

PHYS301
HW # 4
Chapter 5

Pb #5.4

The potential energy corresponding to the force is

$$U = - \int F dx = m k^2 \int \frac{dx}{x^3} = - \frac{m k^2}{2x^2}$$

The force is conservative (central force) and therefore the total energy is constant. Note that the particle starts from rest at point d and moves toward the center.

$$\Rightarrow E_i = E_f$$

$$\frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \frac{k^2}{x^2} = E = \text{Constant}$$

$$E = U_i + K_i^{\rightarrow 0} = - \frac{1}{2} m \frac{k^2}{d^2}$$

$$\Rightarrow \dot{x}^2 - \frac{k^2}{x^2} = - \frac{k^2}{d^2}$$

$$\dot{x}^2 = k^2 \left(\frac{1}{x^2} - \frac{1}{d^2} \right) = \left(\frac{dx}{dt} \right)^2$$

$$\Rightarrow \frac{dx}{dt} = \pm k \sqrt{\frac{1}{x^2} - \frac{1}{d^2}} \Rightarrow dt = \pm \frac{dx}{k \sqrt{\frac{1}{x^2} - \frac{1}{d^2}}}$$

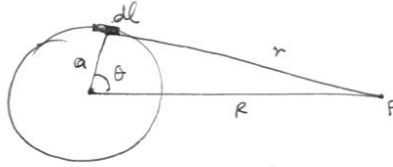
$$\int_0^t dt = \pm \frac{d}{k} \int_d^0 \frac{x dx}{\sqrt{d^2 - x^2}} = \pm \frac{d}{k} \left. \sqrt{d^2 - x^2} \right|_d^0$$

$$\Rightarrow t = \pm \frac{d}{k} (d - 0) = \pm \frac{d^2}{k}$$

Choose + answer (time is positive)

$$\boxed{t = \frac{d^2}{k}}$$

Pb# 5.9



$$\phi = -G \int \frac{\lambda dl}{r}$$

$$dM = \lambda dl$$

$$\lambda = \frac{M}{2\pi a} \quad (\text{kg/m})$$

$$r = \sqrt{R^2 + a^2 - 2aR \cos \theta}$$

and $dl = a d\theta$

$$\phi = - \frac{GM}{2\pi} \int_0^{2\pi} \frac{d\theta}{\sqrt{R^2 + a^2 - 2aR \cos \theta}}$$

If $R \gg a$ then $\sqrt{R^2 + a^2 - 2aR \cos \theta} = R \sqrt{1 + \frac{a^2}{R^2} - \frac{2a \cos \theta}{R}}$

$$= R \left[1 - \underbrace{\left(\frac{2a \cos \theta}{R} - \frac{a^2}{R^2} \right)}_x \right]^{1/2} = R [1 - x]^{1/2}$$

If $x \ll 1 \Rightarrow$

$$\phi = - \frac{GM}{2\pi R} \int_0^{2\pi} [1 - x]^{-1/2} d\theta$$

$$\approx - \frac{GM}{2\pi R} \int_0^{2\pi} \left(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 \right) d\theta$$

$$\approx - \frac{GM}{2\pi R} \int_0^{2\pi} \left\{ 1 + \frac{1}{2} \left(\frac{2a \cos \theta}{R} - \frac{a^2}{R^2} \right) + \frac{3}{8} \left(\frac{2a \cos \theta}{R} - \frac{a^2}{R^2} \right)^2 \right\} d\theta$$

$$\phi \approx - \frac{GM}{2\pi R} \left\{ \int_0^{2\pi} d\theta + \int_0^{2\pi} \frac{a}{R} \cos \theta d\theta - \int_0^{2\pi} \frac{a^2}{2R^2} d\theta + \int_0^{2\pi} \frac{3}{2} \frac{a^2}{R^2} \cos^2 \theta d\theta \right\}$$

Note: We have neglected term of the form $\frac{a^3}{R^3}$ and $\frac{a^4}{R^4}$!

$$\phi \approx - \frac{GM}{2\pi R} \left[2\pi - \pi \frac{a^2}{R^2} + \frac{3}{2} \frac{a^2}{R^2} \pi \right]$$

Note: $\int_0^{2\pi} \cos^2 \theta \, d\theta = \frac{1}{2} \int_0^{2\pi} \sin^2 \theta \, d\theta + \frac{1}{2} \int_0^{2\pi} \cos^2 \theta \, d\theta = \frac{1}{2} \int_0^{2\pi} 1 \, d\theta = \pi$

$$\boxed{\phi \approx - \frac{GM}{R} \left[1 + \frac{1}{4} \frac{a^2}{R^2} \right]}$$

The first term correction is $= - \frac{GMa^2}{4R^3}$

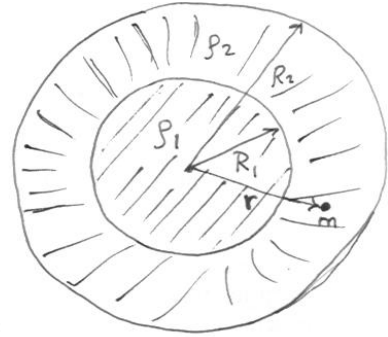
Pb# 5.13

The force on mass m is

$$F = \frac{GMm}{r^2} \text{ (magnitude)}$$

direction is radially inward.

r : distance from the mass m to the center, as shown in the figure.



$$\text{Now } M = \frac{4}{3}\pi R_1^3 \rho_1 + \frac{4}{3}\pi (r^3 - R_1^3) \rho_2$$

$$= \frac{4}{3}\pi R_1^3 (\rho_1 - \rho_2) + \frac{4}{3}\pi r^3$$

$$\Rightarrow F = \frac{4\pi G m}{3} \left[\frac{R_1^3}{r^2} (\rho_1 - \rho_2) + \rho_2 r \right]$$

Pb# 5.14

The work done to assemble a small element of mass dm is

$$dW = dU = - \frac{Gm dm}{r}$$

$$\text{but } dm = \underbrace{(4\pi r^2 dr)}_{dV} \rho$$



$$\Rightarrow dU = - \frac{G}{r} \underbrace{\left(\frac{4\pi r^3}{3} \rho\right)}_m 4\pi r^2 dr \rho$$

$$= - G \left(\frac{4\pi}{3}\right)^2 3 \rho^2 r^4 dr$$

$$U = \int_0^R dU = - G \left(\frac{4\pi}{3}\right)^2 3 \rho^2 \int_0^R r^4 dr$$

$$= - 3G \underbrace{\left(\frac{4\pi}{3} \rho\right)^2}_{M^2} \frac{R^5}{5}$$

$$= - \frac{3G}{5R} \underbrace{\left(\frac{4\pi}{3} \rho R^3\right)^2}_{M^2}$$

$$V = \frac{4\pi}{3} R^3$$

$$\rho = \frac{M}{V} \text{ — total mass}$$

$$M = \rho \times \frac{4\pi}{3} R^3$$

$$\Rightarrow \boxed{U = - \frac{3GM^2}{5R}}$$