Phys301 HW solution
Chapter 3

Pb#2. a) see equation (3.41) in the textbook.

$$|x_{en}| = A e^{-\beta t} = \frac{A}{2} \qquad A: \text{initial amplitude}$$

where $t = 10\,\text{sec}$ $\Rightarrow$ $e^{-10\beta} = \frac{1}{2} \Rightarrow -10\beta = \ln\left(\frac{1}{2}\right)$

$$+10\beta = \ln(2) \Rightarrow \beta = \frac{\ln(2)}{10} = \underline{\underline{0.0693}}\ \text{sec}^{-1}$$

b) $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{10^4}{100}} = 10\ \text{rad/s}$$

$$\Rightarrow \quad \omega_1 = \sqrt{100 - (0.0693)^2} \Rightarrow f_1 = \frac{\omega_1}{2\pi}$$

$$f_1 = \frac{10}{2\pi}\left(1 - 2.4\times10^{-5}\right)$$

$$= \left(\frac{10}{2\pi}\right)\cdot\left[1 - 2.4\times10^{-5}\right] = f_0\left[1 - 2.4\times10^{-5}\right]$$

$$f_1 = f_0(1 - \delta) \qquad \text{where } \delta = 2.4\times10^{-5}$$

$\Rightarrow f_1 \simeq f_0$ sin δ is small. $\left(f_1 \simeq f_0 \simeq 1.59\ s^{-1}\right)$

c) The decrement of the motion is defined to be

$$e^{\beta T_1} = e^{\frac{\beta}{f_1}} = e^{\frac{0.0693 \times 2\pi}{10}} \simeq 1.0455.$$

Pb # 4. For simple harmonic oscillator:

The position $x(t) = A \sin \omega_0 t$

The velocity $\dot{x}(t) = \omega_0 A \cos \omega_0 t$ $\omega_0 = \sqrt{\frac{k}{m}}$.

The time average of the kinetic energy is

$$\langle T \rangle = \frac{\omega_0}{2\pi} \int_0^{2\pi/\omega_0} \frac{1}{2} m \dot{x}^2 \, dt$$

$$\langle T \rangle = \frac{\omega_0}{2\pi} \int_0^{2\pi/\omega_0} \frac{1}{2} m \omega_0^2 A^2 \cos^2 \omega_0 t \, dt$$

$$= \frac{\omega_0^3 m}{4\pi} A^2 \underbrace{\int_0^{2\pi/\omega_0} \cos^2 \omega_0 t \, dt}_{\left(\underbrace{\frac{2\pi}{\omega_0}}_{T\,(\text{period})}\right)\frac{1}{2}}$$

$$\boxed{\langle T \rangle = \frac{\omega_0^2 m A^2}{4}}$$

$$\langle u \rangle = \frac{1}{T} \int_0^T \frac{1}{2} k x^2 \, dt = \frac{1}{T} \frac{k A^2}{2} \underbrace{\int_0^T \sin^2 \omega_0 t \, dt}_{\frac{T}{2}}$$

$$= \frac{k A^2}{4} = \frac{m A^2 \omega_0^2}{4}$$

$$\boxed{\langle u \rangle = \frac{\omega_0^2 m A^2}{4}}$$

$\Rightarrow \boxed{\langle T \rangle = \langle u \rangle}$ because the system is conservative and the total energy of the system is conserved.

$$E = T + u.$$

$$\Rightarrow \qquad \langle T \rangle = \langle U \rangle = \frac{E}{2}$$

b) Space average

Kinetic energy: $\overline{T} = \frac{1}{A}\int_0^A \frac{1}{2} m \dot{x}^2 \, dx$

potential energy: $\overline{U} = \frac{1}{A}\int_0^A \frac{1}{2} k x^2 \, dx$

$$\dot{x}^2 = \omega_0^2 A^2 \cos^2 \omega_0 t = \omega_0^2 A^2 (1 - \sin^2 \omega_0 t)$$
$$= \omega_0^2 \cancel{A}(A^2 - x^2)$$

$$\overline{T} = \frac{m\omega_0^2}{2A}\int_0^A (A^2 - x^2)\, dx$$

$$= \frac{m\omega_0^2}{2A}\left(A^3 - \frac{A^3}{3}\right) = \frac{2m\omega_0^2 A^3}{6A} = \frac{2m\omega_0^2 A^2}{6}$$

