Home Work Solution
Chapter 1.

Pb#4.

a) If $C = AB$ where C, A and B are matrices,

then $C_{ij} = \sum_k A_{ik} B_{kj}$

$$C^t_{ij} = C_{ji} = \sum_k A_{jk} B_{ki} = \sum_k B_{ki} A_{jk}$$

$$= \sum_k B^t_{ik} A^t_{kj}$$

$$\Rightarrow \underbrace{C^t}_{(AB)^t} = B^t A^t \quad \Rightarrow \boxed{(AB)^t = B^t A^t}$$

b) It is know that $(AB)^{-1}(AB) = \mathbb{1}$ —(1)
identity matrix

We need to show that $(AB)^{-1} = B^{-1}A^{-1}$ — (2)

Suppose that (2) is true. Let us use it in equation (1)

$$B^{-1}\underbrace{A^{-1}A}_{\mathbb{1}} B = \underbrace{B^{-1}B}_{\mathbb{1}} = \mathbb{1} \Rightarrow \text{this is equivalent to equation (1)}$$

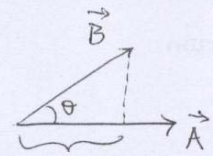
$$\Rightarrow \boxed{(AB)^{-1} = B^{-1}A^{-1}}$$

Pb# 9. $\vec{A} = \hat{i} + 2\hat{j} - \hat{k}$ $\quad \vec{B} = -2\hat{i} + 3\hat{j} + \hat{k}$

a) $\vec{A} - \vec{B} = 3\hat{i} - \hat{j} - 2\hat{k}$

$|\vec{A} - \vec{B}| = \sqrt{14}$

b)



$B\cos\theta =$ component of $\vec{B}$ along $\vec{A}$!

$$\vec{A}\cdot\vec{B} = A\,B\cos\theta \Rightarrow B\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|}$$

$$B\cos\theta = \frac{3}{\sqrt{6}} = \text{component of } \vec{B} \text{ along } \vec{A}.$$

c) $$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{A\,B} = \frac{3}{\sqrt{6}\times\sqrt{14}} \Rightarrow \boxed{\theta = 71^\circ}$$

d) $$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ -2 & 3 & 1 \end{vmatrix} = \boxed{5\hat{i} + 1\hat{j} + 7\hat{k}}$$

e) $$\vec{A}-\vec{B} = 3\hat{i} - \hat{j} - 2\hat{k}$$
$$\vec{A}+\vec{B} = -1\,\hat{i} + 5\hat{j}$$

$$(\vec{A}-\vec{B})\times(\vec{A}+\vec{B}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & -2 \\ -1 & 5 & 0 \end{vmatrix}$$

$$= \boxed{10\hat{i} + 2\hat{j} + 14\hat{k}}$$

Pb # 15.

For orthogonal matrices $\lambda\lambda^t = \mathbb{1}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & -\alpha \\ 0 & \alpha & \alpha \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & \alpha \\ 0 & -\alpha & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2a^2 & 0 \\ 0 & 0 & 2a^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

By identification $\Rightarrow$ $2a^2 = 1$ $\Rightarrow$ $\boxed{a = \pm\sqrt{\frac{1}{2}}}$

Pb # 20.

a) $\sum_{ij} \varepsilon_{ijk} \delta_{ij} \overset{?}{=} 0$

remember that $\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

and $\varepsilon_{ijk} = 0$ if $i = j$, $i = k$, $j = k$

$\Rightarrow$ In all situtions either $\delta_{ij} = 0$ or $\varepsilon_{ijk} = 0$

$$\Rightarrow \sum_{ij} \varepsilon_{ijk} \delta_{ij} = 0.$$

b) $\sum_{jk} \varepsilon_{ijk} \varepsilon_{ljk} \overset{?}{=} 2\delta_{il}$

When $j = k$ $\varepsilon_{ijk} = \varepsilon_{ijj} = 0$ and $\varepsilon_{ljk} = \varepsilon_{ljj} = 0$

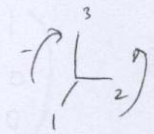
$\Rightarrow$ The remaining terms are

$$\varepsilon_{i12}\,\varepsilon_{l12} + \varepsilon_{i13}\,\varepsilon_{l13} + \varepsilon_{i21}\,\varepsilon_{l21} + \varepsilon_{i31}\,\varepsilon_{l31}$$

$$+ \varepsilon_{i32}\,\varepsilon_{l32} + \varepsilon_{i23}\,\varepsilon_{l23}.$$

See the previous equation !!!

