KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

Department of Physics

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PHYSICS 212.01(002)

Third Major Examination (Ch. 5 & 6) – Closed Book

Name: Key ID. #__________

(30 Pts) 1. Explain briefly what is meant by the following, stating any relevant mathematical relations, and defining symbols used.

(a) Probability density for a particle whose wave function is $\Psi(x,t)$.

$\rho(x,t) = |\Psi(x,t)|^2 = \Psi^*(x,t)\,\Psi(x,t)$

$\rho(x,t)\,dx$ = probability that a particle constrained to move along the x-axis will be found in an interval dx, around point x, at time t.

(b) Expectation value of an observable Q if the wave function is $\Psi(x,t)$..

The expectation value of an observable Q is the average value of the observable Q, at time t = $\langle Q \rangle$

$$\langle Q \rangle = \int_{-\infty}^{\infty} \Psi^*(x,t)\,[Q]\,\Psi(x,t)\,dx$$

where $[Q]$ is the associated operator, for the observable

(c) Time-independent Schrodinger equation.

Separable solutions of the Schrodinger Equation that are of the form $\Psi(x,t) = \psi(x)\,e^{-i\omega t}$, with $\psi(x)$ a time-independent wavefunction satisfying the Time independent Schrodinger equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U(x)\psi(x) = E\psi(x)$$

(d) The Heisenberg uncertainty principle.

for cannonically conjugate variables such as position & linear momentum along the same axis, or energy & time, the uncertainties of the two variables is such that: the product is $\geq \frac{\hbar}{2}$, e.g.

$$\Delta p_x \, \Delta x \geq \frac{\hbar}{2}, \text{ etc}$$

$$\Delta E \, \Delta t \geq \frac{\hbar}{2}.$$

(e) Particle transmission coefficient.

T = Transmission coefficient measures the probability that a particle incident on a barrier will emerge on the other side of the barrier. Or,

$$T = \frac{(\Psi^* \Psi)_{\text{transmitted}}}{(\Psi^* \Psi)_{\text{incident}}}$$

(f) Sharp and fuzzy observables.

Sharp observables are observables where repeated measurements performed on identically prepared systems, always yield the same value.

Fuzzy observables, such as position, are those for which repeated measurements yield different results. Fuzziness is reflected in the spread about an average value as measured by the standard deviation – ΔQ, which can be calculated from $\Delta Q = \sqrt{\langle Q^2 \rangle - \langle Q \rangle^2}$.

(30 Pts) 2. An electron is trapped in an infinitely deep potential well with width L=0.300 nm. The wave function for the electron is $\psi(x) = A \sin\left(\frac{n\pi x}{L}\right)$.

(a) Show that the normalization constant $A = \sqrt{\frac{2}{L}}$

(b) If the electron is in the ground state, what is the probability of finding the electron between x = 0.0 and 0.100 nm.

(c) Repeat the calculation of part (b) for the n = 3 state.

(d) Calculate the energy in electron-Volt for the electron in the n = 2 state.

a) $$1 = \int_0^L |\psi|^2 dx = A^2 \int_0^L \sin^2 \frac{n\pi x}{L} = \frac{A^2}{2}\int_0^L \left(1 - \cos\frac{2n\pi x}{L}\right)dx$$

$$= \frac{A^2}{2}\left(x - \frac{L}{2n\pi}\sin\frac{2n\pi x}{L}\right)\Big|_0^L = \frac{A^2 L}{2} \Rightarrow \underline{\underline{A = \sqrt{\frac{2}{L}}}}$$

b) Given $L = 0.300\,nm$, $n = 1$, $0.100\,nm = \frac{L}{3}$

$$P_{(0-0.100)\,nm} = \frac{2}{L}\int_0^{L/3} \sin^2\frac{\pi x}{L}\,dx = \frac{\cancel{2}}{\cancel{2}L}\int_0^{\frac{L}{3}}\left(1 - \cos\frac{2\pi x}{L}\right)dx$$

$$P_{0-0.1} = \frac{1}{L}\left(x - \frac{L}{2\pi}\sin\frac{2\pi x}{L}\right)\Big|_0^{\frac{L}{3}} = \frac{1}{L}\left(\frac{L}{3} - \frac{L}{2\pi}\sin\frac{2\pi}{3} - 0 + 0\right)$$

$$= \frac{1}{3} - \frac{1}{2\pi}\frac{\sqrt{3}}{2} = \underline{\underline{0.1955}}$$

c) for $n = 3$, $\psi = \frac{2}{L}\sin\frac{3\pi x}{L}$, $0.100 = \frac{L}{3}$

$$P_{0-0.1} = \frac{2}{L}\int_0^{L/3}\sin^2\frac{3\pi x}{L}\,dx = \frac{\cancel{2}}{\cancel{2}L}\int_0^{L/3}\left(1 - \cos\frac{6\pi x}{L}\right)dx$$

