KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
DEPARTMENT OF PHYSCIS

PHYSICS 212 Modern Physics
TERM 102
EXAM #2

Date: Saturday 04 May 2011
Instructor: Dr. A. Mekki
Time: 120 minutes

Name: Key **Id#:**

SHOW THE DETAILS OF YOUR WORK

Question#	Maximum	Your grade
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total	80	
	100	

Q1. (10 points)

(a) Find an expression for the de Broglie wavelength for an electron accelerated through a **large** potential difference V.
(Hint: use relativistic linear momentum and energy.)

$$\lambda = \frac{h}{p}$$

$$E^2 = p^2c^2 + \cancel{(m_0c^2)^2} = (K + m_0c^2)^2 = K^2 + 2Km_0c^2 + \cancel{(m_0c^2)^2}$$

$$\Rightarrow pc = \sqrt{K^2 + 2Km_0c^2} = \sqrt{1 + \frac{K}{2m_0c^2}}\,\sqrt{2Km_0c^2}$$

$$\lambda = \frac{hc}{pc} = \frac{hc}{\sqrt{2Km_0c^2}\sqrt{1+\frac{K}{2m_0c^2}}} = \frac{hc}{\sqrt{2eVm_0c^2}\sqrt{1+\frac{eV}{2m_0c^2}}}$$

$$\lambda = \frac{hc}{\sqrt{2em_0c^2}\sqrt{V}\sqrt{1+\frac{eV}{2m_0c^2}}} = \frac{12.27}{\sqrt{V}}\,\frac{1}{\sqrt{1+\frac{eV}{2m_0c^2}}}\ (\mathring{A})$$

$$\frac{hc}{\sqrt{2em_0c^2}} = \frac{12400\,\text{eV}.\mathring{A}}{\left(2\times1.6\times10^{-19}\times 0.511\times10^{6}/1.6\times10^{-19}\right)^{1/2}} = 12.27$$

(b) Calculate the de Broglie wavelength for a potential difference of 10 MV.

$$V = 10\,\text{MV} = 10^7\,\text{V}$$

$$\lambda = \frac{12.27\,\text{eV}.\mathring{A}}{\sqrt{10^7}}\,\frac{1}{\sqrt{1+\frac{10\,\text{MeV}}{2\times0.511\,\text{MeV}}}} = 1.18\times10^{-3}\,\mathring{A}$$

(c) Calculate the percentage error introduced when the classical de Broglie wavelength is used.

$$\lambda = \frac{h}{\sqrt{2m_0eV}} = \frac{hc}{\sqrt{2mc^2eV}} = \frac{12400\,\text{eV}.\mathring{A}}{\sqrt{2\times0.511\times10^{6}\times10^{7}}}$$

$$= 3.88\times10^{-3}\,\mathring{A}$$

$$\%\ \text{error} = \frac{3.88-1.18}{1.18} = 229\%$$

Q2. (10 points)

(a) What is the difference between phase velocity and group velocity?

The phase velocity is the velocity of the Components waves making the group of waves.

The group velocity is the velocity of the wave packet representing the particle.

The angular frequency of the surface waves in a liquid is given in terms of the wave number k by $\omega = \sqrt{\frac{Tk^3}{\rho}}$ where ρ is the density of the liquid and T is the surface tension (which gives an upward force on an element of the surface liquid). Find

(b) the **phase velocity**

$$v_p = \frac{\omega}{k} = \sqrt{\frac{T}{\rho}} \frac{k\sqrt{k}}{k} = \sqrt{\frac{Tk}{\rho}}$$

(c) and **the group velocity** for these surface waves

$$v_g = \frac{d\omega}{dk} = \sqrt{\frac{T}{\rho}} \frac{d}{dk}\left(k^{3/2}\right) = \sqrt{\frac{T}{\rho}} \frac{3}{2} k^{1/2}$$

$$= \frac{3}{2}\sqrt{\frac{Tk}{\rho}} = \frac{3}{2} v_p$$

(d) Do we have dispersion in this case? Explain.

Since v_p depends on k, we have dispersion.

Q3. (10 points)

Using the nonrelativistic Doppler formula ($\frac{v}{c}=\frac{\Delta f}{f}$) calculate the Doppler broadening, $\Delta\lambda$, of a 400 nm line emitted by a Hydrogen atom at 2000 K. Calculate the broadening by considering the atom to be moving directly toward an observer. Take the kinetic energy to be $K=\frac{3}{2}k_B T$, where k_B is Boltzman constant and T is the temperature of the atom.