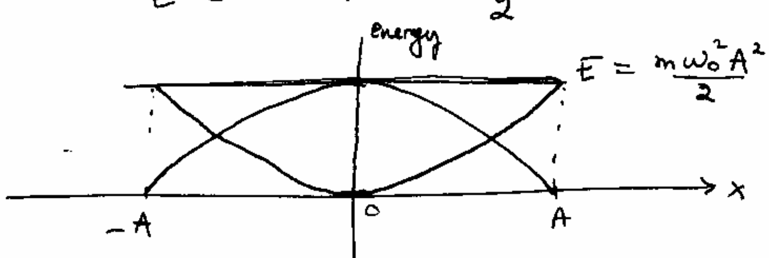
$$\boxed{\overline{T} = \frac{2m\omega_0^2 A^2}{6}}$$

$$\overline{U} = \frac{k}{2A}\int_0^A x^2\, dx = \frac{m\omega_0^2}{2A}\frac{A^3}{3} = \frac{m\omega_0^2 A^2}{6A}$$

$$\boxed{\overline{U} = \frac{m\omega_0^2 A^2}{6A}}$$

$$\Rightarrow \boxed{\overline{T} = 2\overline{U}}$$

$$E = T + U = \frac{m\omega_0^2 A^2}{2}.$$



Area under T vs. x is twice the

Pb# 10. Take the underdamped situation.

$$X(t) = A e^{-\beta t} \cos(\omega_1 t - \delta)$$

The amplitude $A e^{-\beta t}$ decreases. $A e^{-1}$ of its initial value.

$$A e^{-1} = A e^{-\beta n T}$$

$$\Rightarrow \quad \beta n T = 1 = \beta n \frac{2\pi}{\omega_1} \Rightarrow \omega_1 = 2\pi \beta n$$

but: $\omega_1^2 = \omega_0^2 - \beta^2$

$$\Rightarrow \quad \cancel{\omega_0^2 = \beta^2 \left[1 + 4\pi^2 n^2\right]} =$$

$$\omega_1^2 = \omega_0^2 - \frac{\omega_1^2}{4\pi^2 n^2} \Rightarrow \omega_1^2 \left(1 + \frac{1}{4\pi^2 n^2}\right) = \omega_0^2$$

$$\frac{\omega_1}{\omega_0} = \sqrt{1 + \frac{1}{4\pi^2 n^2}} \cong 1 - \frac{1}{8\pi^2 n^2}.$$

$$\boxed{\frac{\omega_1}{\omega_0} = 1 - \left(8\pi^2 n^2\right)^{-1}}$$

Pb# 15.

We need to plot

$$X(t) = A\,e^{-\beta t} \cos(\omega_1 t - \delta)$$

and $V(t) = -A\,e^{-\beta t}\left[\beta \cos(\omega_1 t - \delta) + \omega_1 \sin(\omega t - \delta)\right]$

