KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
Physics Department

PHYS 301
Final Exam (27 May, 2001) – Term 002

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Exam Time: 180 minutes
Attempt all problems and show the details of your work

Problem #	Grade
1	/15
2	/1~~0~~5
3	/10
4	/~~20~~15
5	/15
6	/15
7	/15
Total:	/100

Q1. (15 points)

A particle of mass m is thrown vertically upward with a speed v_0 into the air where it experiences a frictional force $f_R = -m\alpha v$. ($\alpha > 0$ and constant, also assume constant gravitational field during the motion of the particle).

(a) Show that the following relationship holds between the velocity of the particle and the time:

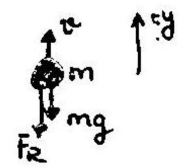
$$m\frac{dv}{dt} = -mg - m\alpha v \qquad (1)$$

and integrate Eq. (1) to find $v(t)$, knowing that $v(0) = v_0$.

(b) Find the time needed for the particle to reach maximum height.

(c) Find the maximum height (take $y(0) = 0$).

(d) Study the limit when $\alpha \to 0$ (no air resistance) for both parts (b) and (c).



(a) Newton's second law:

$$F_{net} = ma$$

$$m\frac{dv}{dt} = -mg - m\alpha v$$

$$dv = -(g + \alpha v)\,dt$$

$$\int_{v_0}^{v} \frac{dv}{g+\alpha v} = -\int_0^t dt \Rightarrow \frac{1}{\alpha}\ln(g+\alpha v)\Big|_{v_0}^{v} = -t$$

$$-\alpha t = \ln\left(\frac{g+\alpha v}{g+\alpha v_0}\right) \Rightarrow g+\alpha v = (g+\alpha v_0)e^{-\alpha t}$$

$$\Rightarrow v = -\frac{g}{\alpha} + \left(\frac{g+\alpha v_0}{\alpha}\right)e^{-\alpha t}$$

$$\boxed{v = -\frac{g}{\alpha} + \left(\frac{g}{\alpha} + v_0\right)e^{-\alpha t}} \quad \text{—— (1)}$$

(b) For maximum height $v = 0$

$$\text{Eq. 1} \Rightarrow \quad \frac{g}{\alpha} = \left(\frac{g}{\alpha} + v_0\right) e^{-\alpha t}$$

$$\Rightarrow \quad e^{-\alpha t} = \frac{g}{g + \alpha v_0} \quad \Rightarrow \quad \alpha t = \ln\left(1 + \frac{\alpha v_0}{g}\right)$$

$$\Rightarrow \quad \boxed{t_{max} = \frac{1}{\alpha} \ln\left(1 + \frac{\alpha v_0}{g}\right)}$$

(c) $$v = \frac{dy}{dt} = -\frac{g}{\alpha} + \left(\frac{g}{\alpha} + v_0\right) e^{-\alpha t}$$

$$\int dy = \int v\, dt \quad \Rightarrow \quad y = -\frac{g}{\alpha} t - \frac{1}{\alpha}\left(\frac{g}{\alpha} + v_0\right) e^{-\alpha t} + C$$

$$y(0) = -\frac{1}{\alpha}\left(\frac{g}{\alpha} + v_0\right) + C = 0 \Rightarrow C = \frac{1}{\alpha}\left(\frac{g}{\alpha} + v_0\right)$$

$$y(t) = -\frac{g}{\alpha} t - \frac{1}{\alpha}\left(\frac{g}{\alpha} + v_0\right) e^{-\alpha t} + \frac{1}{\alpha}\left(\frac{g}{\alpha} + v_0\right)$$

$$y_{max}(t_{max}) = -\frac{g}{\alpha}\left(\frac{1}{\alpha} \ln\left(1 + \frac{\alpha v_0}{g}\right)\right) - \frac{\cancel{(\alpha v_0 + g)}}{\alpha^2} \frac{g}{\cancel{\alpha v_0 + g}} + \frac{1}{\alpha}\left(\frac{g}{\alpha} + v_0\right)$$

$$= -\frac{g}{\alpha^2} \ln\left(1 + \frac{\alpha v_0}{g}\right) - \cancel{\frac{g}{\alpha^2}} + \cancel{\frac{g}{\alpha^2}} + \frac{v_0}{\alpha^2}$$

$$\boxed{y_{max} = -\frac{g}{\alpha^2} \ln\left(1 + \frac{\alpha v_0}{g}\right) + \frac{v_0}{\alpha^2}}$$

