Phys 301

Home Work Solution

Chapter 2

Pb#9.

(a) No resisting force

motion along the y-axis : $F = ma = m\frac{dv}{dt} = -mg$

$$\int dv = -g\int dt \Rightarrow \boxed{v = -gt + v_0} \quad \text{—— (1)}$$

v_0 is the initial velocity of the particle at $t = 0$.

At maximum height $v = 0 = -gt_m + v_0 \Rightarrow \boxed{t_m = \frac{v_0}{g}}$ — (2)

(b) Resistive force $F_r = -mkv$

motion along the y- axis: $F = ma = m\frac{dv}{dt} = -mg - kmv$

$$\int \frac{dv}{kv+g} = -\int dt$$

$$\frac{1}{k}\ln(kv+g) = -t + C$$

$$\Rightarrow kv + g = e^{-kt + kC}$$

$t=0 \quad v = v_0 \quad \Rightarrow kv_0 + g = 1\, e^{kC}$

$$\Rightarrow \quad kC = \ln(kv_0 + g)$$

$$\Rightarrow \quad C = \frac{1}{k}\ln(kv_0 + g)$$

$$\Rightarrow \quad kv + g = e^{-kt}(kv_0 + g)$$

$$\boxed{v = -\frac{g}{k} + \frac{(kv_0+g)}{k} e^{-kt}} \qquad \text{—— (3)}$$

at maximum height $v = 0$

$$-\frac{g}{k} + \frac{(kv_0+g)}{k} e^{-kt_m} = 0$$

$$\Rightarrow \quad \frac{g}{kv_0+g} = e^{-kt_m}$$

$$\Rightarrow \quad k\,t_m = \ln\left(\frac{kv_0}{g} + 1\right)$$

For small values of k $\quad \ln\left(1 + \frac{kv_0}{g}\right) = \frac{kv_0}{g} - \frac{1}{2}\left(\frac{kv_0}{g}\right)^2 + \frac{1}{3}\left(\frac{kv_0}{g}\right)^3$

$$\Rightarrow \quad \boxed{t_m = \frac{v_0}{g}\left[1 - \frac{kv_0}{2g} + \frac{1}{3}\left(\frac{kv_0}{g}\right)^2 + \cdots\right]} \qquad \text{—— (4)}$$

when $k \to 0 \qquad t_m \to \frac{v_0}{g}$ (equation 2).

you can also integrate t from $0 \to t$ and v from $v_0 \to 0$

$$\int_{v_0}^{0} \frac{dv_y}{g + kv} = -\int_0^t dt$$

Pb # 23 $\quad F(t) = k t e^{-\alpha t}$

$$F = m a \Rightarrow a = \frac{F}{m} = \frac{kt}{m} e^{-\alpha t}$$

$$a = \frac{dv}{dt} \Rightarrow v = \int_0^t a\, dt = \frac{k}{m} \int_0^t t e^{-\alpha t} dt$$

integrating by part setting $u = t \quad dv = e^{-\alpha t} dt$

$$\int u\, dv = uv - \int v\, du$$

$$\int t e^{-\alpha t} dt = -\frac{t}{\alpha} e^{-\alpha t} + \frac{1}{\alpha} \int_0^t e^{-\alpha t} dt$$

$$= -\frac{t}{\alpha} e^{-\alpha t} - \frac{1}{\alpha^2} e^{-\alpha t} \Big|_0^t$$

$$= -\frac{t}{\alpha} e^{-\alpha t} - \frac{1}{\alpha^2} e^{-\alpha t} + \frac{1}{\alpha^2}$$

$$\Rightarrow \boxed{v(t) = \frac{k}{m} \left[\frac{1}{\alpha^2} - \frac{1}{\alpha} e^{-\alpha t} \left(t + \frac{1}{\alpha} \right) \right]}$$

$$x(t) = \int_0^t v(t)\, dt = \frac{k}{m} \left[\frac{1}{\alpha^2} \int_0^t dt - \frac{1}{\alpha} \underbrace{\int_0^t t e^{-\alpha t} dt}_{\text{as before}} - \frac{1}{\alpha^2} \int_0^t e^{-\alpha t} dt \right]$$