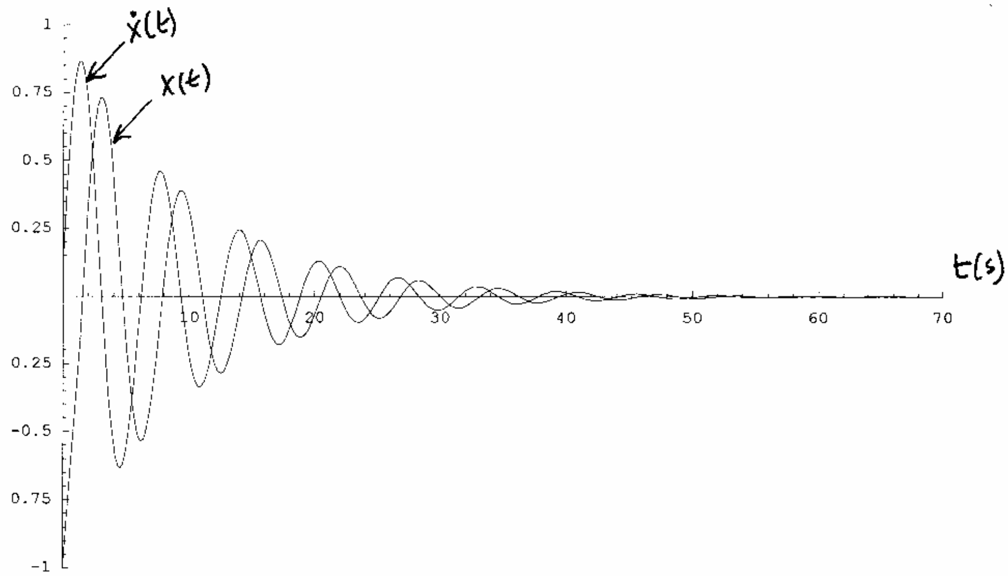
with $A = 1$ cm, $\omega_0 = 1$ rad/s, $\beta = 0.1\ s^{-1}$ and $\delta = \pi$ rad.

The plots $\underline{(i)}$ are $X(t)$ and $V(t)$ vs. t on the same graph $\underline{(ii)}$ $\dot{X}(t)$ vs. $X(t)$ (phase diagram).

In[7]:=

(i)

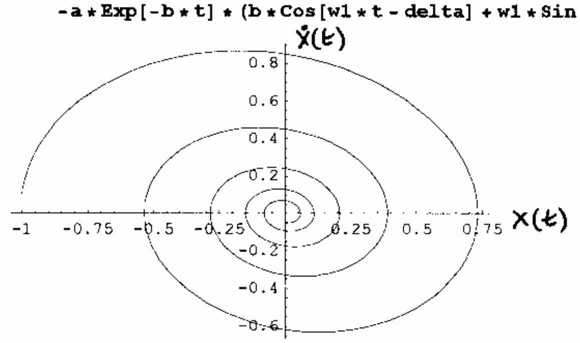
```
a = 1; w0 = 1; b = 0.1; delta = Pi; w1 = √(w0^2 - b^2)
Plot[{a * Exp[-b * t] * Cos[w1 * t - delta],
   -a * Exp[-b * t] * (b * Cos[w1 * t - delta] + w1 * Sin[w1 * t - delta])},
  {t, 0, 70}, PlotRange -> {{0, 70}, {-1, 1}}]
```



Out[8]= - Graphics -

(ii) In[16]:=

```
ParametricPlot[{a * Exp[-b * t] * Cos[w1 * t - delta],
   -a * Exp[-b * t] * (b * Cos[w1 * t - delta] + w1 * Sin[w1 * t - delta])}, {t, 0, 30}]
```



Out[16]= - Graphics -

Pb # 21.

We want to plot $X(t) = (A + Bt)\, e^{-\beta t}$

and $V(t) = \left[B - \beta(A+Bt)\right] e^{-\beta t}$

$X(0) = \boxed{A = X_o}$

$V(0) = B - \beta A = V_o \implies \boxed{B = V_o + \beta X_o}$

We can easily see that as $t \to \infty$ $X(t) = Bt\, e^{-\beta t}$

and $V(t) = -\beta B t\, e^{-\beta t}$

$\implies \cancel{V(t) = -\beta X(t)}\ \boxed{\dot{X}(t) = -\beta X(t)}$

- This is the equation of a line with slope $-\beta$ on the phase diagram.