* Now let $i = l = 1$



$$\sum_{jk} \varepsilon_{ijk}\, \varepsilon_{ljk} = \underset{(1)}{\varepsilon_{123}} \times \underset{(1)}{\varepsilon_{123}} + \underset{(-1)}{\varepsilon_{132}} \times \underset{(-1)}{\varepsilon_{132}} = 2$$

* $i = l = 2$

$$\sum_{jk} \varepsilon_{ijk}\, \varepsilon_{ljk} = \underset{(-1)}{\varepsilon_{213}} \times \underset{(-1)}{\varepsilon_{213}} + \underset{(1)}{\varepsilon_{231}} \times \underset{(1)}{\varepsilon_{231}} = 2$$

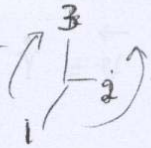
* $i = l = 3$

$$\sum_{jk} \varepsilon_{ijk}\, \varepsilon_{ljk} = \varepsilon_{312}\, \varepsilon_{312} + \varepsilon_{321}\, \varepsilon_{321} = 2$$

* $$\left.\begin{aligned} &i=1, \ l=2 \\ &i=1, \ l=3 \\ &i=2, \ l=1 \\ &i=2, \ l=3 \\ &i=3, \ l=1 \\ &i=3, \ l=2 \end{aligned}\right\} \sum_{jk} \varepsilon_{ijk}\, \varepsilon_{ljk} = 0$$

$\Rightarrow$ in general: $\boxed{\sum_{jk} \varepsilon_{ijk}\, \varepsilon_{ljk} = 2\,\delta_{il}}$

c) $\displaystyle\sum_{ijk} \varepsilon_{ijk}\,\varepsilon_{ijk} \overset{?}{=} 6$



$$\sum_{ijk} \varepsilon_{ijk}\,\varepsilon_{ijk} = \overset{1}{\varepsilon_{123}}\,\overset{1}{\varepsilon_{123}} + \overset{1}{\varepsilon_{231}}\,\overset{1}{\varepsilon_{231}} + \underset{1}{\varepsilon_{312}}\,\underset{1}{\varepsilon_{312}}$$

$$+ \underset{-1}{\varepsilon_{132}}\,\underset{-1}{\varepsilon_{132}} + \underset{-1}{\varepsilon_{321}}\,\underset{-1}{\varepsilon_{321}} + \underset{-1}{\varepsilon_{213}}\,\underset{-1}{\varepsilon_{213}}$$

$$\Rightarrow \boxed{\sum_{ijk} \varepsilon_{ijk}\,\varepsilon_{ijk} = 6}$$

Pb # 25

In spherical coordinates

$$\hat{e}_r = \sin\theta\cos\phi\ \hat{e}_1 + \sin\theta\sin\phi\ \hat{e}_2 + \cos\theta\ \hat{e}_3 \qquad \left[\text{from } \vec{r} = r\,\hat{e}_r\right]$$

$$\hat{e}_\theta = \cos\theta\cos\phi\ \hat{e}_1 + \cos\theta\sin\phi\ \hat{e}_2 - \sin\theta\ \hat{e}_3 \qquad \left[\text{from } \hat{e}_\theta = \frac{\frac{d\vec{r}}{d\theta}}{\left|\frac{d\vec{r}}{d\theta}\right|}\right]$$

$$\hat{e}_\phi = -\sin\phi\ \hat{e}_1 + \cos\phi\ \hat{e}_2 \qquad \left[\text{From } \hat{e}_\phi = \frac{\frac{d\vec{r}}{d\phi}}{\left|\frac{d\vec{r}}{d\phi}\right|}\right]$$