$$= \frac{1}{L}\left(x - \frac{L}{6\pi}\sin\frac{6\pi x}{L}\right)\Big|_0^{L/3} = \frac{1}{L}\left(\frac{L}{3} - \frac{L}{6\pi}\sin 2\pi - 0 + 0\right)$$

$$= \underline{\underline{\frac{1}{3}}} = \underline{\underline{0.333}}$$

d) For $n = 2$ state, we have $2\frac{\lambda}{2} = L$, $\lambda = \frac{h}{p} \Rightarrow p = \frac{h}{L}$

$$K = \frac{p^2}{2m} = \frac{h^2}{2mL^2} = \frac{(6.62\times10^{-34}\,J\cdot s)^2}{2\times 9.11\times10^{-31}\,kg\,(0.300\times10^{-9}\,m)^2}$$

$$= 2.68\times10^{-18}\,J = \underline{\underline{16.7\,eV}}$$

Problem 2:

A particle moving along the $x-$axis is in a stationary state characterized by the wave function

$$\psi(x) = \begin{cases} 0 & x < -a \\ A\left(1 + \cos\frac{\pi x}{a}\right) & -a < x < a \\ 0 & x > a \end{cases}$$

a) Sketch $\psi(x)$.

b) Is this wave function physically acceptable? Explain why.

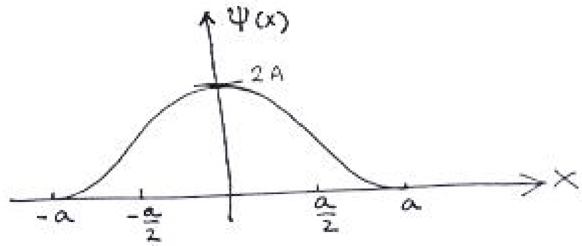
c) Calculate A so that $\psi(x)$ is normalized.

d) Evaluate $< x >$, $< p >$, Δx, Δp.

Problem 2:

$$\psi(x) = \begin{cases} 0 & x < -a,\ x > a \\ A\left(1+\cos\frac{\pi x}{a}\right) & -a < x < a \end{cases}$$

a)



b) This wave function is indeeded physically acceptable because it is square integrable, single-valued, continuous and has a continuous first derivative.

c)
$$1 = \int_{-a}^{a} |\psi(x)|^2 dx = |A|^2 \int_{-a}^{a} \left[1 + 2\cos\left(\frac{\pi x}{a}\right) + \cos^2\left(\frac{\pi x}{a}\right)\right] dx$$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$= |A|^2 \int_{-a}^{a} \left[\frac{3}{2} + 2\cos\left(\frac{\pi x}{a}\right) + \frac{1}{2}\cos\left(\frac{2\pi x}{a}\right)\right] dx$$

$$= |A|^2 \left[\frac{3}{2}x + \frac{2a}{\pi}\sin\left(\frac{\pi x}{a}\right) + \frac{a}{4\pi}\sin\left(\frac{2\pi x}{a}\right)\right]_{-a}^{a}$$

$$= 3aA^2 \Rightarrow \boxed{A = \frac{1}{\sqrt{3a}}}$$

d) since $\psi(x)$ is even $\Rightarrow$ $\langle x \rangle = \int_{-a}^{a} x\,|\psi(x)|^2 dx = 0$

since $\psi(x)$ is real $\Rightarrow$ $\langle p \rangle = 0$ $\quad \langle p \rangle = -\int_{-a}^{a} \psi^* i\hbar \frac{\partial \psi}{\partial x}\, dx$

$$\langle x^2 \rangle = A^2 \int_{-a}^{a} x^2 \left(1 + \cos\frac{\pi x}{a}\right)^2 dx = A^2 \int_{-a}^{a} x^2 \left[\frac{3}{2} + 2\cos\left(\frac{\pi x}{a}\right) + \frac{1}{2}\cos\left(\frac{2\pi x}{a}\right)\right] dx$$

$$= \frac{a^2}{3}\left(1 - \frac{15}{4\pi^2}\right) \Rightarrow \boxed{\Delta x = \sqrt{\langle x^2 \rangle} = a\sqrt{\frac{1}{3} - \frac{5}{4\pi^2}}}$$

$$\langle p^2 \rangle = A^2 \int_{-a}^{a} \left(1 + \cos\frac{\pi x}{a}\right)\left(-\hbar^2 \frac{d^2}{dx^2}\right)\left(1 + \cos\frac{\pi x}{a}\right) dx$$

$$= \frac{\hbar^2\pi^2}{a^2} A^2 \int_{-a}^{a} \left(\left[\cos\left(\frac{\pi x}{a}\right) + \cos^2\left(\frac{\pi x}{a}\right)\right] dx = \frac{\hbar^2\pi^2}{a^2}\frac{1}{3a}\, a\right.$$

$$\Rightarrow \boxed{\Delta p = \sqrt{\langle p^2 \rangle} = \frac{\hbar\pi}{\sqrt{3}\,a}}$$

