KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
DEPARTMENT OF PHYSCIS

PHYSICS 212 *Modern Physics*
TERM 042
EXAM #2

Date: Saturday 07 May 2005
Instructor: Dr. A. Mekki
Time of the exam: 90 minutes

Name: Key **Id#:** ____________________

SHOW THE DETAILS OF YOUR WORK

Question#	Maximum	Your grade
1	5	
2	5	
3	10	
4	5	
5	10	
6	10	
7	10	
8	5	
Total	60	
	15	

Q1. (5 points)
Consider a proton moving with a speed of 3×10^7 m/s along the x-axis. If the uncertainty in its position is 0.01×10^{-10} m, calculate the minimum uncertainty in
(a) its momentum

$v = 3.0 \times 10^{7}$ m/s $\qquad \Delta x = 0.01 \times 10^{-10}$ m

$$\Delta x \cdot \Delta p_{x_{min}} = \frac{\hbar}{2}$$

$$\Delta p_{x_{min}} = \frac{\hbar}{2\Delta x} = \frac{1.05 \times 10^{-34}}{2 \times 0.01 \times 10^{-10}} = \boxed{5.25 \times 10^{-23}\ \mathrm{Kg} \cdot \frac{\mathrm{m}}{\mathrm{s}}}$$

(b) its speed

$$p_x = m v_x \Rightarrow \Delta p_x = m \Delta v_x \Rightarrow \Delta v_x = \frac{\Delta p_x}{m}$$

$$\Rightarrow \Delta v_x = \frac{5.25 \times 10^{-23}}{1.67 \times 10^{-27}} = \boxed{3.14 \times 10^{4}\ \mathrm{m/s}}$$

(c) its kinetic energy

$$K = \frac{1}{2} m v^2 \Rightarrow dK = \frac{1}{2} 2 m v\, dv$$

$$\Rightarrow \Delta K = m v \Delta v = 1.67 \times 10^{-27} \times 3 \times 10^{7} \times 3.14 \times 10^{4}$$

$$\Rightarrow \boxed{\Delta K = 1.58 \times 10^{-15}\ \mathrm{J}}$$

Q2. (5 points)
Calculate the de Broglie wavelength of an electron accelerated through a potential difference of 5 MV.
Hint: treat the problem relativistically, that is, use the relativistic total and kinetic energies of the electron.

$$\lambda = \frac{h}{p}$$

$$E^2 = p^2c^2 + \cancel{m_o^2c^4} = (K + m_oc^2)^2$$

$$= K^2 + 2Km_oc^2 + \cancel{m_o^2c^4}$$

$$\Rightarrow \quad pc = \sqrt{K^2 + 2Km_oc^2} = K\sqrt{1 + \frac{2m_oc^2}{K}}$$

$$p = \frac{K}{c}\sqrt{1 + \frac{2m_oc^2}{K}}$$

$$\Rightarrow \quad \lambda = \frac{h}{p} = \frac{hc}{K}\left(1 + \frac{2m_oc^2}{K}\right)^{-1/2} \quad \text{and } K = eV$$

$$\lambda = \frac{hc}{eV}\left(1 + \frac{2m_oc^2}{eV}\right)^{-1/2}$$

$$= \frac{12400 \text{ eV Å}}{5\times10^6 \text{ eV}}\left(1 + \frac{2\times511\times10^3\text{eV}}{5\times10^6\text{eV}}\right)^{-1/2}$$

$$\boxed{\lambda = 2.3\times10^{-3} \text{ Å}}$$

Q3. (10 points)

An electron is trapped in a one-dimensional region of length 1.0×10^{-10} m (a typical atomic diameter).