$$v = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{e}_r) = \dot{r}\,\hat{e}_r + r\,\dot{\hat{e}}_r$$

$$\dot{\hat{e}}_r = \dot{\theta}\cos\theta\cos\phi\ \hat{e}_1 - \dot{\phi}\sin\theta\sin\phi\ \hat{e}_1 + \dot{\theta}\cos\theta\sin\phi\ \hat{e}_2$$

$$+ \dot{\phi}\sin\theta\cos\phi\ \hat{e}_2 - \dot{\theta}\sin\theta\ \hat{e}_3$$

$$= \dot{\theta}\left(\cos\theta\cos\phi\ \hat{e}_1 + \cos\theta\sin\phi\ \hat{e}_2 - \sin\theta\ \hat{e}_3\right) +$$

$$\dot{\phi}\sin\theta\left(-\sin\phi\ \hat{e}_1 + \cos\phi\ \hat{e}_2\right) = \dot{\theta}\,\hat{e}_\theta + \dot{\phi}\sin\theta\ \hat{e}_\phi$$

$$\boxed{\dot{\hat{e}}_r = \dot{\theta}\ \hat{e}_\theta + \dot{\phi}\sin\theta\ \hat{e}_\phi}$$

$$\Rightarrow \vec{v} = \dot{r}\,\hat{e}_r + r\left(\dot{\phi}\sin\theta\,\hat{e}_\phi + \dot{\theta}\,\hat{e}_\theta\right)$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \ddot{r}\,\hat{e}_r + \dot{r}\,\dot{\hat{e}}_r + \dot{r}\,\dot{\phi}\sin\theta\,\hat{e}_\phi + r\,\ddot{\phi}\sin\theta\,\hat{e}_\phi$$
$$+ r\,\dot{\phi}\,\dot{\theta}\cos\theta\,\hat{e}_\phi + r\dot{\phi}\sin\theta\,\dot{\hat{e}}_\phi + r\ddot{\theta}\,\hat{e}_\theta + r\dot{\theta}\,\dot{\hat{e}}_\theta + \dot{r}\dot{\theta}\,\hat{e}_\theta$$

$$= \ddot{r}\,\hat{e}_r + \dot{r}\left(\dot{\theta}\,\hat{e}_\theta + \dot{\phi}\sin\theta\,\hat{e}_\phi\right) + \dot{r}\,\dot{\phi}\sin\theta\,\hat{e}_\phi + r\,\ddot{\phi}\sin\theta\,\hat{e}_\phi$$
$$+ r\dot{\phi}\dot{\theta}\cos\theta\,\hat{e}_\phi + r\dot{\phi}\sin\theta\,\underset{\uparrow \text{find it!}}{\dot{\hat{e}}_\phi} + r\,\ddot{\theta}\,\hat{e}_\theta + \dot{r}\,\dot{\theta}\,\hat{e}_\theta + r\dot{\theta}\,\underset{\uparrow \text{find it!}}{\dot{\hat{e}}_\theta}$$

Now

$$\dot{\hat{e}}_\theta = -\dot{\theta}\sin\theta\cos\theta\,\hat{e}_1 - \dot{\phi}\cos\theta\sin\phi\,\hat{e}_1$$
$$-\dot{\theta}\sin\theta\sin\phi\,\hat{e}_2 + \dot{\phi}\cos\theta\cos\phi\,\hat{e}_2$$
$$-\dot{\theta}\cos\theta\,\hat{e}_3$$
$$= -\dot{\theta}\,\underbrace{\left(\sin\theta\cos\phi\,\hat{e}_1 + \sin\theta\sin\phi\,\hat{e}_2 + \cos\theta\,\hat{e}_3\right)}_{\hat{e}_r}$$
$$+\dot{\phi}\cos\theta\,\underbrace{\left(-\sin\theta\,\hat{e}_1 + \cos\theta\,\hat{e}_2\right)}_{\hat{e}_\phi}$$