(Hint: Find first the speed of the atom)

$$KE = \frac{1}{2} m v^2 = \frac{3}{2} k_B T$$

$$v = \sqrt{\frac{3 k_B T}{m}} = \sqrt{\frac{3 \times 1.38\times10^{-23} \times 2000}{1.67\times10^{-27}}} = 7041 \text{ m/s}$$

$$\frac{\Delta f}{f} = \frac{v}{c} = \frac{\Delta f \lambda}{c}$$

$$\Rightarrow \quad \Delta f = \frac{v}{\lambda} = \frac{7041}{400\times10^{-9}} = 1.76\times10^{10} \text{ Hz}$$

$$f = \frac{c}{\lambda} \Rightarrow df = -\frac{c}{\lambda^2} d\lambda$$

$$\Delta f = \frac{c}{\lambda^2} \Delta\lambda$$

$$\Delta\lambda = \frac{\Delta f \lambda^2}{c} = \frac{1.76\times10^{10} \times (400\times10^{-9})^2}{3\times10^8} = 0.094\times10^{-10} \text{ m}$$

$$\boxed{\Delta\lambda = 0.094 \text{ Å}}$$

Q4. (10 points)
Consider a proton moving with a speed of 3.0 x 10^7 m/s along the x-axis. If the uncertainty in its position is 0.01 x 10^{-10} m, calculate the minimum uncertainty in
(a) its momentum

$$\Delta x \, \Delta p_x = \frac{\hbar}{2}$$

$$\Delta p_x = \frac{\hbar}{2\Delta x} = \frac{1.05\times10^{-34}}{2\times 0.01\times10^{-10}} = 5.25\times10^{-23}\ \mathrm{kg}\cdot\frac{\mathrm{m}}{\mathrm{s}}$$

(b) its speed

$$p_x = m v_x \Rightarrow \Delta p_x = m\,\Delta v_x$$

$$\Delta v_x = \frac{\Delta p_x}{m} = \frac{5.25\times10^{-23}}{1.67\times10^{-27}} = 3.14\times10^{4}\ \mathrm{m/s}$$

(c) its kinetic energy

$$K = \frac{1}{2} m v_x^2$$

$$dK = m v_x \, dv_x$$

$$\Delta K = m v_x \, \Delta v_x$$

$$\Delta K = 1.67\times10^{-27} \times 3\times1\times10^{7} \times 3.14\times10^{4}$$

$$\Delta K = 1.57\times10^{-15}\ \mathrm{J}$$

Q5. (10 points)
An electron is trapped in a one-dimensional region of length 1.0 x 10^{-10} m (a typical atomic diameter).
(a) How much energy (in eV) must be supplied to excite the electron from the ground state to the third excited state?

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2}$$

$$\Delta E = E_4 - E_1 = \frac{(16-1)\pi^2\hbar^2}{2mL^2} = \frac{15\pi^2\hbar^2}{2mL^2}$$

$$= \frac{15 \times \pi^2 \times (1.05\times10^{-34})^2}{2\times 9.1\times10^{-31}\times(1\times10^{-10})^2} = 8.9\times10^{-17} J = 561\,eV$$

(b) In the third excited state, what is the probability of finding the electron in the region from x = 0.10 × 10^{-10} m to x = 0.20 × 10^{-10} m?

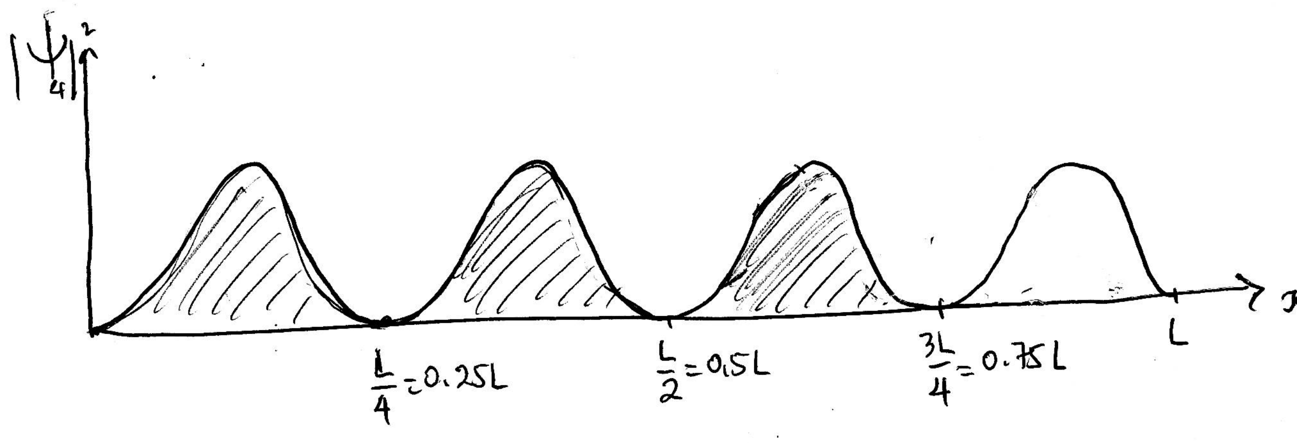
$$P = \int_{0.1L}^{0.2L} |\psi_4|^2 dx = \frac{2}{L}\int_{0.1L}^{0.2L} \sin^2\left(\frac{4\pi}{L}x\right) dx$$

