KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
Physics Department

PHYS 301
First Major Exam (Saturday 22 October, 2005) – Term 051

Instructor's Name: Dr. A. Mekki

Student's Name: ______________________________

I.D. No: ____________________

Exam Time: 90 minutes
Show the details of your work

Problem #	Grade
1	/10
2	/10
3	/10
4	/10
5	/10
Total:	/50
Total:	/20

Q.1

(a) Calculate $\frac{d}{dt}\left[\vec{r}\cdot\left(\vec{v}\times\vec{a}\right)\right]$, where r, v, and a are the position, velocity and acceleration of a particle.

(b) Show that $\vec{\nabla} f(r) = \frac{\vec{r}}{r}\frac{df}{dr}$

(a) $$\frac{d}{dt}\left[\vec{r}\cdot(\vec{v}\times\vec{a})\right] = \frac{d\vec{r}}{dt}\cdot(\vec{v}\times\vec{a}) + \vec{r}\cdot\left(\frac{d\vec{v}}{dt}\times\vec{a}\right) + \vec{r}\cdot\left(\vec{v}\times\frac{d\vec{a}}{dt}\right)$$

$$= \vec{v}\cdot(\vec{v}\times\vec{a}) + \vec{r}\cdot\left(\cancelto{0}{\vec{a}\times\vec{a}}\right) + \vec{r}\cdot\left(\vec{v}\times\frac{d\vec{a}}{dt}\right)$$

$\Downarrow$

$$= \vec{a}\cdot\left(\cancelto{0}{\vec{v}\times\vec{v}}\right) + \vec{r}\cdot\left(\vec{v}\times\frac{d\vec{a}}{dt}\right) = \vec{r}\cdot(\vec{v}\times\dot{\vec{a}})$$

(b) $$\vec{\nabla} f(r) = \sum_i \frac{\partial f(r)}{\partial x_i}\,\hat{e}_i$$

$$= \sum_i \frac{\partial f}{\partial r}\frac{\partial r}{\partial x_i}\,\hat{e}_i$$

$$= \sum_i \frac{\partial f}{\partial r}\frac{\partial\left(\sum_j x_j^2\right)^{1/2}}{\partial x_i}\,\hat{e}_i$$

$$= \frac{\partial f}{\partial r}\sum_i \frac{1}{2}\,2x_i\,\frac{1}{\left(\sum_j x_j^2\right)^{1/2}}\,\hat{e}_i$$

$$= \frac{\partial f}{\partial r}\frac{1}{r}\sum_i x_i\hat{e}_i = \frac{\partial f}{\partial r}\frac{1}{r}\vec{r}$$

$$\Rightarrow \quad \vec{\nabla} f(r) = \frac{\partial f}{\partial r}\frac{\vec{r}}{r}$$

Q.2 A particle of mass m is thrown vertically upward, with an initial speed v_0. It experiences a frictional force proportional to v^2, i.e., $f_r = -\alpha v^2$.

(a) Write Newton's second law and integrate it to find $v(t)$. (set $\beta = \frac{mg}{\alpha}$)

(b) Find the time needed to reach the maximum height H and study the limit when $\alpha \to 0$ (no air resistance).

Given that:

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$$

$$\tan^{-1} x \cong x \; as \; x \to 0$$

(a) $m \frac{dv}{dt} = -mg - \alpha v^2$ $\quad \frac{m}{\alpha} dv = -\left(\frac{mg}{\alpha} + v^2\right) dt$

$$\Rightarrow \int_{v_0}^{v} \frac{dv}{\frac{mg}{\alpha} + v^2} = -\frac{\alpha}{m}\int_0^t dt \qquad \text{let } \beta = \frac{mg}{\alpha}$$

$$\Rightarrow \int_{v_0}^{v} \frac{dv}{\beta + v^2} = -\frac{\alpha}{m} t = \frac{1}{\sqrt{\beta}} \tan^{-1}\left(\frac{v}{\sqrt{\beta}}\right)\Big|_{v_0}^{v}$$

$$\Rightarrow \tan^{-1}\frac{v}{\sqrt{\beta}} - \tan^{-1}\frac{v_0}{\sqrt{\beta}} = -\frac{\alpha\sqrt{\beta}}{m} t$$

$$\tan^{-1}\frac{v}{\sqrt{\beta}} = \tan^{-1}\frac{v_0}{\sqrt{\beta}} - \frac{\alpha\sqrt{\beta}}{m} t$$

$$\frac{v}{\sqrt{\beta}} = \tan\left[\tan^{-1}\frac{v_0}{\sqrt{\beta}} - \frac{\alpha\sqrt{\beta}}{m} t\right]$$

$$\boxed{v(t) = \sqrt{\beta}\tan\left[\tan^{-1}\frac{v_0}{\sqrt{\beta}} - \frac{\alpha\sqrt{\beta}}{m} t\right]}$$

b) Maximum height $\Rightarrow v(t) = 0$

$$\tan \gamma = 0 \Rightarrow \gamma = 0$$

$$\Rightarrow \tan^{-1}\frac{v_0}{\sqrt{\beta}} - \frac{\alpha\sqrt{\beta}}{m} t = 0 \Rightarrow \boxed{t = \frac{m}{\alpha\sqrt{\beta}}\tan^{-1}\frac{v_0}{\sqrt{\beta}}}$$

No air resistance $\alpha \to 0$ $\beta \to \infty$

let $x = \frac{v_0}{\beta}$ when $\beta \to \infty$ $x \to 0$

$\tan^{-1}(x) = x$ as $x \to 0$

$$\tan^{-1}\left(\frac{v_0}{\sqrt{\beta}}\right) = \frac{v_0}{\sqrt{\beta}}$$

$$\Rightarrow t = \frac{m}{\alpha\sqrt{\beta}} \times \frac{v_0}{\sqrt{\beta}} = \frac{m v_0}{\alpha \beta} = \frac{m v_0}{\alpha \frac{mg}{\alpha}} = \frac{v_0}{g}$$

which is the result expected when there is no air resistance.