If $\beta = 1\ s^{-1}$ we take as initial conditions:

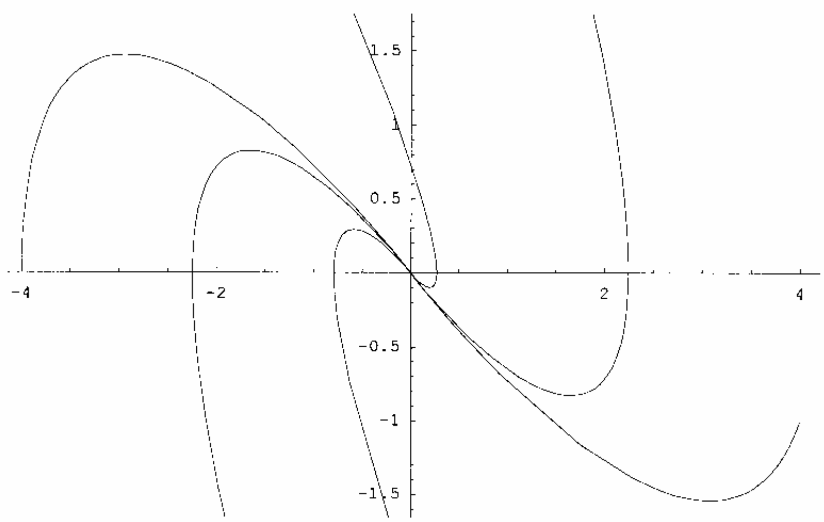
$X_o, V_o = (-2, 4), (1, 4), (4, -4)$ (above the line)

$X_o, V_o = (-1, -4), (1, -4), (-4, 0)$ (below the line)

- The graphs are shown below.

```
In[187]:= x02 = -2; v02 = 4; x03 = 1; v03 = 4; x04 = 4; v04 = -1;
          x05 = 1; v05 = -4; x06 = -1; v06 = -4; x07 = -4; v07 = 0; b = 1
          x2[t_] := (x02 + (v02 + b * x02) * t) * Exp[-b * t]
          v2[t_] := ((v02 + b * x02) - b * (x02 + (v02 + b * x02) * t)) * Exp[-b * t]
          x3[t_] := (x03 + (v03 + b * x03) * t) * Exp[-b * t]
          v3[t_] := ((v03 + b * x03) - b * (x03 + (v03 + b * x03) * t)) * Exp[-b * t]
          x4[t_] := (x04 + (v04 + b * x04) * t) * Exp[-b * t]
          v4[t_] := ((v04 + b * x04) - b * (x04 + (v04 + b * x04) * t)) * Exp[-b * t]
          x5[t_] := (x05 + (v05 + b * x05) * t) * Exp[-b * t]
          v5[t_] := ((v05 + b * x05) - b * (x05 + (v05 + b * x05) * t)) * Exp[-b * t]
          x6[t_] := (x06 + (v06 + b * x06) * t) * Exp[-b * t]
          v6[t_] := ((v06 + b * x06) - b * (x06 + (v06 + b * x06) * t)) * Exp[-b * t]
          x7[t_] := (x07 + (v07 + b * x07) * t) * Exp[-b * t]
          v7[t_] := ((v07 + b * x07) - b * (x07 + (v07 + b * x07) * t)) * Exp[-b * t]
          ParametricPlot[{{x2[t], v2[t]}, {x3[t], v3[t]}, {x4[t], v4[t]},
            {x5[t], v5[t]}, {x6[t], v6[t]}, {x7[t], v7[t]}}, {t, 0, 40}]

Out[187]= 1
```




```
Out[200]= - Graphics -
```

Pb # 22.