(a) How much energy (in eV) must be supplied to excite the electron from the ground state to the third excited state?

$$\Delta E = E_4 - E_1 = \frac{\pi^2 \hbar^2}{2mL^2}\left((4)^2 - (1)^2\right) = \frac{15\pi^2\hbar^2}{2mL^2}$$

$$= 8.9 \times 10^{-17}\,\text{J} = \boxed{561\,\text{eV}}$$

(b) In the third excited state, what is the probability of finding the electron in the region from $x = 0.1 \times 10^{-10}$ m to $x = 0.2 \times 10^{-10}$ m?

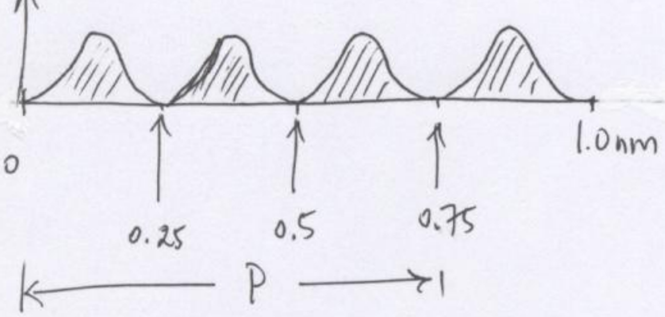
$$P = \int_{0.1\,\text{nm}}^{0.2\,\text{nm}} |\psi_4|^2\, dx = \frac{2}{L}\int_{0.1L}^{0.2L} \sin^2\left(\frac{4\pi}{L}x\right) dx$$

$$= \frac{1}{L}\int_{0.1L}^{0.2L}\left[1 - \cos\left(\frac{8\pi}{L}x\right)\right] dx = \frac{1}{L}\left\{ x\Big|_{0.1L}^{0.2L} - \frac{L}{8\pi}\sin\left(\frac{8\pi}{L}x\right)\Big|_{0.1L}^{0.2L}\right\}$$

$$= \frac{1}{\cancel{L}}\left\{0.1\cancel{L} - \frac{\cancel{L}}{8\pi}\left[\underbrace{\sin(1.6\pi)}_{-0.95} - \underbrace{\sin(0.8\pi)}_{0.59}\right]\right\}$$

$$= 0.16 = \boxed{16\%}$$

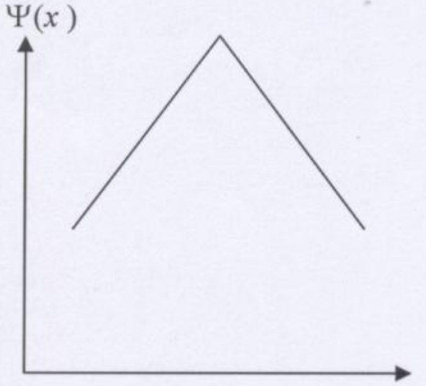
(c) Draw a diagram showing the probability density for the third excited state and deduce the probability of finding the electron in the region $x = 0$ and $x = 0.75 \times 10^{-10}$ m.



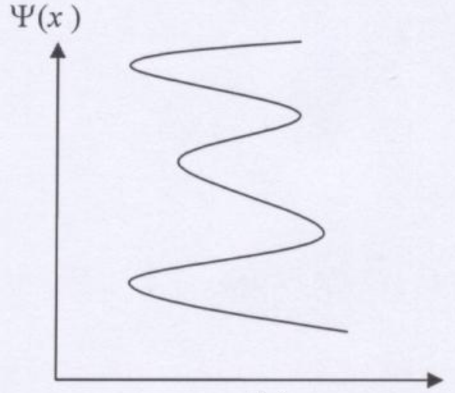
$$P = \frac{3 \times 1}{4} = 0.75 = \boxed{75\%}$$

Q4. (5 points)

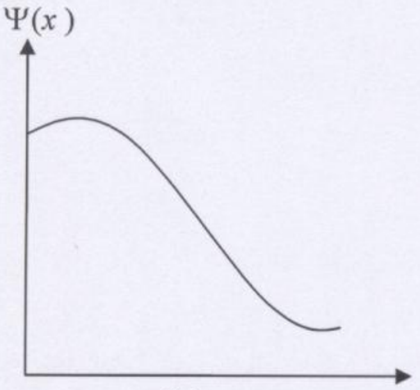
(a) Determine whether each of the following functions is acceptable or not as a wavefunction over the indicated interval and explain the reason for those not acceptable.