(d) $$\alpha \to 0 \qquad \frac{\alpha v_0}{g} \to 0 \qquad \ln\left(1 + \frac{\alpha v_0}{g}\right) = \frac{\alpha v_0}{g} - \frac{\alpha^2 v_0^2}{g^2}$$

$$\Rightarrow \quad y_{max} = -\frac{g}{\alpha^2}\left(\frac{\alpha v_0}{g} - \frac{\alpha^2 v_0^2}{g^2}\right) + \frac{v_0}{\alpha^2} = \boxed{\frac{v_0^2}{g}}$$

and $$t_{max} = \frac{1}{\alpha}\left(\frac{\alpha v_0}{g}\right) = \boxed{\frac{v_0}{g}}$$

Q2. (15 points)

A simple pendulum consists of a mass m = 20 g, suspended from a fixed point by a weightless string of length l = 20 cm. The mass moves within a viscous medium with a retarding force of $f_R = 2m\sqrt{gl}\dot{\theta}$. The mass is pulled and angle of 10^0 and then released from rest.

(a) Find the general expression for the angular position of the mass, θ(t) in the case of small oscillations.

(b) Using the initial conditions specified in the problem, calculate the constants involved in θ(t).

(c) Calculate the speed of the mass at t = 2 s.

(a) $F = ma = m\ell\ddot{\theta} = -mg\sin\theta - 2m\sqrt{g/\ell}\,(\dot{\theta}\ell)$

$\Rightarrow \ddot{\theta} + \frac{g}{\ell}\sin\theta + 2\sqrt{\frac{g}{\ell}}\,\dot{\theta} = 0$

or $\ddot{\theta} + 2\beta\dot{\theta} + \omega_0^2\sin\theta = 0$; $\beta = \sqrt{\frac{g}{\ell}}$ and $\omega_0 = \sqrt{\frac{g}{\ell}}$

For small oscillations $\sin\theta \simeq \theta$

$\Rightarrow \ddot{\theta} + 2\beta\dot{\theta} + \omega_0^2\theta = 0$ —— (1)

since $\beta = \omega_0$ we have the case of critical damping

the solution of eq. 1 is $\boxed{\theta(t) = (A + Bt)\,e^{-\beta t}}$ — (2)

(b) $\theta(0) = 10^\circ = 0.175$ rad $= A$ —— (3)

$\dot{\theta}(t) = B e^{-\beta t} - (A+Bt)\beta e^{-\beta t}$

$\dot{\theta}(0) = B - A\beta = 0 \Rightarrow B = \beta A$ —— (4)

$\beta = \sqrt{\frac{g}{\ell}} = 7$ /s $\Rightarrow B = 1.22$ rad/s

$\Rightarrow \boxed{\theta(t) = (0.175 + 1.22t)\,e^{-7t}}$ —— (5)

(c) $\dot{\theta}(t) = 1.22\, e^{-7t} - 7(0.175 + 1.22t)\, e^{-7t}$

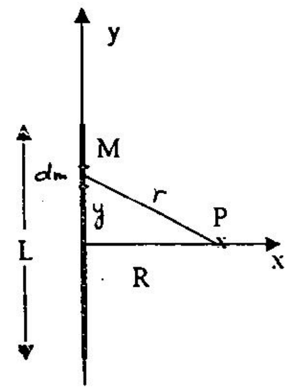
$\dot{\theta}(2s) = 1.22 \times e^{-14} - 7(0.175 + 2.44)\, e^{-14}$

$= -1.42 \times 10^{-5}$ rad/s

$$\boxed{v = \dot{\theta} l = -2.84 \times 10^{-6} \text{ m/s}}$$

Q3. (10 points)

Calculate the **gravitational potential** at point P located a distance R from the axis of a thin rod of mass M and length L (see the figure).



