PHYS301
HW # 4
Chapter 5

Pb #5.4

The potential energy corresponding to the force is

$$U = -\int F\,dx = mk^2 \int \frac{dx}{x^3} = -\frac{mk^2}{2x^2}$$

The force is conservative (central force) and therefore the total energy is constant. Note that the particle start from rest at point d and moves toward the center.

$$\Rightarrow E_i = E_f$$

$$\frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \frac{k^2}{x^2} = E = \text{Constant}$$

$$E = U_i + \cancel{K_i}^{\,0} = -\frac{1}{2} m \frac{k^2}{d^2}$$

$$\Rightarrow \dot{x}^2 - \frac{k^2}{x^2} = -\frac{k^2}{d^2}$$

$$\dot{x}^2 = k^2\left(\frac{1}{x^2} - \frac{1}{d^2}\right) = \left(\frac{dx}{dt}\right)^2$$

$$\Rightarrow \frac{dx}{dt} = \pm k\sqrt{\frac{1}{x^2} - \frac{1}{d^2}} \Rightarrow dt = \pm \frac{dx}{k\sqrt{\frac{1}{x^2} - \frac{1}{d^2}}}$$

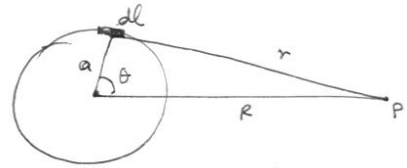
$$\int_0^t dt = \pm \frac{d}{k} \int_d^0 \frac{x\,dx}{\sqrt{d^2 - x^2}} = \pm \frac{d}{k} \sqrt{d^2 - x^2}\Big|_d^0$$

$$\Rightarrow t = \pm \frac{d}{k}(d - 0) = \pm \frac{d^2}{k}$$

Choose + answer (time is positive)

$$\boxed{t = \frac{d^2}{k}}$$

Pb# 5.9



$$\phi = -G \int \frac{\lambda \, dl}{r}$$

$dM = \lambda \, dl$

$\lambda = \frac{M}{2\pi a}$ (kg/m)

$$r = \sqrt{R^2 + a^2 - 2aR\cos\theta}$$

and $dl = a \, d\theta$

$$\phi = -\frac{GM}{2\pi} \int_0^{2\pi} \frac{d\theta}{\sqrt{R^2 + a^2 - 2aR\cos\theta}}$$

If $R \gg a$ then $\sqrt{R^2 + a^2 - 2aR\cos\theta} = R\sqrt{1 + \frac{a^2}{R^2} - \frac{2a}{R}\cos\theta}$

$$= R\left[1 - \underbrace{\left(\frac{2a}{R}\cos\theta - \frac{a^2}{R^2}\right)}_{x}\right]^{1/2} = R\left[1 - x\right]^{1/2}$$

if $x \ll 1 \Rightarrow$

$$\phi = -\frac{GM}{2\pi R} \int_0^{2\pi} \left[1 - x\right]^{-1/2} d\theta$$

$$\simeq -\frac{GM}{2\pi R} \int_0^{2\pi} \left(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3\right) d\theta$$

$$\simeq -\frac{GM}{2\pi R} \int_0^{2\pi} \left\{1 + \frac{1}{2}\left(2\frac{a}{R}\cos\theta - \frac{a^2}{R^2}\right) + \frac{3}{8}\left(\frac{2a}{R}\cos\theta - \frac{a^2}{R^2}\right)^2\right\} d\theta$$

$$\phi \simeq -\frac{GM}{2\pi R}\left\{\int_0^{2\pi} d\theta + \int_0^{2\pi} \frac{a}{R}\cos\theta \, d\theta - \int_0^{2\pi} \frac{a^2}{2R^2} d\theta + \int_0^{2\pi} \frac{3}{2}\frac{a^2}{R^2}\cos^2\theta \, d\theta\right\}$$

Note: We have neglected term of the form $\frac{a^3}{R^3}$ and $\frac{a^4}{R^4}$!

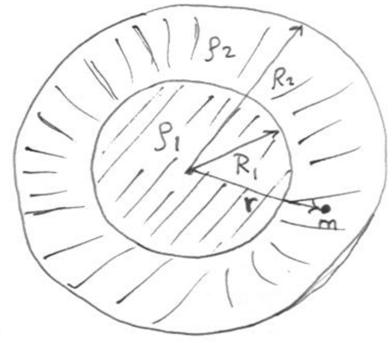
$$\phi \simeq -\frac{GM}{2\pi R}\left[2\pi - \pi\frac{a^2}{R^2} + \frac{3}{2}\frac{a^2}{R^2}\pi\right]$$

Note : $\displaystyle\int_0^{2\pi} \cos^2\theta \, d\theta = \frac{1}{2}\cancelto{0}{\sin\theta\cos\theta}\Big|_0^{2\pi} + \frac{1}{2}\theta\Big|_0^{2\pi} = \pi$

$$\boxed{\phi \simeq -\frac{GM}{R}\left[1 + \frac{1}{4}\frac{a^2}{R^2}\right]}$$

The first term correction is $\simeq -\dfrac{GMa^2}{4R^3}$

Pb# 5.13



The force on mass m is

$$F = \frac{GMm}{r^2} \quad \text{(magnitude)}$$

direction is radially inward.

r: distance from the mass m to the center, as shown in the figure.

Now

$$M = \frac{4}{3}\pi R_1^3 \rho_1 + \frac{4}{3}\pi (r^3 - R_1^3)\rho_2$$

$$= \frac{4}{3}\pi R_1^3 (\rho_1 - \rho_2) + \frac{4}{3}\pi r^3$$

$$\Rightarrow \boxed{F = \frac{4\pi G m}{3}\left[\frac{R_1^3}{r^2}(\rho_1 - \rho_2) + \rho_2 r\right]}$$

Pb # 5.14

The work done to assemble a small element of mass dm is

$$dW = dU = -\frac{G\,m\,dm}{r}$$

but $dm = \underbrace{(4\pi r^2 dr)}_{dV}\,\rho$

$$\Rightarrow \quad dU = -\frac{G}{r}\underbrace{\left(\frac{4}{3}\pi r^3 \rho\right)}_{m} 4\pi r^2 dr\,\rho$$

$$= -G\left(\frac{4\pi}{3}\right)^2 3\rho^2 r^4\,dr$$

$$U = \int_0^R dU = -G\left(\frac{4}{3}\pi\right)^2 3\rho^2 \int_0^R r^4\,dr$$

$$= -3G\underbrace{\left(\frac{4}{3}\pi\rho\right)^2}\frac{R^5}{5}$$

$V = \frac{4}{3}\pi R^3$

$\rho = \frac{M}{V}$ — total mass / total volume

$M = \rho \times \frac{4}{3}\pi R^3$

$$= -\frac{3G}{5R}\underbrace{\left(\frac{4}{3}\pi\rho R^3\right)^2}_{M^2}$$

$$\Rightarrow \quad \boxed{U = -\frac{3GM^2}{5R}}$$