$$= \frac{2}{L}\left[\frac{x}{2}\Big|_{0.1L}^{0.2L} - \frac{L}{16\pi}\sin\left(\frac{8\pi}{L}x\right)\Big|_{0.L}^{0.2L}\right]$$

$$= 0.1 - \frac{1}{8\pi}\left(\sin(1.6\pi) - \sin(0.8\pi)\right) = 0.16 = 16\%$$

$P = 16\%$

(c) Draw a diagram showing the probability density for the third excited state and deduce from the diagram the probability of finding the electron in the region x = 0.0 and x = 0.75 x 10^{-10} m.



$P = \frac{3}{4} = 0.75 = 75\%$

Q6. (10 points)
Consider the wave function of the ground state of a simple harmonic oscillator. Calculate
(a) <x>

$$\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\frac{\alpha x^2}{2}} \qquad \alpha = \frac{m\omega}{\hbar}$$

$$\langle x \rangle = \int_{-\infty}^{+\infty} x\,|\psi_0|^2\,dx = \left(\frac{\alpha}{\pi}\right)^{1/2} \int_{-\infty}^{+\infty} \underset{\uparrow\atop\text{odd}}{x}\; \underset{\uparrow\atop\text{even}}{e^{-\alpha x^2}}\,dx = 0$$

(b) <x^2>

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} x^2 |\psi_0|^2\,dx = \left(\frac{\alpha}{\pi}\right)^{1/2} 2 \underbrace{\int_0^{+\infty} x^2 e^{-\alpha x^2}\,dx}_{\frac{1}{4\alpha}\sqrt{\frac{\pi}{\alpha}}}$$

$$= 2\sqrt{\frac{\alpha}{\pi}}\,\frac{1}{4\alpha}\sqrt{\frac{\pi}{\alpha}} = \frac{1}{2\alpha}$$

$$\langle x^2 \rangle = \frac{\hbar}{2m\omega}$$

(c) <p>

$$\langle p \rangle = -i\hbar \left(\frac{\pi}{\alpha}\right)^{1/2} \int_{-\infty}^{+\infty} e^{-\frac{\alpha x^2}{2}} \frac{\partial}{\partial x} e^{-\frac{\alpha x^2}{2}}\,dx$$

$$= -i\hbar \left(\frac{\pi}{\alpha}\right)^{1/2} \int_{-\infty}^{+\infty} e^{-\frac{\alpha x^2}{2}} (-\alpha x)\, e^{-\frac{\alpha x^2}{2}}\,dx$$

$$= -i\hbar \left(\frac{\pi}{\alpha}\right)^{1/2} \alpha \int_{-\infty}^{+\infty} \underset{\uparrow\atop\text{odd}}{x}\; \underset{\uparrow\atop\text{even}}{e^{-\alpha x^2}}\,dx = 0$$

(d) $<p^2>$

$$\langle K\rangle + \langle U\rangle = \langle E\rangle$$

$$\frac{\langle p^2\rangle}{2m} + \frac{1}{2}\underbrace{k}_{m\omega^2}\underbrace{\langle x^2\rangle}_{\frac{\hbar}{2m\omega}} = \frac{1}{2}\hbar\omega$$

$$\frac{\langle p^2\rangle}{2m} + \frac{\hbar\omega}{4} = \frac{1}{2}\hbar\omega$$

$$\frac{\langle p^2\rangle}{2m} = \frac{1}{4}\hbar\omega \quad \Rightarrow \quad \langle p^2\rangle = \frac{m\hbar\omega}{2}$$

(e) Δx Δp

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{\hbar}{2m\omega}}$$

$$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\frac{m\hbar\omega}{2}}$$

$$\Delta x \cdot \Delta p = \frac{\hbar}{2}$$

(f) Is the uncertainty principal violated? Explain.