$$d\phi = -G\frac{dm}{r} = -G\frac{\lambda\, dy}{r}$$

$$\phi = -G\lambda\int_{-\ell/2}^{\ell/2}\frac{dy}{r}$$

$$= -G\lambda\int_{-\ell/2}^{\ell/2}\frac{dy}{\sqrt{y^2+R^2}}$$

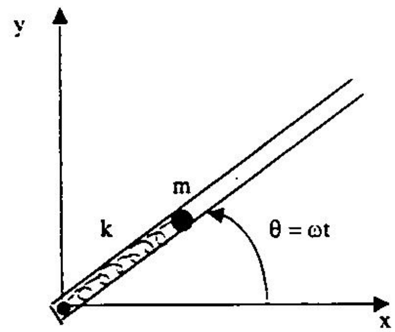
$$= -G\frac{M}{\ell}\ln\left(y+\sqrt{y^2+R^2}\right)\Big|_{-\ell/2}^{\ell/2}$$

$$\phi = -G\frac{M}{\ell}\ln\left\{\frac{\left[\frac{\ell}{2}+\sqrt{\frac{\ell^2}{4}+R^2}\right]}{\left[-\frac{\ell}{2}+\sqrt{\frac{\ell^2}{4}+R^2}\right]}\right\}$$

$$\boxed{\phi = -G\frac{M}{\ell}\ln\left(\frac{\ell+\sqrt{\ell^2+4R^2}}{-\ell+\sqrt{\ell^2+4R^2}}\right)}$$

Q4. (15 points)

A particle of mass m is attached to the end of a spring with spring constant k and unstretched length b. The mass is constrained to move inside a thin hallow frictionless tube. The tube is raised to an inclination angle θ at a constant rate ω ($\theta = 0$ at $t = 0$) as seen in the figure.



(a) Find the Lagrangian of the system.

(b) Show that the equation of motion of the mass m is given by

$$\ddot{r} + \Omega^2 r = \omega_o^2 b - g\sin\omega t\,; \qquad \Omega^2 > 0$$

Identify Ω^2, ω_0^2, and state under what condition is $\Omega^2 > 0$.

(c) Solve the previous equation for the special case $\omega = \omega_0$ with the same initial conditions $r = b$ and $\dot{r} = 0$ at $t = 0$.

(a) $L = T - U$

$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2)$

$U_s = \frac{1}{2} k(r-b)^2 \qquad \theta = \omega t \Rightarrow \dot{\theta} = \omega$

$U_g = mgr\sin\omega t$

$$L = \frac{1}{2} m(\dot{r}^2 + r^2\omega^2) - mgr\sin\omega t - \frac{1}{2} k(r-b)^2 \qquad \text{—— (1)}$$

(b) Lagrange equation $\Rightarrow \frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = 0$ —— (2)

$\frac{\partial L}{\partial r} = mr\omega^2 - mg\sin\omega t - k(r-b)$

$$\frac{\partial L}{\partial \dot r} = m\dot r \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) = m\ddot r$$

$$(2) \Rightarrow \quad m r \omega^2 - mg\sin\omega t - k(r-b) - m\ddot r = 0$$

or $$\ddot r + \left(\frac{k}{m} - \omega^2\right) r = \frac{k}{m} b - g\sin\omega t$$

$$\Rightarrow \quad \Omega^2 = \frac{k}{m} - \omega^2 = \omega_0^2 - \omega^2$$

$$\Rightarrow \boxed{\ddot r + \Omega^2 r = \omega_0^2 b - g\sin\omega t}$$

$\Omega^2 > 0$ if $\omega_0 > \omega$.

(c) If $\omega = \omega_0 \Rightarrow \Omega^2 = 0$

$$\Rightarrow \ddot r = \omega_0^2 b - g\sin\omega t$$

$$\dot r = \omega_0^2 b t + \frac{g}{\omega}\cos\omega t + c'$$

$$r = \frac{1}{2}\omega_0^2 b t^2 + \frac{g}{\omega^2}\sin\omega t + c't + c$$

let us find c' and c!

$$r(0) = c = b$$

$$\dot r(0) = \frac{g}{\omega} + c' = 0 \Rightarrow c' = -\frac{g}{\omega}$$

$$\Rightarrow \boxed{r(t) = b + \frac{g}{\omega_0^2}\sin\omega_0 t - \frac{g}{\omega_0} t + \frac{1}{2}\omega_0^2 b t^2}$$

Q5. (15 points)

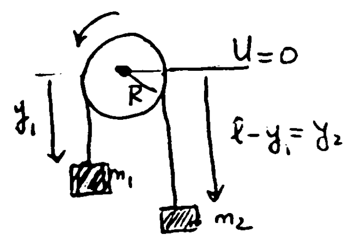
Consider an Atwood machine consisting of a pulley of moment of inertia $I = (1/2)MR^2$ and two masses m_1 and m_2 hanging from a massless rope of fixed length l.

