t, 0, 40}]

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Out[187]= 1



Out[200]= - Graphics -

Pb #22.

a) For the overdamped oscillator

$$x(t) = e^{-\beta t} [A_1 e^{\omega_2 t} + A_2 e^{-\omega_2 t}]$$
$$= A_1 e^{-\beta_1 t} + A_2 e^{-\beta_2 t}$$

where $\beta_1 = \beta - \omega_2$ and $\beta_2 = \beta + \omega_2$.

and $v(t) = -\beta_1 A_1 e^{-\beta_1 t} - \beta_2 A_2 e^{-\beta_2 t}$

at $t=0$ $x(0) = x_0$ and $v(0) = v_0$

$$\Rightarrow x_0 = A_1 + A_2 \quad \text{--- (1)}$$

$$v_0 = -\beta_1 A_1 - \beta_2 A_2 \quad \text{--- (2)}$$

$$(1) \Rightarrow A_2 = x_0 - A_1$$

$$(2) \Rightarrow v_0 = -\beta_1 A_1 - \beta_2 (x_0 - A_1) = A_1 (\beta_2 - \beta_1) - \beta_2 x_0$$

$$\Rightarrow \boxed{A_1 = \frac{v_0 + \beta_2 x_0}{\beta_2 - \beta_1}}$$

$$\text{and } A_2 = \frac{x_0(\beta_2 - \beta_1) + v_0 + \beta_2 x_0}{\beta_2 - \beta_1} = \boxed{\frac{x_0 \beta_1 + v_0}{\beta_2 - \beta_1}}$$

b) when $A_1 = 0$ and $A_2 \neq 0$

$$x(t) = - \frac{\beta_2 x_0 + v_0}{\beta_2 - \beta_1} e^{-\beta_2 t}$$

$$\dot{x}(t) = + \frac{\beta_1 x_0 + v_0}{\beta_2 - \beta_1} \beta_2 e^{-\beta_2 t} = -\beta_2 x(t) \Rightarrow \boxed{\dot{x}(t) = -\beta_2 x(t)}$$

So the phase paths are along the dashed line z with slope $-\beta_1$.
(Fig. 3(10))

Otherwise ($A_1 \neq 0$ and $A_2 \neq 0$)

$$x(t) = A_1 e^{-\beta_1 t} + A_2 e^{-\beta_2 t}$$

$$t \rightarrow \infty \quad x(t) = A_1 e^{-\beta_1 t} \quad \text{because } \beta_2 > \beta_1.$$

$$\dot{x}(t) = -\beta_1 A_1 e^{-\beta_1 t} - \beta_2 A_2 e^{-\beta_2 t}$$

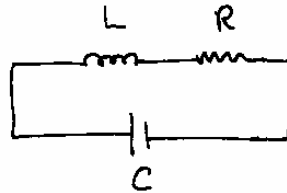
$$t \rightarrow \infty \quad \dot{x}(t) \approx e^{-\beta_2 t} \rightarrow 0$$

$$\dot{x}(t) = -\beta_1 A_1 e^{-\beta_1 t} = -\beta_1 x(t)$$

$$\boxed{\dot{x}(t) = -\beta_1 x(t)}$$

The phase paths are along the other dashed line with slope $-\beta_1$.

Pb # 27.



$$L = 0.1 \text{ H}$$

$$C = 10 \text{ pF}$$

$$R = 100 \text{ } \Omega$$

The circuit equation is

$$L \ddot{q} + R \dot{q} + \frac{1}{C} q = 0 \quad \text{--- (1)}$$

This is similar to the damped mechanical oscillations

$$m \ddot{x} + b \dot{x} + kx = 0$$

$$\text{or } \ddot{x} + 2\beta \dot{x} + \omega_0^2 x = 0 \quad \text{--- (2)}$$

$$\text{Compare (1) \& (2)} \Rightarrow 2\beta = \frac{R}{L} \Rightarrow \beta = \frac{R}{2L}$$

$$\text{and } \omega_0^2 = \frac{1}{LC}$$

$$\omega_1^2 = \omega_0^2 - \beta^2 = \frac{1}{LC} - \frac{R^2}{4L^2} = 75 \times 10^4 \text{ (s}^{-1}\text{)}^2$$

since $\omega_0^2 > \beta^2$ we have underdamped situation.

$$\boxed{\omega_1 = 866 \text{ s}^{-1}}$$