No, we found $\Delta x . \Delta p = \frac{\hbar}{2}$!

Q7. (10 points)

Consider an electron held in a three dimensional infinite potential well in the form

$$U(x,y,z)=\begin{cases} 0 & 0 < x < L, 0 < y < 2L, 0 < z < 2L \\ \infty & \text{otherwise} \end{cases}$$

(a) Find the energies of the five lowest states

$$E_n = \frac{\pi^2\hbar^2}{2m_e}\left[\left(\frac{n_1}{L}\right)^2 + \left(\frac{n_2}{2L}\right)^2 + \left(\frac{n_3}{2L}\right)^2\right]$$

$$= \underbrace{\frac{\pi^2\hbar^2}{8m_eL^2}}_{\equiv E_0}\left(4n_1^2 + n_2^2 + n_3^2\right) = E_0\left(4n_1^2 + n_2^2 + n_3^2\right)$$

$$E_{111} = 6E_0$$

$$E_{121} = E_{112} = 9E_0$$

$$E_{122} = 12E_0$$

$$E_{131} = E_{113} = 14E_0$$

$$E_{123} = E_{132} = 17E_0$$

(b) What are their corresponding normalized wavefunctions?

$$\psi_{111} = \frac{\sqrt{2}}{L^{3/2}}\sin\left(\frac{\pi}{L}x\right)\sin\left(\frac{\pi}{2L}y\right)\sin\left(\frac{\pi}{2L}z\right)$$

$$\psi_{121} = \frac{\sqrt{2}}{L^{3/2}}\sin\left(\frac{\pi}{L}x\right)\sin\left(\frac{\pi}{L}y\right)\sin\left(\frac{\pi}{2L}z\right)$$

$$\psi_{112} = \frac{\sqrt{2}}{L^{3/2}}\sin\left(\frac{\pi}{L}x\right)\sin\left(\frac{\pi}{2L}y\right)\sin\left(\frac{\pi}{L}z\right)$$

$$\psi_{122} = \frac{\sqrt{2}}{L^{3/2}}\sin\left(\frac{\pi}{L}x\right)\sin\left(\frac{\pi}{L}y\right)\sin\left(\frac{\pi}{L}z\right)$$

$$\psi_{131} = \frac{\sqrt{2}}{L^{3/2}}\sin\left(\frac{\pi}{L}x\right)\sin\left(\frac{3\pi}{2L}y\right)\sin\left(\frac{\pi}{2L}z\right)$$

(c) State which states are degenerate?

$$\psi_{113} = \frac{\sqrt{2}}{L^{3/2}}\sin\left(\frac{\pi}{L}x\right)\sin\left(\frac{\pi}{2L}y\right)\sin\left(\frac{3\pi}{2L}z\right)$$

$$\psi_{123} = \frac{\sqrt{2}}{L^{3/2}}\sin\left(\frac{\pi}{L}x\right)\sin\left(\frac{2\pi}{2L}y\right)\sin\left(\frac{3\pi}{2L}z\right)$$

$$\psi_{132} = \sqrt{2}/L^{3/2}\sin\left(\frac{\pi}{L}x\right)\sin\left(\frac{3\pi}{2L}y\right)\sin\left(\frac{\pi}{L}z\right)$$

$E_{121} = E_{112}$ is doubly degenerate

similarly $E_{131} = E_{113}$ and $E_{123} = E_{132}$

Q8. (10 points)

(a) Write the radial wave function for the *2p* states of the hydrogen atom.

$$R_{2p}(r) = \left(\frac{1}{2a_0}\right)^{3/2} \frac{r}{a_0\sqrt{3}} e^{-r/2a_0}$$

a_0 : Bohr radius

(b) Calculate the energy of the electron in this state.

$$E_n = -\frac{13.6}{n^2}$$

$$n=2 \qquad E_2 = -\frac{13.6}{4} = -3.4\ \text{eV}$$

(c) Calculate the *average value* of r for an electron in the *2p* of the hydrogen atom.\

$$\langle r \rangle = \int_0^\infty r\, P_{2p}(r)\, dr = \int_0^\infty r^3 R_{2p}^2(r)\, dr$$

$$= \left(\frac{1}{2a_0}\right)^3 \frac{1}{3a_0^2} \int_0^\infty r^5 e^{-r/a_0}\, dr \qquad \text{let } z = \frac{r}{a_0}$$

$$\langle r \rangle = \frac{1}{24 a_0^5} a_0^6 \underbrace{\int_0^\infty z^5 e^{-z}\, dz}_{5!=} = \frac{120}{24} a_0 = 5a_0$$

(d) Find the most probable distance of the electron from the proton when the electron is in this state.