$$\Rightarrow \boxed{\dot{\hat{e}}_\theta = -\dot{\theta}\,\hat{e}_r + \dot{\phi}\cos\theta\,\hat{e}_\phi}$$

and
$$\dot{\hat{e}}_\phi = -\dot{\phi}\cos\phi\,\hat{e}_1 - \dot{\phi}\sin\phi\,\hat{e}_2 = -\dot{\phi}\left(\cos\phi\,\hat{e}_1 + \sin\phi\,\hat{e}_2\right)$$

but:

$$\sin\theta\,\hat{e}_r = \sin^2\theta\cos\phi\,\hat{e}_1 + \sin^2\theta\sin\phi\,\hat{e}_2 + \sin\theta\cos\theta\,\hat{e}_3$$
$$+$$
$$\cos\theta\,\hat{e}_\theta = \cos^2\theta\cos\phi\,\hat{e}_1 + \cos^2\theta\sin\phi\,\hat{e}_2 - \sin\theta\cos\theta\,\hat{e}_3$$

$$\text{add} = \cos\phi\,\hat{e}_1 + \sin\phi\,\hat{e}_2$$

$$\Rightarrow \boxed{\dot{\hat{e}}_\phi = -\dot{\phi}\left(\sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta\right)}$$

$$\Rightarrow \vec{a} = \ddot{r}\,\hat{e}_r + \dot{r}(\dot{\theta}\,\hat{e}_\theta + \dot{\phi}\sin\theta\,\hat{e}_\phi) + \dot{r}\,\dot{\phi}\sin\theta\,\hat{e}_\phi + r\ddot{\phi}\sin\theta\,\hat{e}_\phi$$
$$+ r\dot{\phi}\dot{\theta}\cos\theta\,\hat{e}_\phi - r\dot{\phi}^2\sin\theta(\sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta) + r\ddot{\theta}\,\hat{e}_\theta$$
$$+ \dot{r}\dot{\theta}\,\hat{e}_\theta + r\dot{\theta}\,(-\dot{\theta}\,\hat{e}_r + \dot{\phi}\cos\theta\,\hat{e}_\phi)$$

$$\boxed{\vec{a} = (\ddot{r} - r\dot{\phi}^2\sin^2\theta - r\dot{\theta}^2)\,\hat{e}_r + (2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta + r\ddot{\theta})\,\hat{e}_\theta + (2\dot{r}\dot{\phi}\sin\theta + r\ddot{\phi}\sin\theta + 2r\dot{\phi}\dot{\theta}\cos\theta)\,\hat{e}_\phi}$$

This is the acceleration vector in spherical Coordinates!

Pb # 30.

$$\vec{\nabla}(\phi\psi) \overset{?}{=} \phi\vec{\nabla}\psi + \psi\vec{\nabla}\phi \quad \text{where } \phi, \psi \text{ are scalars!}$$

The gradiant operator $\vec{\nabla} = \sum_i \frac{\partial}{\partial x_i}\hat{e}_i$; $\vec{\nabla} = \frac{\partial}{\partial x_1}\hat{e}_1 + \frac{\partial}{\partial x_2}\hat{e}_2 + \frac{\partial}{\partial x_3}\hat{e}_3$

$$\Rightarrow \vec{\nabla}(\phi\psi) = \sum_i \frac{\partial(\phi\psi)}{\partial x_i}\hat{e}_i = \sum_i \phi\frac{\partial\psi}{\partial x_i}\hat{e}_i + \sum_i \psi\frac{\partial\phi}{\partial x_i}\hat{e}_i$$
$$= \phi\underbrace{\sum_i \frac{\partial\psi}{\partial x_i}\hat{e}_i}_{\vec{\nabla}\psi} + \psi\underbrace{\sum_i \frac{\partial\phi}{\partial x_i}\hat{e}_i}_{\vec{\nabla}\phi} = \phi\vec{\nabla}\psi + \psi\vec{\nabla}\phi$$

