KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
Physics Department

PHYS 301
First Major Exam (March 24, 2001) – Term 002

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Student's Name: Key

I.D. No: ______________

Exam Time: 90 minutes
Show the details of your work

Problem #	Grade
1	/10
2	/10
3	/10
4	/10
5	/10
Total:	/50

Q.1

(a) Prove the following vectorial relation: $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}^2\vec{A}$

(b) Consider the following matrices

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 1 & 3 \end{pmatrix}$$

calculate:

(i) $|AB|$

(ii) $AB - B^t A^t$

(a) $$\left[\vec{\nabla}\times\vec{\nabla}\times\vec{A}\right]_i = \sum_{j,k} \varepsilon_{ijk}\, \nabla_j\, (\vec{\nabla}\times\vec{A})_k = \sum_{j,k} \varepsilon_{ijk}\, \nabla_j \sum_{\ell,m} \varepsilon_{k\ell m} \nabla_\ell A_m$$

$$= \sum_{j,k} \sum_{\ell,m} \varepsilon_{ijk}\, \varepsilon_{k\ell m}\, \nabla_j\, \nabla_\ell\, A_m$$

$$= \sum_{j,k} \sum_{\ell,m} \left(\delta_{i\ell}\, \delta_{jm} - \delta_{im}\, \delta_{j\ell} \right) \nabla_j\, \nabla_\ell\, A_m$$

$$= \sum_{\substack{j,k \\ \ell,m}} \delta_{i\ell}\, \delta_{jm}\, \nabla_j\, \nabla_\ell\, A_m - \delta_{im}\, \delta_{j\ell}\, \nabla_j\, \nabla_\ell\, A_m$$

$$= \nabla_i\, (\vec{\nabla}_m A_m) - \nabla_\ell \nabla_\ell\, A_m$$

$$= \nabla_i\, (\nabla A)_m - \nabla_\ell^2\, A_m$$

In general: $$\vec{\nabla}\times\vec{\nabla}\times\vec{A} = \vec{\nabla}\cdot(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$$

(b) $$AB = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 1 & 3 \end{pmatrix}$$

$$|AB| = 1 \begin{vmatrix} -2 & 9 \\ 3 & 3 \end{vmatrix} - (-2) \begin{vmatrix} 1 & 9 \\ 5 & 3 \end{vmatrix} + 1 \begin{vmatrix} 1 & -2 \\ 5 & 3 \end{vmatrix} = -33 - 84 + 13$$

$$= \underline{\underline{-104}}$$

$$B^t A^t = (AB) = \begin{pmatrix} -2 & -2 & 3 \\ 1 & 9 & 3 \end{pmatrix}$$

$$AB - (AB)^t = \begin{pmatrix} 0 & -3 & -4 \\ 3 & 0 & 6 \\ 4 & -6 & 0 \end{pmatrix}$$

Q.2

Consider the following force $\vec{F} = 2xz\,\hat{i} + 2yz\,\hat{j} + (x^2 + y^2)\,\hat{k}$.

(a) Calculate $\vec{\nabla} \times \vec{F}$

(b) Find potential energy, U(x,y,z), associated with this force.

(a) $$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz & 2yz & (x^2+y^2) \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(x^2+y^2) - \frac{\partial}{\partial z}(2yz)\right] - \hat{j}\left[\frac{\partial}{\partial x}(x^2+y^2) - \frac{\partial}{\partial z}(2xz)\right] + \hat{k}\left[\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial y}(2xz)\right]$$

$$= \hat{i}\,(2y - 2y) - \hat{j}\,(2x - 2x) + \hat{k}\,(0 - 0) = 0$$

Therefore the force is conservative

(b) $$2xz\,\hat{i} + 2yz\,\hat{j} + (x^2+y^2)\,\hat{k} = -\frac{\partial U}{\partial x}\hat{i} - \frac{\partial U}{\partial y}\hat{j} - \frac{\partial U}{\partial z}\hat{k}$$

$$\Rightarrow \frac{\partial U}{\partial x} = -2xz \Rightarrow U(\vec{r}) = -x^2 z + f(y,z) \quad \text{—— (1)}$$

$$\frac{\partial U}{\partial y} = -2yz \Rightarrow U(\vec{r}) = -y^2 z + g(x,z) \quad \text{—— (2)}$$

$$\frac{\partial U}{\partial z} = -(x^2+y^2) \Rightarrow U(\vec{r}) = -z\,(x^2+y^2) + h(x,y) \quad \text{—— (3)}$$

Combining (1), (2) & (3)

$$\Rightarrow \boxed{U(\vec{r}) = -z(x^2+y^2) + C}$$

Q.3

A particle of mass m is acted upon by a force whose potential energy is given by $U(x) = ax^2 - bx^3$, where a and b are positive constants.

(a) Find the force F producing this potential.

(b) Find the equilibrium positions, and state whether they are stable or unstable.

(c) Sketch the potential energy curve.