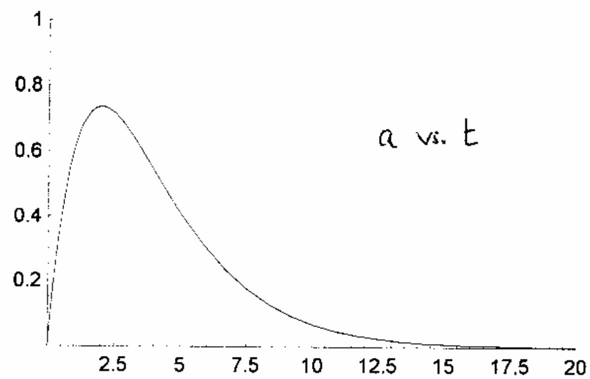
$$x(t) = \frac{k}{m} \left[\frac{t}{\alpha^2} - \frac{1}{\alpha} \left(\frac{1}{\alpha^2} - \frac{1}{\alpha} e^{-\alpha t} \left(1 + \frac{1}{\alpha}\right) \right) + \frac{1}{\alpha^3} \left(e^{-\alpha t} - 1 \right) \right]$$

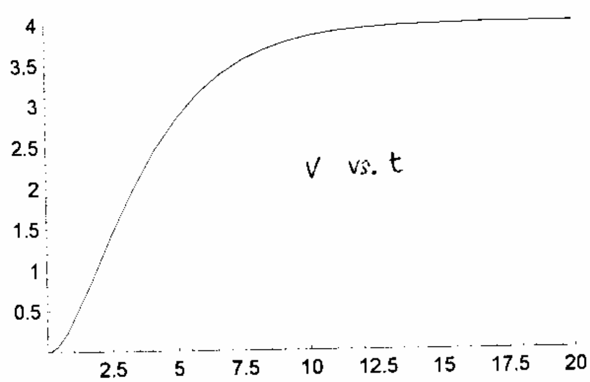
$$\boxed{x(t) = \frac{k}{m} \left[\frac{t}{\alpha^2} - \frac{2}{\alpha^3} + \frac{1}{\alpha^2} e^{-\alpha t} \left(t + \frac{2}{\alpha} \right) \right]}$$

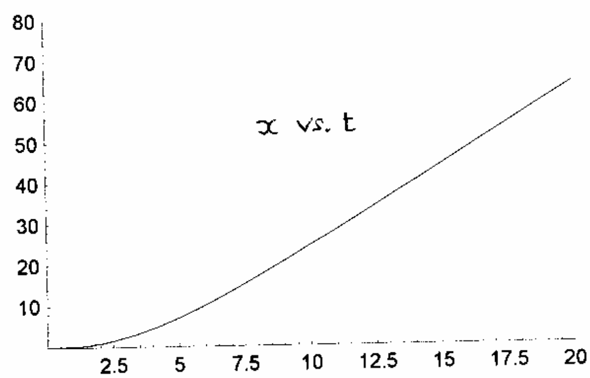
Substitute $m = 1\ kg$ $k = 1\ N/s$ and $\alpha = 0.5\ s^{-1}$

$$\begin{cases} a(t) = t\,e^{-\frac{t}{2}} & \text{— (1)} \\ v(t) = 4 - 2(t+2)\,e^{-\frac{t}{2}} & \text{— (2)} \\ x(t) = -16 + 4t + 4(t+4)\,e^{-\frac{t}{2}} & \text{— (3)} \end{cases}$$

You can plot these functions using any plotting software. I have used Mathematica to plot these graphs.







Pb #31

See example 2.10 for details.