Problem 2: (5 points)

Consider a particle of mass m which moves inside an infinite one-dimensional box potential

$$V(x) = \begin{cases} \infty & x < -\frac{a}{2} \\ 0 & -\frac{a}{2} \leq x \leq \frac{a}{2} \\ \infty & x > \frac{a}{2} \end{cases}$$

Assume that the wave function of this particle is

$$\psi(x) = \begin{cases} 0 & x < -\frac{a}{2} \\ B \sin\left(\frac{2\pi x}{a}\right) e^{-iEt/\hbar} & -\frac{a}{2} \leq x \leq \frac{a}{2} \\ 0 & x > \frac{a}{2} \end{cases}$$

a)Find the energy of this particle.

b) Sketch the probability $P(x,t)$ for this particle.

c) Calculate B so that $\psi(x)$ is normalized.

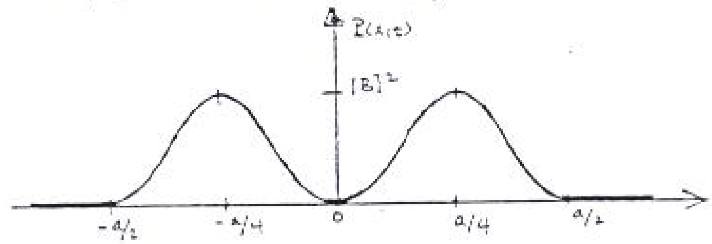
d) Evaluate $< x >$, $< p >$, $< x^2 >$ and $< p^2 >$.

e) Evaluate $\Delta x \Delta p$. Is this result compatible with the uncertainty principle?

a) $E_2 = ?$

$$H\psi_2 = -\frac{\hbar^2}{2m}\frac{\partial^2\psi_2}{\partial x^2} = \left(\frac{2\pi}{a}\right)^2\frac{\hbar^2}{2m}\psi_2 \rightarrow \boxed{E_2 = \frac{2}{m}\left(\frac{\pi\hbar}{a}\right)^2}$$

b) $P(x,t) = |\psi_2(x,t)|^2 = |B|^2 \sin^2\left(\frac{2\pi x}{a}\right)$



c) $B = ?$

Normalization: $$\int_{-\infty}^{+\infty}|\psi_2(x,t)|^2 dx = 1 \rightarrow |B|^2\int_{-a/2}^{a/2}\sin^2\left(\frac{2\pi x}{a}\right)dx = \frac{|B|^2}{2}\int_{-a/2}^{a/2}dx$$

$$\boxed{B = \sqrt{2/a} = A}$$

d) $\langle x\rangle = ?$

$$\langle x\rangle = \int_{-\infty}^{+\infty}\psi_2^* \, x \, \psi_2 \, dx = |B|^2\int_{-a/2}^{a/2} x \sin^2\left(\frac{2\pi x}{a}\right)dx = 0$$

e) $\langle p\rangle = ?$ $$\langle p\rangle = \frac{2\pi}{a}\frac{\hbar}{i}|B|^2\int_{-a/2}^{a/2}\sin\left(\frac{2\pi x}{a}\right)\cos\left(\frac{2\pi x}{a}\right)dx = 0$$

f) $\Delta x = ?$ $$\Delta x = \sqrt{\langle x^2\rangle} = \left\{\frac{2}{a}\int_{-a/2}^{a/2}x^2\sin^2\left(\frac{2\pi x}{a}\right)dx\right\}^{1/2} = \frac{a}{2}\sqrt{\frac{1}{3}-\frac{1}{2\pi^2}}$$

g) $\Delta p = ?$ $$\Delta p = \sqrt{\langle p^2\rangle} = \left\{\left(\frac{2\pi}{a}\right)^2\hbar^2\int_{-a/2}^{a/2}|\psi|^2 dx\right\}^{1/2} = \frac{2\pi\hbar}{a}$$

h) $\Delta x \cdot \Delta p = ?$

$$\boxed{\Delta x \cdot \Delta p = \pi\hbar\sqrt{\frac{1}{3}-\frac{1}{2\pi^2}} \simeq 1.7\,\hbar} > (\Delta x \cdot \Delta p)_1$$

this is due to the fact that when the energy increases the uncertainty on the position Δx,