Homework Solution # 6

Chapter 7

Solve problems # 4, 7, 12, 15, 25 & 28 from the textbo

Pb# 7.4

Let the particle move in the x-y plane but since $f = f(r)$

use polar coordinates (r, θ) to describe the motion of the particle

$$L = T - U$$

$$T = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right) = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right)$$

$$\vec{F} = -\vec{\nabla} U \Rightarrow f = -\frac{\partial U}{\partial r} \Rightarrow U = -\int F_r \, dr = +\int A r^{\alpha - 1} dr$$

$$U = \frac{A}{\alpha} r^{\alpha} + C \qquad \text{but } U(r=0) = 0 \Rightarrow C = 0$$

$$\Rightarrow U = \frac{A}{\alpha} r^{\alpha}$$

$$L = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) - \frac{A}{\alpha} r^{\alpha} \qquad \text{——— (1)}$$

Two Lagrange's equations:

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}} \right) = 0 \Rightarrow -A r^{\alpha - 1} + m r \dot{\theta}^2 - m \ddot{r} = 0$$

or $$\boxed{m\ddot{r} - m r \dot{\theta}^2 + A r^{\alpha - 1} = 0} \qquad \text{——— (2)}$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}} \right) = 0 \Rightarrow \frac{d}{dt}\left(m r^2 \dot{\theta} \right) = 0 \Rightarrow \boxed{m r^2 \dot{\theta} = \text{Const} = l} \qquad (2)$$

Yes, the angular momentum about the origin is conserved.

Multiplying (2) by $\dot{r}$, we get

$$m\dot{r}\ddot{r} - \frac{\dot{r}\ell^2}{mr^3} + A\dot{r}r^{\alpha-1} = 0$$

$$\frac{d}{dt}\Big(\underbrace{\tfrac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2}}_{T} + \underbrace{\frac{A}{\alpha}r^{\alpha}}_{U}\Big) = 0$$

$$\frac{d}{dt}(E) = 0 \implies E = \text{Const}$$

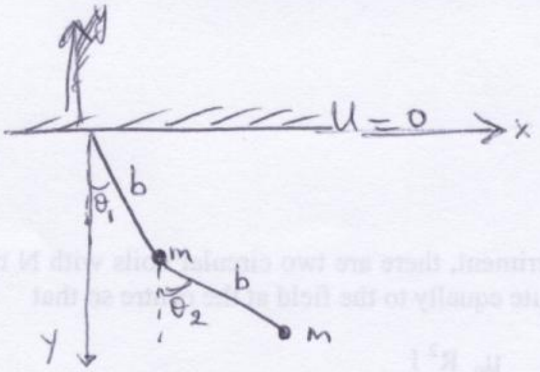
The total energy is conserved.

$$\frac{d}{dt}\Big(\underbrace{\tfrac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2}}_{T} + \underbrace{\frac{A}{\alpha}r^{\alpha}}_{U}\Big)$$

$$m\dot{r}\ddot{r} \cancel{+} \frac{\ell^2\dot{r}}{\cancel{2}mr^3} + Ar^{\alpha-1}\dot{r}$$

$$= \dot{r}\underbrace{\Big(m\ddot{r} - \frac{\ell^2}{mr^3} + Ar^{\alpha-1}\Big)}_{=0} = \quad (\text{from eq. (2)})$$

Pb# 7.7



Take θ_1 and θ_2 ~~are~~ as the generalized coordinates

$$T = \frac{1}{2} m (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m (\dot{x}_2^2 + \dot{y}_2^2) \quad (1)$$

$$U = -m\, b g \cos\theta_1 - m\, 2b\, g \cos\theta_2 \quad \text{———} \quad (2)$$

$$\begin{cases} x_2 = b \sin\theta_1 + b \sin\theta_2 \\ y_2 = b \cos\theta_1 + b \cos\theta_2 \end{cases} \qquad \begin{cases} x_1 = b \sin\theta_1 \\ x_2 = b \cos\theta_1 \end{cases}$$

$$(1) \Rightarrow T = \frac{1}{2} m b^2 \left[2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1 \dot{\theta}_2 (\sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2) \right]$$

$$= \frac{1}{2} m b^2 \left[2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right]$$

$$U = -mgb\,(2\cos\theta_1 + \cos\theta_2)$$

Therefore the Lagrangian L is: $L = T - U$

$$\boxed{L = m b^2 \left[\dot{\theta}_1^2 + \frac{\dot{\theta}_2^2}{2} + \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right] + mgb(2\cos\theta_1 + \cos\theta_2)}$$

$$\frac{\partial L}{\partial \theta_1} = -mb^2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - mgb\,(2\sin\theta_1)\cancel{-\sin\theta_2})$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = 2mb^2 \dot{\theta}_1 + mb^2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$\frac{\partial L}{\partial \theta_2} = +mb^2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - mgb \sin\theta_2$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = mb^2 \dot{\theta}_2 + \dot{\theta}_1 mb^2 \cos(\theta_1 - \theta_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = 2mb^2\ddot{\theta}_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = mb^2\ddot{\theta}_2$$