Solutions are given by equation 2.78

$$x - x_0 = A \cos\alpha t + B \sin\alpha t$$

$$y - y_0 = \dot{y}_0 t$$

$$z - z_0 = -B \cos\alpha t + A \sin\alpha t$$

where $\alpha = \frac{qB_0}{m}$

at $t = 0$ $\quad \dot{z} = \dot{z}_0$ and $\dot{x} = \dot{x}_0$

$$\Rightarrow \quad \alpha B = \dot{x}_0$$

$$\alpha A = \dot{z}_0$$

$$\Rightarrow \begin{cases} x - x_0 = \frac{\dot{z}_0}{\alpha} \cos\alpha t + \frac{\dot{x}_0}{\alpha} \sin\alpha t \\ y - y_0 = \dot{y}_0 t \\ z - z_0 = -\frac{\dot{x}_0}{\alpha} \cos\alpha t + \frac{\dot{z}_0}{\alpha} \sin\alpha t \end{cases}$$

Note that $(x - x_0)^2 + (z - z_0)^2 = \left(\frac{\dot{z}_0^2}{\alpha^2} + \frac{\dot{x}_0^2}{\alpha^2} \right) \underbrace{\left(\cos^2\alpha t + \sin^2\alpha t \right)}_{1}$

In the x-z plane the motion is circular with a radius $r = \frac{1}{\alpha} \sqrt{\dot{z}^2 + \dot{x}_0^2} = \frac{m}{qB_0} \sqrt{\dot{x}_0^2 + \dot{z}_0^2}$

Pb # 38.

$$v(x) = \alpha\, x^{-n}$$

$v(x) = 0$ when $x = 0$ at $t = 0$

(a) $$F(x) = m\,a = m\,\frac{dv}{dt} = m\left(\frac{dv}{dx}\right)\left(\frac{dx}{dt}\right) = m\left(\frac{dv}{dx}\right)v$$

$$= m\left(-n\,\alpha\, x^{-n-1}\right)\left(\alpha\, x^{-n}\right)$$

$$\boxed{F(x) = -m\,n\,\alpha^{2}\, x^{-(2n+1)}}$$

(b) $$v(x) = \frac{dx}{dt} = \alpha\, x^{-n} \Rightarrow x^{n}\,dx = \alpha\,dt$$

$$\int x^{n}\,dx = \int \alpha\,dt \Rightarrow \frac{x^{n+1}}{n+1} = \alpha t + \cancel{C}$$

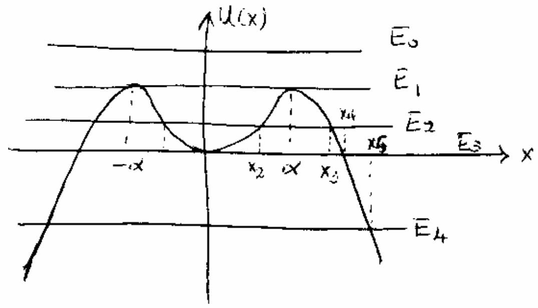
$$x^{n+1} = (n+1)\alpha t \Rightarrow \boxed{x(t) = \left[(n+1)\,\alpha\, t\right]^{\frac{1}{n+1}}}$$

(c) $$\boxed{F(t) = -\,m\,n\,\alpha^{2}\left[(n+1)\alpha t\right]^{-\frac{2n+1}{n+1}}}$$

Pb # 43. $F(x) = -kx + k\dfrac{x^3}{d^2}$ $\quad k > 0$

$$F(x) = -\frac{\partial U}{\partial x} \Rightarrow U(x) = -\int F(x)\,dx$$

$$U(x) = +\frac{1}{2}kx^2 - \frac{1}{4}k\frac{x^4}{\alpha^2}$$



plot of $U(x)$ vs. x

For small x values
$U(x) \longrightarrow \frac{1}{2}kx^2$

For large x values
$U(x) \longrightarrow -\frac{1}{4}k\frac{x^4}{\alpha^2}$

Let us look for equilibrium points:

$$\frac{\partial U}{\partial x} = kx - \frac{kx^3}{\alpha^2} = 0 \Rightarrow x\,k\left(1 - \frac{x^2}{\alpha^2}\right) = 0$$

$$\Rightarrow x = 0 \quad \text{or} \quad x^2 = \alpha^2 \Rightarrow x = \pm\alpha$$