(a) Show that Lagrange equation can be written in the form

$$L = \frac{1}{2}(m_1 + m_2 + \frac{I}{R^2})\dot{y}^2 + m_1 g y + m_2 g(l - y)$$

(b) Write the Hamiltonian of the system and Hamilton's equations of motion.

(c) Calculate the acceleration of each mass if $m_1 = 3$ kg, $m_2 = 1$ kg, and $M = 5$ kg.



(a) $T = \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{y}_2^2 + \frac{1}{2} I \omega^2$

$y_2 = l - y_1 \Rightarrow \dot{y}_2 = -\dot{y}_1$;

$\omega = \frac{\dot{y}_1}{R}$

call $y_1 = y$!

$T = \frac{1}{2} m_1 \dot{y}^2 + \frac{1}{2} m_2 \dot{y}^2 + \frac{1}{2}\frac{I}{R^2}\dot{y}^2$

$U = -m_1 g y_1 - m_2 g y_2 = -m_1 g y - m_2 g (l - y)$

$$\boxed{L = \frac{1}{2}\left(m_1 + m_2 + \frac{I}{R^2}\right)\dot{y}^2 + m_1 g y + m_2 g (l - y)}$$

(b) $H = P_y \dot{y} - \frac{1}{2}\left(m_1 + m_2 + \frac{I}{R^2}\right)\dot{y}^2 - m_1 g y - m_2 g (l - y)$

$p_y = \frac{\partial L}{\partial \dot{y}} = \left(m_1 + m_2 + \frac{I}{R^2}\right)\dot{y} \Rightarrow \dot{y} = \frac{P_y}{\left(m_1 + m_2 + \frac{I}{R^2}\right)}$

$H = \frac{P_y^2}{\left(m_1 + m_2 + \frac{I}{R^2}\right)} - \frac{(m_1 + m_2 + I/R)\, P_y^2}{2\left(m_1 + m_2 + \frac{I}{R^2}\right)^2} - m_1 g y - m_2 g (l - y)$

$H = H(q_k, P_k, t)$!!! No $\dot{q}_k$.

$$\Rightarrow \boxed{H = \frac{p_y^2}{2\left(m_1+m_2+\frac{I}{R^2}\right)} - m_1 g y - m_2 g(\ell - y)}$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m_1+m_2+\frac{I}{R^2}}$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = m_1 g - m_2 g = g(m_1 - m_2)$$

$$\Rightarrow \quad \ddot{y} = \frac{\dot{p}_y}{m_1+m_2+\frac{I}{R^2}} = \frac{g(m_1-m_2)}{m_1+m_2+\frac{I}{R^2}}$$

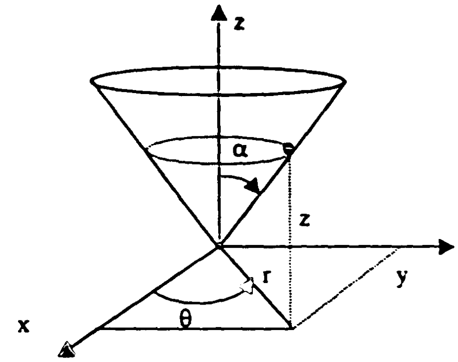
$$I = \frac{1}{2} M R^2$$

$$\Rightarrow \boxed{\ddot{y} = \frac{g(m_1-m_2)}{m_1+m_2+\frac{M}{2}}}$$

Calculation $\Rightarrow \ddot{y} = 3.0 \ m/s^2$

Q6. (15 points)

A particle of mass m is constrained to move on the inside surface of a smooth cone of half angle α, i.e., $z = r \text{ cotan } \alpha$.



(a) Calculate the total mechanical energy of the system and show that it can be written in the form

$$E = \frac{1}{2} m^* \dot{r}^2 + V_{eff}$$

where $m^* = \dfrac{m}{\sin^2 \alpha}$

and identify the expression of V_{eff}.