$\Rightarrow$ Lagrange equations for θ_1 and θ_2 are:

$$\dot{\theta}_1\dot{\theta}_2 \sin(\theta_1-\theta_2) + \frac{2g}{b}\sin\theta_1 + 2\ddot{\theta}_1 = 0$$

$$-\dot{\theta}_1\dot{\theta}_2 \sin(\theta_1-\theta_2) + \frac{g}{b}\sin\theta_2 + 2\ddot{\theta}_2 = 0$$

Pb #7.12

$$\alpha = \frac{d\theta}{dt} = \dot{\theta} = \text{Const.}$$

Use polar coordinates (r, θ)

$$L = T - U \qquad T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right)$$

$$U = mg\, r \sin\theta$$

$$L = \frac{1}{2} m \left(\dot{r}^2 + r^2 \alpha^2 \right) - mg\, r \sin\theta$$

Lagrange equation for r is

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}} \right) = 0$$

$$m r \alpha^2 - mg \sin\theta - m\ddot{r} = 0$$

$$\ddot{r} - \alpha^2 r = -g \sin\alpha t \qquad (\theta = \alpha t)$$

The general solution is of the form $r(t) = r_c(t) + r_p(t)$

with $\quad r_c = A e^{\alpha t} + B e^{-\alpha t}$

For r_p, try a solution of the form $r_p = C \sin\alpha t$

$$\ddot{r}_p = - C \alpha^2 \sin\alpha t$$

$$\Rightarrow -C\alpha^2 \cancel{\sin\alpha t} - \alpha^2 C \cancel{\sin\alpha t} = -g \cancel{\sin\alpha t}$$

$$C = \frac{g}{2\alpha^2}$$

$$\Rightarrow r(t) = A e^{\alpha t} + B e^{-\alpha t} + \frac{g}{2\alpha^2} \sin\alpha t$$

We can determine A and B using initial Conditions

$$r(0) = r_0$$

$$\dot{r}(0) = 0$$

$$r_0 = A + B$$

$$0 = \alpha A - \alpha B + \frac{g\alpha}{2\alpha^2} \Rightarrow A - B + \frac{g}{2\alpha^2} = 0$$

$$A = B - \frac{g}{2\alpha^2}$$

$$r_0 = 2B - \frac{g}{2\alpha^2} \Rightarrow B = \frac{1}{2}\left(r_0 + \frac{g}{2\alpha^2}\right)$$

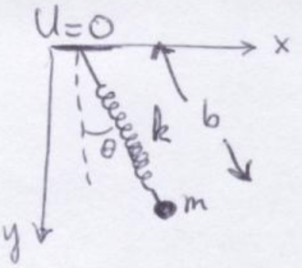
$$A = \frac{1}{2}\left(r_0 - \frac{g}{2\alpha^2}\right)$$

$$\Rightarrow r(t) = \frac{1}{2}\left(r_0 - \frac{g}{2\alpha^2}\right)e^{\alpha t} + \frac{1}{2}\left(r_0 + \frac{g}{2\alpha^2}\right)e^{-\alpha t} + \frac{g}{2\alpha^2}\sin\alpha t$$

$$= r_0\left(\frac{e^{\alpha t} + e^{-\alpha t}}{2}\right) + \frac{g}{2\alpha^2}\left(\frac{-e^{\alpha t} + e^{-\alpha t}}{2}\right) + \frac{g}{2\alpha^2}\sin\alpha t$$

$$\boxed{r(t) = r_0 \cosh\alpha t + \frac{g}{2\alpha^2}\left[\sin\alpha t - \sinh\alpha t\right]}$$

Pb # 7.15



b: unextended length of spring

r: variable length of spring

$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2)$ use polar coordinates

$U = \frac{1}{2} k (r - b)^2 - mg\, r \cos\theta$

$L = T - U = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - \frac{1}{2} k (r-b)^2 + mg r \cos\theta$

Lagrange's equations for r and θ :