(d) Discuss the motion if $E_1 = 2E_0$, $E_2 = \frac{E_0}{2}$ and $E_3 = -E_0$ where $E_0 = \frac{4a^3}{27b^2}$

(b) $\frac{\partial U}{\partial x} = 2ax - 3bx^2 = x(2a - 3bx) = 0$

extrema are $x_0 = 0$

$x_1 = \frac{2a}{3b}$

$\left.\frac{\partial^2 U}{\partial x^2}\right|_0 = \left. 2a - 6bx \right|_0 = 2a > 0$ stable

$\left.\frac{\partial^2 U}{\partial x^2}\right|_{\frac{2a}{3b}} = 2a - \cancel{6}^{2}\cancel{b} \times \frac{2a}{\cancel{3}\cancel{b}} = -2a < 0$ unstable

(a) $F = -\nabla U = -\frac{\partial U}{\partial x} = -2ax + 3bx^2$.

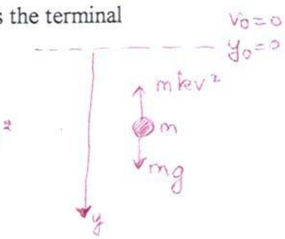
(c) $x \to -\infty$ $U(x) \to +\infty$ $x = 0$ $U(x) = 0$

$x \to +\infty$ $U(x) \to -\infty$

$x = \frac{2a}{3b}$ $U(x) = a\left(\frac{2a}{3b}\right)^2 - b\left(\frac{2a}{3b}\right)^3 = \frac{4a^3}{27b^2}$

Q.4

Consider a particle of mass m whose motion starts from rest in a constant gravitational field. If a resisting force proportional to the square of the velocity (mkv^2) is acting along the y-direction, find the distance the particle falls while accelerating from $v = 0$ to $v = 0.5\ v_t$ where v_t is the terminal velocity of the particle.



Newton's second law : $m \frac{d\mathbf{v}}{dt} = mg - mkv^2$

$$\frac{dv}{dt} = g - kv^2 \qquad \frac{dv}{dt} = \frac{dv}{dy}\underbrace{\frac{dy}{dt}}_{v} = g - kv^2$$

$$\cancel{\frac{dv}{g-kv^2} = dt} \qquad \Rightarrow \int_0^v \frac{v\,dv}{g-kv^2} = \int_0^y dy$$

$$-\frac{1}{2k}\ln\left(g-kv^2\right)\Big|_0^v = y$$

$$y = +\frac{1}{2k}\ln\left(\frac{g}{g-kv^2}\right)$$

To calculate v_t set $\frac{dv}{dt} = 0 = g - kv_t^2 \Rightarrow v_t = \sqrt{\frac{g}{k}}$

v_t when $v = 0.5\ v_t = \frac{1}{2}\sqrt{\frac{g}{k}}$

$$y = +\frac{1}{2k}\ln\left(\frac{g}{g-\frac{kg}{4k}}\right) = +\frac{1}{2k}\ln\left(\frac{4}{3}\right) = \frac{0.14}{k}$$

Q.5

A rocket traveling through the atmosphere experiences ONLY a linear air resistance force given by $-kv$. Find the final speed of the rocket in terms of the u (the exhaust speed), m (the final mass), m_0 (the initial mass) and α (a constant given by $\frac{dm}{dt} = -k\alpha$).

$$\Delta P = F\,dt$$

as derived in class $\Delta P = m\,dv + u\,dm$

$$\Rightarrow m\,dv + u\,dm = -kv\,dt\text{; but } dt = -\frac{dm}{k\alpha}$$

$$\Rightarrow m\,dv + u\,dm = +\frac{kv\,dm}{k\alpha} = \frac{v}{\alpha}\,dm$$

$$\Rightarrow m\,dv = \left(\frac{v}{\alpha} - u\right)dm$$

$$\Rightarrow \frac{m}{dm} = \alpha\frac{dv}{(v - u\alpha)}$$

integrate on both sides $\Rightarrow \int_{m_0}^{m} \frac{dm}{dm} = \int_0^v \frac{\alpha\,dv}{(v - \alpha u)}$

$$\ln\left(\frac{m}{m_0}\right) = \alpha \ln(v - \alpha u)\Big|_0^v = \alpha \ln\left(\frac{v - \alpha u}{-\alpha u}\right)$$

$$\Rightarrow \ln\left(\frac{m}{m_0}\right)^{1/\alpha} = \ln\left(\frac{v - \alpha u}{-\alpha u}\right)$$

$$\Rightarrow -\frac{v - \alpha u}{\alpha u} = \left(\frac{m}{m_0}\right)^{1/\alpha}$$

$$\frac{v}{\alpha u} = 1 - \left(\frac{m}{m_0}\right)^{1/\alpha}$$

$$\Rightarrow \boxed{v = \alpha u\left[1 - \left(\frac{m}{m_0}\right)^{1/\alpha}\right]}$$