Q.3 Consider a particle moving in the region $x > 0$ under the influence of a potential

$$U(x) = U_0\left(\frac{a}{x} + \frac{x}{a}\right)$$

Where $U_0 = 1$ J and $a = 2$ m., find

(a) the equilibrium points, and determine whether they are stable or unstable.

(b) Plot the potential U(x).

a) Equilibrium points are found by setting $\frac{dU}{dx} = 0$

$$\Rightarrow \quad U_0\left(-\frac{a}{x^2} + \frac{1}{a}\right) = 0 \quad \Rightarrow \quad x^2 = a^2 \quad \Rightarrow \boxed{x = a}$$

stability :

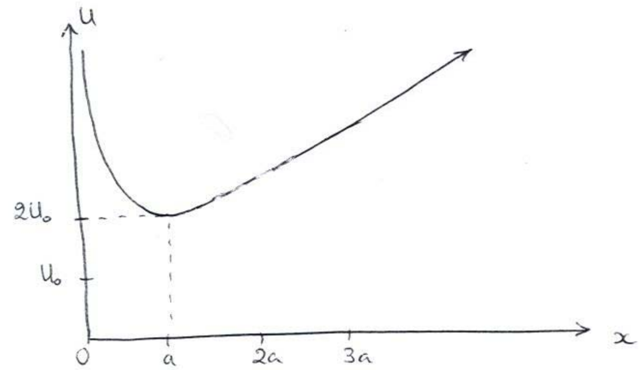
$$\frac{d^2U}{dx^2} = U_0\left(\frac{2a}{x^3}\right)\Bigg|_{x=a} = \frac{2U_0}{a^2} > 0$$

$\Rightarrow$ Stable equilibrium

b) $U(a) = 2U_0$

$x \to +\infty \qquad U \to +\infty$

$x \to 0 \qquad U \to +\infty$



Q.4 Consider a single stage rocket taking off from Earth under the influence of gravity. Find

(a) the velocity of the rocket as a function of its initial mass m_0, final mass m, exhaust speed u and $\alpha = -\frac{dm}{dt}$, the fuel burn rate)

(b) its position as a function of time.

a) The change in momentum $P_f - P_i = m\,dv + u\,dm = F_{ext}\,dt$

$$\Rightarrow \quad m\,dv + u\,dm = -mg\,dt$$

$$\int_0^v dv = -\frac{u}{m}\int_{m_0}^{m} dm - g\int_0^t dt \Rightarrow v = -gt - u\ln\left(\frac{m}{m_0}\right)$$

$$\Rightarrow \quad \boxed{v(t) = -gt + u\ln\left(\frac{m_0}{m}\right)}$$

b) $$v = \frac{dy}{dt} \Rightarrow \int_0^y dy = \int_0^t \left[-gt + u\ln\left(\frac{m_0}{m}\right)\right] dt$$

$$y = -\frac{1}{2}gt^2 - \frac{u}{\alpha}\int_{m_0}^{m} \ln\left(\frac{m_0}{m}\right) dm$$

$$= -\frac{1}{2}gt^2 - \frac{u}{\alpha}\left[m\left(1 + \ln\left(\frac{m_0}{m}\right)\right)\right]_{m_0}^{m}$$

$$= -\frac{1}{2}gt^2 - \frac{u}{\alpha}\left[m\left(1 + \ln\left(\frac{m_0}{m}\right)\right) - m_0\right]$$

$$= -\frac{1}{2}gt^2 - \frac{u}{\alpha}(m - m_0) - \frac{um}{\alpha}\ln\left(\frac{m_0}{m}\right)$$

$$\boxed{y(t) = -\frac{1}{2}gt^2 + ut - \frac{um}{\alpha}\ln\left(\frac{m_0}{m}\right)}$$

note : $\alpha = -\frac{dm}{dt}$ $\qquad -\int_{m_0}^{m} \frac{dm}{\alpha} = \int_0^t dt$

$$t = -\frac{(m - m_0)}{\alpha}$$

Q.5 A force $\vec{F} = 4x^2\hat{i} + 2y\hat{j} + 2\hat{k}$ acts on a 5 kg particle changing only its kinetic energy.

(a) How much work is done on the particle as it moves from coordinates (2 m, 5 m , 3 m) to (5 m, 0 m , -4 m).

(b) What is the final speed of the particle if it starts from rest.

a) $W = \int \vec{F} \cdot d\vec{r}$

$= \int (4x^2\hat{i} + 2y\hat{j} + 2\hat{k}) \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$

$= \int_2^5 4x^2\,dx + \int_5^0 2y\,dy + \int_3^{-4} 2\,dz$

$= 4\frac{x^3}{3}\Big|_2^5 + 2\frac{y^2}{2}\Big|_5^0 + 2z\Big|_3^{-4} =$

$= \frac{4}{3}(5^3 - 2^3) + (0 - 5^2) + 2(-4 - 3)$

$= 156 - 39 = \boxed{117\text{ J}}$

b) $\Delta K = W \Rightarrow \frac{1}{2} m v_f^2 - 0 = 117$

$\Rightarrow v_f^2 = 46.8\ m^2/s^2 \Rightarrow \boxed{v_f = 6.8\ m/s}$