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = 0$$

$$-k(r-b) + mg \cos\theta - m\ddot{r} + m r \dot{\theta}^2 = 0$$

L.E gives: $\boxed{\ddot{r} - r\dot{\theta}^2 + \frac{k}{m}(r-b) - g\cos\theta = 0}$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$$

$$\frac{\partial L}{\partial \theta} = -mgr\sin\theta$$

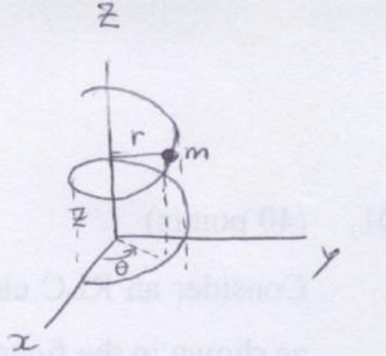
$$\frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta}$$

L.E gives $\boxed{\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} + \frac{g}{r}\sin\theta = 0}$

Pb # 7.25

Use cylindrical coordinates

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2 \right)$$



$$U = mgz$$

$$H = \cancel{p_r \dot{r}} + p_z \dot{z} + \cancel{p_\theta \dot{\theta}} - L \qquad \text{since } r = \text{Const.}$$

Use the relation $z = k\theta \Rightarrow \dot{z} = k\dot{\theta}$

$$T = \frac{1}{2} m \left(\frac{r^2}{k^2} \dot{z}^2 + \dot{z}^2 \right) \Longrightarrow L = \frac{1}{2} m \left(\frac{r^2}{k^2} + 1 \right) \dot{z}^2 - mgz$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m \left(\frac{r^2}{k^2} + 1 \right) \dot{z}$$

$$\Rightarrow H = \frac{p_z^2}{m\left(\frac{r^2}{k^2}+1\right)} - \frac{p_z^2}{2m\left(\frac{r^2}{k^2}+1\right)} + mgz$$

$$H = \frac{p_z^2}{2m\left(\frac{r^2}{k^2}+1\right)} + mgz$$

Hamilton's equation gives:

$$\dot{p}_z = -\frac{\partial H}{\partial z} \quad \text{and} \quad \dot{z} = \frac{\partial H}{\partial p_z}$$

$$\Rightarrow \dot{p}_z = -mg$$

$$\dot{z} = \frac{p_z}{m\left(\frac{r^2}{k^2}+1\right)} \qquad \text{already found previously.}$$

$$\ddot{z} = \frac{\dot{p}_z}{m\left(\frac{r^2}{k^2}+1\right)} = -\frac{gm}{\left(\frac{r^2}{k^2}+1\right)m}$$

$$\Rightarrow \boxed{\ddot{z} = \frac{-g}{r^2/k^2+1}} \Rightarrow \dot{z} = \frac{-g}{\frac{r^2}{k^2}+1} t + c \Rightarrow z = \cdots$$

Pb # 7.28

$$F_r = \frac{k}{r^2} \qquad U = -\int F_r \, dr = -\int \frac{k\,dr}{r^2}$$

$$U = -\frac{k}{r} + C$$

$$r \to \infty \qquad U \to 0 \quad \Longrightarrow C = 0$$

$$\Longrightarrow U = -\frac{k}{r}$$

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2\right)$$

$$L = T - U = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2\right) + \frac{k}{r}$$

$$H = p_r \dot{r} + p_\theta \dot{\theta} - L$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad \Longrightarrow \dot{r} = \frac{p_r}{m}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} \quad \Longrightarrow \dot{\theta} = \frac{p_\theta}{m r^2}$$

$$H = \frac{p_r^2}{m} + \frac{p_\theta^2}{m r^2} - \frac{1}{2} m\left(\frac{p_r^2}{m^2} + r^2 \frac{p_\theta^2}{m^2 r^4}\right) - \frac{k}{r}$$

$$= \frac{p_r^2}{m} + \frac{p_\theta^2}{m r^2} - \frac{p_r^2}{2m} - \frac{p_\theta^2}{2 m r^2} - \frac{k}{r}$$

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2 m r^2} - \frac{k}{r}$$

Hamilton's equations of motion are:

$$\boxed{\begin{array}{ll} \dot{r} = \dfrac{\partial H}{\partial p_r} = \dfrac{p_r}{\cancel{2}m} \; ; & \dot{p}_r = -\dfrac{\partial H}{\partial r} = \dfrac{p_\theta^2}{m r^3} - \dfrac{k}{r^2} \\[2ex] \dot{\theta} = \dfrac{\partial H}{\partial p_\theta} = \dfrac{p_\theta}{m r^2} \; ; & \dot{p}_\theta = -\dfrac{\partial H}{\partial \theta} = 0 \end{array}}$$