Pb # 31

a) $\vec{\nabla} r^n \overset{?}{=} n\, r^{n-2}\,\vec{r}$

remember that $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ is magnitude of vector $\vec{r}$.

$$= \sqrt{\sum_i x_i^2}$$

$$\Rightarrow r^n = \left(\sum x_i^2\right)^{n/2}$$

$$\Rightarrow \quad \vec{\nabla} r^n = \sum_i \frac{\partial}{\partial x_i}\Big(\sum_j x_j^2\Big)^{n/2} \hat{e}_i$$

$$= \frac{n}{\cancel{2}} \sum_i \cancel{2} x_i \Big(\sum_j x_j^2\Big)^{\frac{n}{2}-1} \hat{e}_i$$

$$= n \sum_i x_i \underbrace{\Big[\sum_j x_j^2\Big]^{\frac{n-2}{2}}}_{r^{n-2}} \hat{e}_i$$

$$= n\, r^{n-2} \underbrace{\sum_i x_i \hat{e}_i}_{\vec{r}} = n\, r^{n-2} \vec{r}$$

$$\Rightarrow \boxed{\vec{\nabla} r^n = n\, r^{n-2}\, \vec{r}}$$

b) $\vec{\nabla} f(\vec{r}) \overset{?}{=} \frac{\vec{r}}{|\vec{r}|} \frac{df}{dr}$

$$\vec{\nabla} f(r) = \sum_i \frac{\partial f}{\partial x_i} \hat{e}_i = \sum_i \frac{\partial f}{\partial r} \frac{\partial r}{\partial x_i} \hat{e}_i$$

$$= \frac{df}{dr} \sum_i \frac{\partial}{\partial x_i}\Big(\sum_j x_j^2\Big)^{1/2} \hat{e}_i$$

$$= \frac{df}{dr} \sum_i \frac{1}{\cancel{2}} \cancel{2} x_i \underbrace{\Big(\sum_j x_j^2\Big)^{-1/2}}_{\frac{1}{|\vec{r}|}} \hat{e}_i$$

$$= \frac{df}{dr} \frac{1}{|\vec{r}|} \underbrace{\sum_i x_i \hat{e}_i}_{\vec{r}} = \frac{df}{dr} \frac{\vec{r}}{|\vec{r}|}$$

$$\Rightarrow \boxed{\vec{\nabla} f(r) = \frac{df}{dr} \frac{\vec{r}}{|\vec{r}|}}$$

c) $\nabla^2 (\operatorname{Ln} r) \overset{?}{=} \frac{1}{r^2}$

$$\nabla^2 = \sum_i \frac{\partial^2}{\partial x_i^2}$$

$$\Rightarrow \nabla^2(\ln r) = \sum_i \frac{\partial}{\partial x_i} \frac{\partial \ln\left(\sum_j x_j^2\right)^{1/2}}{\partial x_i}$$

$$= \sum_i \frac{\partial}{\partial x_i} \frac{1}{\cancel{2}} \cancel{2} x_i \left(\sum_j x_j^2\right)^{-1/2} \frac{1}{\left(\sum_j x_j^2\right)^{1/2}}$$

$$= \sum_i \frac{\partial}{\partial x_i} x_i \left(\sum_j x_j^2\right)^{-1}$$

$$= \sum_i \underbrace{\frac{\partial x_i}{\partial x_i}}_{3} \underbrace{\left(\sum_j x_j^2\right)^{-1}}_{r^{-2}} + \sum_i \underbrace{(-x_i)(2x_i)}_{-2r^2} \underbrace{\left(\sum_j x_j^2\right)^{-2}}_{\frac{1}{r^4}}$$

remember $|\vec{r}| = \sqrt{\sum_j x_j^2} = \left(\sum_j x_j^2\right)^{1/2}$

$$\Rightarrow \nabla^2(\ln r) = \frac{3}{r^2} - \frac{2}{r^2} = \frac{1}{r^2}$$

$$\Rightarrow \boxed{\nabla^2(\ln r) = \frac{1}{r^2}}$$