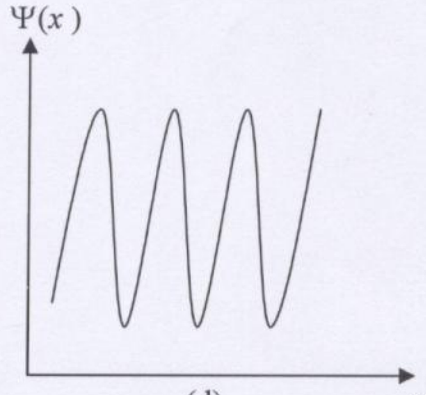
(a)



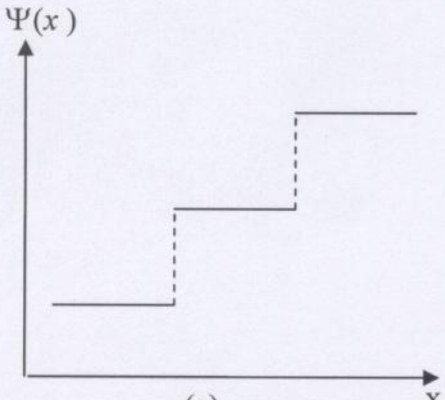
(b)



(c)



(d)



(e)

(a) No, Not smooth

(b) No, Not single valued

(c) Yes.

(d) Yes.

(e) No, Not Continuous

Q5. (10 points)

Consider the wavefunction of the ground state of a simple harmonic oscillator given by $\psi_0(x) = (\frac{\alpha}{\pi})^{1/4} e^{-\alpha x^2/2}$ where $\alpha = \frac{m\omega}{\hbar}$. Calculate

(a) $\langle x \rangle$

$$\langle x \rangle = \int_{-\infty}^{+\infty} x \, |\psi_0|^2 \, dx = \left(\frac{\alpha}{\pi}\right)^{1/2} \int_{-\infty}^{+\infty} x \, e^{-\alpha x^2} \, dx = 0$$

odd × even

(b) $\langle x^2 \rangle$

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} x^2 \, |\psi_0|^2 \, dx = \left(\frac{\alpha}{\pi}\right)^{1/2} \underbrace{\int_{-\infty}^{+\infty} x^2 e^{-\alpha x^2} \, dx}_{\frac{1}{4\alpha}\sqrt{\frac{\pi}{\alpha}}}$$

$$= \cancel{\left(\frac{\alpha}{\pi}\right)^{1/2}} \times \frac{1}{4\alpha} \cancel{\left(\frac{\pi}{\alpha}\right)^{1/2}}$$

$$= \frac{1}{4\alpha} = \frac{\hbar}{4m\omega}$$

(c) $\langle p \rangle$

$$\langle p \rangle = -i\hbar \left(\frac{\pi}{\alpha}\right)^{1/2} \int_{-\infty}^{+\infty} e^{-\frac{\alpha x^2}{2}} \frac{\partial}{\partial x}\left(e^{-\alpha \frac{x^2}{2}}\right) dx$$

$$= -i\hbar \left(\frac{\pi}{\alpha}\right)^{1/2} \int_{-\infty}^{+\infty} e^{-\alpha \frac{x^2}{2}} (-\alpha x) \, e^{-\alpha \frac{x^2}{2}} \, dx$$

$$= i\alpha\hbar \left(\frac{\pi}{\alpha}\right)^{1/2} \int_{-\infty}^{+\infty} x \, e^{-\alpha x^2} \, dx = 0$$