(b) Study the approximate behavior of V_{eff} (look for extrema) and plot the curve V_{eff} versus r for $r > 0$.

(c) Discuss the motion of the particle when $E_1 > V_{eff}^{min}$ and when $E_2 = V_{eff}^{min}$. What happens when $E < V_{eff}^{min}$?

(a) $E = T + U$

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2 \right)$$

$$\therefore \quad U = mgz = mg \cdot r \cdot \text{cotan}\,\alpha$$

$$\Rightarrow E = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2 \right) + mgr \text{ cotan}\,\theta$$

$$= \frac{1}{2} m \left(\dot{r}^2 + \dot{r}^2 \text{cotan}^2 \alpha \right) + \frac{1}{2} m r^2 \dot{\theta}^2 + mgr \text{ cotan}\,\theta$$

$$= \frac{1}{2} m \dot{r}^2 \left(1 + \text{cotan}^2 \alpha \right) + \frac{1}{2} m r^2 \frac{\ell^2}{m^2 r^4} + mgr \text{ cotan}\,\theta$$

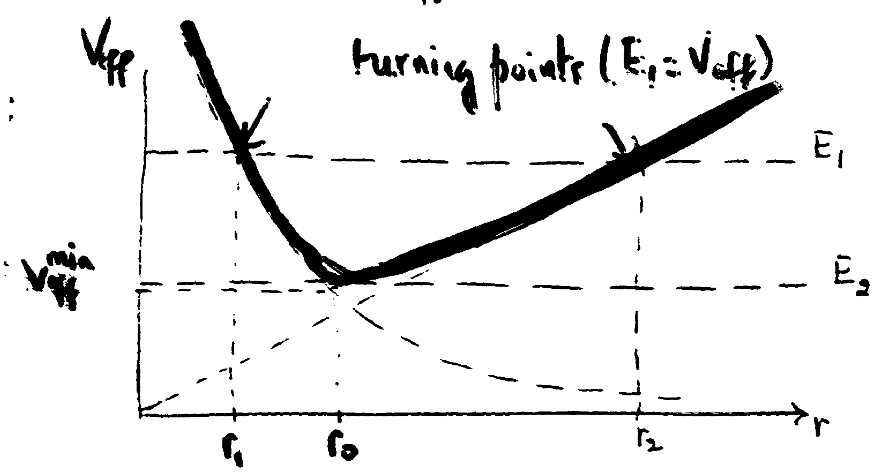
$$E = \frac{1}{2} m \dot{r}^2 \frac{1}{\sin^2\alpha} + \frac{1}{2}\frac{\ell^2}{mr^2} + mgr\cot\alpha$$

$$E = \frac{1}{2} m^* \dot{r}^2 + V_{eff}$$

where $m^* = \dfrac{m}{\sin^2\alpha}$ and $V_{eff} = \dfrac{\ell^2}{2mr^2} + mgr\cot\alpha$

b) $$\frac{dV_{eff}}{dr} = -\frac{\ell^2}{mr^3} + mg\cot\alpha = 0 \Rightarrow r_0 = \left[\frac{\ell^2}{m^2 g\cot\alpha}\right]^{1/3}$$

$$\left.\frac{d^2V_{eff}}{dr^2}\right|_{r_0} = \left.\frac{3\ell^2}{mr^4}\right|_{r_0} > 0 \Rightarrow \text{stable equilibrium (minimum)}$$



(c) For $E_1 > V_{eff}^{min}$ we have a periodic motion between r_1 and r_2, the two turning points.

For $E_2 = V_{eff}^{min}$ $\dot{r} = 0$ and $r = r_0$, the particle will move in a circular orbit of radius r_0.

For $E < V_{eff}^{min}$ $\dot{r}^2 < 0$ (impossible).

Q7. (15 points)

An object of unit mass orbits in a central potential U(r). Its orbit is $r = ae^{-b\theta}$, where a and b are constants.

(a) Find the potential U as a function of r.

(b) Find θ as a function of time.