Three equilibrium points: $x_0 = 0$
$x_1 = \alpha$
$x_2 = -\alpha$

Are these equilibrium points stable or unstable?

$$\left.\frac{\partial^2 U}{\partial x^2}\right|_0 = \left. k - 3k\frac{x^2}{\alpha^2}\right|_0 = k > 0 \Rightarrow \text{stable}$$

$$\left.\frac{\partial^2 U}{\partial x^2}\right|_{\pm\alpha} = k - 3k\frac{\alpha^2}{\alpha^2} = -2k < 0 \Rightarrow \text{unstable}$$

For $E = E_0$, the motion is unbounded.

For $E = E_1$, if the particle is at a point of unstable equilibrium it may remain there but if perturbed slightly it will move away from equilibrium.

Let us calculate the value of E_1!

$$\frac{dU}{dx} = kx - k\frac{x^3}{\alpha^2} = 0 \Rightarrow x_{1,2} = \pm\alpha \qquad x_0 = 0$$

$$U(\pm\alpha) = \frac{1}{2}k\alpha^2 - \frac{1}{4}k\alpha^2 = \frac{1}{4}k\alpha^2 = E_1$$

For $E = E_2$ the particle is either bounded and oscillates between $-x_2$ and x_2 or it comes from $\pm\infty$ to $\pm x_3$ and returns to $\pm\infty$.

For $E = E_3$, the particle is either at the stable equilibrium point $x = 0$ or beyound $\pm x_4$.

For $E = E_4$, the particle comes from $\pm\infty$ to $\pm x_5$ and returns to $\pm\infty$.

Pb#52.

Since this is a vertical motion under gravity, then

$$v = -gt + u \ln\left(\frac{m_0}{m}\right)$$ (see text book or notes)

$$v = \frac{dy}{dt} = -gt + u \ln\left(\frac{m_0}{m}\right)$$

$$\Rightarrow \int_0^y dy = -g\int_0^t t\,dt + u\int_0^t \ln\left(\frac{m_0}{m}\right) dt$$

but $dt = -\frac{dm}{\alpha}$ (α: burn out rate)

$$\Rightarrow y = -\frac{1}{2}gt^2 - \frac{u}{\alpha}\int_{m_0}^{m} \ln\left(\frac{m_0}{m}\right) dm \qquad (y_0 = 0)$$

$\int \ln\left(\frac{a}{x}\right) dx = x\left[1 + \ln\left(\frac{a}{x}\right)\right]$ so we have

$$y = -\frac{1}{2}gt^2 - \frac{u}{\alpha}\left[m + m\ln\left(\frac{m_0}{m}\right)\right]_{m_0}^{m}$$

$$= -\frac{1}{2}gt^2 - \frac{u}{\alpha}\left[m + m\ln\left(\frac{m_0}{m}\right) - m_0\right]$$

$$y = -\frac{1}{2}gt^2 - u\underbrace{\frac{(m-m_0)}{\alpha}}_{t} - \frac{u}{\alpha}m\ln\left(\frac{m_0}{m}\right)$$

$$\boxed{y = -\frac{1}{2}gt^2 + ut - \frac{um}{\alpha}\ln\left(\frac{m_0}{m}\right)}$$

This is equation 2.133 in the text book!

at burn out $t = t_b$ and $y = y_b$

At the top of the path

$$v_f = 0 = v_B - g\,t_f \Rightarrow t_f = \frac{v_B}{g}$$

$$y_f - \cancel{y_0}^{\to 0} = v_B\,t_f - \frac{1}{2} g t_f^2$$

$$y_f = \frac{v_B^2}{g} - \frac{1}{2} g \frac{v_B^2}{g^2} = \frac{v_B^2}{2g}$$

— t_f
y_f

— t_b
y_b