a) For the overdamped oscillator

$$x(t) = e^{-\beta t}\left[A_1 e^{\omega_2 t} + A_2 e^{-\omega_2 t}\right]$$

$$= A_1 e^{-\beta_1 t} + A_2 e^{-\beta_2 t}$$

where $\beta_1 = \beta - \omega_2$ and $\beta_2 = \beta + \omega_2$.

and $$v(t) = -\beta_1 A_1 e^{-\beta_1 t} - \beta_2 A_2 e^{-\beta_2 t}$$

at $t=0$ $x(0) = x_0$ and $v(0) = V_0$

$$\Rightarrow \quad x_0 = A_1 + A_2 \qquad \text{— (1)}$$

$$V_0 = -\beta_1 A_1 - \beta_2 A_2 \qquad \text{— (2)}$$

$$(1) \Rightarrow A_2 = x_0 - A_1$$

$$(2) \Rightarrow V_0 = -\beta_1 A_1 - \beta_2(x_0 - A_1) = A_1(\beta_2 - \beta_1) - \beta_2 x_0$$

$$\Rightarrow \boxed{A_1 = \frac{V_0 + \beta_2 x_0}{\beta_2 - \beta_1}}$$

and $$A_2 = \frac{x_0(\beta_2 - \beta_1) - V_0 - \beta_2 x_0}{\beta_2 - \beta_1} = \boxed{-\frac{x_0\beta_1 + V_0}{\beta_2 - \beta_1}}$$

b) when $A_1 = 0$ and $A_2 \neq 0$

$$x(t) = -\frac{\beta_1 x_0 + V_0}{\beta_2 - \beta_1} e^{-\beta_2 t}$$

$$\dot{x}(t) = +\frac{\beta_1 x_0 + V_0}{\beta_2 - \beta_1}\beta_2 e^{-\beta_2 t} = -\beta_2 x(t) \Rightarrow \boxed{\dot{x}(t) = -\beta_2 x(t)}$$

So the phase paths are along the dashed line ~~in~~ with slope[illegible]

(Fig. 3.10)

otherwise ($A_1 \neq 0$ and $A_2 \neq 0$)

$$x(t) = A_1 e^{-\beta_1 t} + A_2 e^{-\beta_2 t}$$

$$t \to \infty \qquad x(t) = A_1 e^{-\beta_1 t} \qquad \text{because } \beta_2 > \beta_1 .$$

$$\dot{x}(t) = -\beta_1 A_1 e^{-\beta_1 t} - \beta_2 A_2 e^{-\beta_2 t}$$

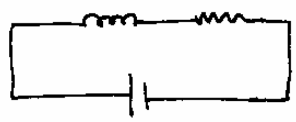
$$t \to \infty \qquad e^{-\beta_2 t} \to 0$$

$$\dot{x}(t) = -\beta_1 A_1 e^{-\beta_1 t} = -\beta_1 x(t)$$

$$\boxed{\dot{x}(t) = -\beta_1 x(t)}$$

The phase paths are along the other dashed line with slope $-\beta_1$.

Pb # 27.



$L = 0.1\ H$

$C = 10\ \mu F$

$R = 100\ \Omega$

The circuit equation is

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = 0 \qquad \text{—— (1)}$$

This is similar to the damped mechanical oscillations

$$m\ddot{x} + b\dot{x} + kx = 0$$

$$\text{or} \quad \ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \qquad \text{—— (2)}$$

Compare (1) & (2) $\Rightarrow \quad 2\beta = \frac{R}{L} \Rightarrow \beta = \frac{R}{2L}$

$$\text{and} \quad \omega_0^2 = \frac{1}{LC}$$

$$\omega_1^2 = \omega_0^2 - \beta^2 = \frac{1}{LC} - \frac{R^2}{4L^2} = 75 \times 10^4 \ \cancel{\text{rad}}(s^{-1})^2$$

since $\omega_0^2 > \beta^2$ we have underdamped situation.

$$\boxed{\omega_1 = 866\ \cancel{\text{Hz}}\ s^{-1}}$$