(a) $$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{r^2}{\ell^2} F(r)$$

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{b}{a} e^{b\theta} \quad ; \quad \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{b^2}{a} e^{b\theta}$$

$$\frac{b^2}{a} e^{b\theta} + \frac{e^{b\theta}}{a} = -\frac{a^2 e^{-2b\theta}}{\ell^2} F(r)$$

$$(b^2+1)\frac{e^{b\theta}}{a} = -\frac{a^2 e^{-2b\theta}}{\ell^2} F(r)$$

$$\Rightarrow F(r) = -\frac{\ell^2(b^2+1)}{a^3 e^{-3b\theta}} = -\frac{\ell^2(b^2+1)}{r^3}$$

$$\Rightarrow F(r) = -\frac{\ell^2(b^2+1)}{r^3} = -\frac{dU}{dr}$$

$$U(r) = -\int_{\infty}^{r} F(r)\,dr = -\frac{\ell^2(b^2+1)}{2r^2}$$

$$\boxed{U(r) = -\frac{\ell^2(b^2+1)}{2r^2}}$$

(b) $$\dot{\theta} = \frac{d\theta}{dt} = \frac{\ell^2}{r^2} = \frac{\ell^2}{a^2 e^{-2b\theta}} = \frac{\ell^2}{a^2} e^{2b\theta}$$

$$\int e^{-2b\theta}\, d\theta = \int \frac{\ell^2}{a^2}\, dt$$

$$-\frac{1}{2b}\,e^{-2b\theta} = \frac{\ell^2}{a^2}\,t + C_1$$

$$e^{-2b\theta} = -\frac{2b\ell^2}{a^2}\,t + C_2$$

$$-2b\,\theta = \ln\left(-\frac{2b\ell^2}{a^2}\,t + C_2\right)$$

$$\boxed{\theta(t) = -\frac{1}{2b}\ln\left(-\frac{2b\ell^2}{a^2}\,t + C_2\right)}$$

Formula Sheet

$x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta)$ $\qquad$ $\omega_1 = \sqrt{\omega_o^2 - \beta^2}$

$x(t) = e^{-\beta t}(A_1 e^{\omega_2 t} + A_2 e^{-\omega_2 t})$ $\qquad$ $\omega_2 = \sqrt{\beta^2 - \omega_o^2}$

$x(t) = (A + Bt)e^{-\beta t}$ $\qquad$ $\omega_o = \beta_c$

$x_p(t) = D\cos(\omega t - \delta)$ $\qquad$ $D = \dfrac{A}{\sqrt{(\omega_o^2 - \omega^2)^2 + 4\omega^2\beta^2}}$

$\Phi = -G\int \frac{\rho}{r} dv$ or $\Phi = -G\int \frac{\sigma}{r} dA$ or $\Phi = -G\int \frac{\lambda}{r} dl$

$g = -G\int \frac{\rho}{r^2} dv$ or $g = -G\int \frac{\sigma}{r^2} dA$ or $g = -G\int \frac{\lambda}{r^2} dl$

$\oint \vec{g}\cdot d\vec{A} = -4\pi G M_{encl}$

$\dfrac{\partial f}{\partial y_i} = \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'_i}\right) = 0 \qquad i = 1, 2, 3 \ldots, n$; $\dfrac{\partial f}{\partial x} - \dfrac{d}{dx}\left(f - y'\dfrac{\partial f}{\partial y'}\right) = 0$

$L = T - U$; $\dfrac{\partial L}{\partial q_i} - \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_i} = 0$; $H = \sum_i p_i \dot{q}_i - L$; $p_k = \dfrac{\partial L}{\partial \dot{q}_k}$

$\dot{q}_k = \dfrac{\partial H}{\partial p_k}$; $\dfrac{\partial H}{\partial q_k} = -\dot{p}_k$; $-\dfrac{\partial L}{\partial t} = \dfrac{\partial H}{\partial t}$; $\dfrac{d^2}{d\theta^2}\left(\dfrac{1}{r}\right) + \dfrac{1}{r} = -\dfrac{\mu r^2}{l^2}F(r)$

$E = T + U$ $\qquad$ $\vec{F} = -\vec{\nabla} U$ $\qquad$ $l = mr^2\dot{\theta} = const$

$Ln(1 + x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \cdots$; $\int \dfrac{dy}{\sqrt{a^2 + y^2}} = Ln(y + \sqrt{y^2 + a^2})$