Most probable distance $\Rightarrow \frac{dP_{2p}}{dr} = 0$ (P_{2p} is maximum).

$$\frac{d}{dr}\left(r^4 e^{-r/a_0}\right) = 4r^3 e^{-r/a_0} - \frac{r^4}{a_0} e^{-r/a_0} = 0$$

$$\left(4r^3 - \frac{r^4}{a_0}\right) e^{-r/a_0} = 0$$

$$r^3 e^{-r/a_0}\left(4 - \frac{r}{a_0}\right) = 0 \Rightarrow r = 4a_0$$

FORMULA SHEET

$E = \gamma m_o c^2 = K + m_o c^2 \qquad p = \gamma m_o u$

$\lambda = \frac{h}{p} \qquad v_g = \frac{dw}{dk} \qquad \Delta p_x \cdot \Delta x \geq \frac{\hbar}{2} \qquad \Delta E \cdot \Delta t \geq \frac{\hbar}{2}$

$\int_{+\infty}^{-\infty} |\psi|^2 \, dx = 1 \qquad \langle Q \rangle = \int_{-\infty}^{+\infty} \psi^* [Q] \psi \, dx$

$\psi_n(x) = \sqrt{\frac{2}{L}} \sin(\frac{n\pi}{L} x) \qquad E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2}$

$\psi_0(x) = (\frac{m\omega}{\pi\hbar})^{1/4} e^{-m\omega x^2 / 2\hbar} \qquad E_n = (n + \frac{1}{2})\hbar\omega$

$\Delta Q = \sqrt{\langle Q^2 \rangle - \langle Q \rangle^2}$

$[p_x] = \frac{\hbar}{i} \frac{\partial}{\partial x} \qquad [E] = i\hbar \frac{\partial}{\partial t} \qquad [K] = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \qquad [H] = -\frac{\hbar 2}{2m} \frac{\partial^2}{\partial x^2} + U(x)$

$T(E) = \left\{ 1 + \frac{1}{4} \left[\frac{U^2}{E(U-E)} \right] \sinh^2 \alpha L \right\}^{-1} \qquad \alpha = \frac{\sqrt{2m(U-E)}}{\hbar}$

$E = \frac{\pi^2 \hbar^2}{2mL^2} (n_1^2 + n_2^2 + n_3^2) \qquad |\vec{L}| = \sqrt{l(l+1)}\, \hbar \qquad L_z = m_l \hbar$

$\psi_{nlm}(r,\theta,\phi) = R_{nl}(r) Y_l^{ml}(\theta,\phi) \qquad E_n = (-13.6\, eV) \frac{Z^2}{n^2}$

$P(r) = r^2 |R_{nl}(r)|^2 \qquad \int_0^\infty P(r) dr = 1 \qquad \langle f \rangle = \int_0^\infty f(r) P(r) dr$

$\int_0^\infty x^n e^{-x} dx = n! \qquad \int_0^\infty z^2 e^{-az^2} dz = \frac{1}{4a}\sqrt{\frac{\pi}{a}} \quad a > 0 \qquad \int_0^\pi \alpha^2 \cos 2\alpha \, d\alpha = \frac{\pi}{2} \qquad \int \sin^2 ax \, dx = \frac{x}{2} - \frac{1}{4a} \sin 2ax$

Constants:

$e = 1.6 \times 10^{-19}\ C \qquad m_e = 9.1 \times 10^{-31}\ kg \qquad \hbar = 1.05 \times 10^{-34}\ J \cdot s$

$m_p = 1.67 \times 10^{-27}\ kg \qquad c = 3 \times 10^8\ m/s \qquad 1u = 1.66 \times 10^{-27}\ kg$

$k_B = 1.38 \times 10^{-23}\ J/K \qquad k = 9 \times 10^9\ N \cdot m^2 C^2 \qquad m_e c^2 = 0.511\ MeV \qquad m_p c^2 = 938\ MeV$

$a_0 = 0.05292\ nm$