odd even

(d) $\langle p^2 \rangle$

$$< K > + < U > = < E >$$

$$\frac{<p^2>}{2m} + \frac{1}{2} k <x^2> = \frac{1}{2} \hbar \omega \; ; \quad \text{but } k = m\omega^2$$

$$\frac{<p^2>}{2m} + \frac{1}{2} \omega^{\cancel{2}} \cancel{m} \frac{\hbar}{2 \cancel{m} \cancel{\omega}} = \frac{1}{2} \hbar \omega$$

$$\frac{<p^2>}{2m} = \frac{1}{2} \hbar \omega - \frac{1}{4} \hbar \omega = \frac{1}{4} \hbar \omega$$

$$\Rightarrow \boxed{<p^2> = \frac{1}{2} m \hbar \omega}$$

Another method is $<p^2> = \hbar^2 \left(\frac{\alpha}{\pi} \right)^{1/2} \int_{-\infty}^{+\infty} e^{-\frac{\alpha x^2}{2}} \frac{\partial^2}{\partial x^2} e^{-\frac{\alpha x^2}{2}} dx$

(e) $\Delta x \cdot \Delta p$

$$\Delta x = \sqrt{<x^2> - \cancel{<x>^2}} = \sqrt{\frac{\hbar}{2m\omega}}$$

$$\Delta p = \sqrt{<p^2> - \cancel{<p>^2}} = \sqrt{\frac{1}{2} m \hbar \omega}$$

$$\Delta x . \Delta p = \sqrt{\frac{\hbar^2}{4}} = \frac{\hbar}{2}$$

(f) Is the uncertainty principle violated? Explain.

No, since $\Delta x . \Delta p \geqslant \frac{\hbar}{2}$ is satisfied

Q6. (5 points)
Consider an electron held in a two dimensional infinite potential well in the form

$$U(x,y)=\begin{cases}0 & 0\langle x\langle L,\, 0\langle y\langle 2L\\ \infty & \text{otherwise}\end{cases}$$

Find the energies of the four lowest states and their corresponding normalized wavefunctions. State which of these states are degenerate.

$$E_{n_1,n_2} = \frac{\pi^2\hbar^2}{2m}\left(\frac{n_1^2}{L^2}+\frac{n_2^2}{4L^2}\right) = \underbrace{\frac{\pi^2\hbar^2}{8mL^2}}_{E_0}\left(4n_1^2+n_2^2\right)$$

- ground state: $n_1 = n_2 = 1$ $\quad E_{11} = 5E_0$

- first excited state: $n_1 = 1$ $n_2 = 2$ $\quad E_{12} = 8E_0$

- Second excited state: $n_1 = 1$ $n_2 = 3$ $\quad E_{13} = 13E_0$

- fourth excited state: $n_1 = 2$ $n_3 = 1$ $\quad E_{21} = 17E_0$

The Wavefunctions are:

$$\psi_{11} = \sqrt{\frac{2}{L}}\sqrt{\frac{2}{2L}}\sin\left(\frac{\pi}{L}x\right)\sin\left(\frac{\pi}{2L}y\right)$$
$$= \frac{\sqrt{2}}{L}\sin\left(\frac{\pi}{L}x\right)\sin\left(\frac{\pi}{2L}y\right)$$

$$\psi_{12} = \frac{\sqrt{2}}{L}\sin\left(\frac{\pi}{L}x\right)\sin\left(\frac{\pi}{L}y\right)$$

$$\psi_{13} = \frac{\sqrt{2}}{L}\sin\left(\frac{\pi}{L}x\right)\sin\left(\frac{3\pi}{2L}y\right)$$

$$\psi_{21} = \frac{\sqrt{2}}{L}\sin\left(\frac{2\pi}{L}x\right)\sin\left(\frac{\pi}{2L}y\right)$$

None of these states are degenerate.

Q7. (10 points)
The radial part of the wavefunction for the hydrogen atom in the *2p* state is given by
$R_{2p} = (\frac{1}{2a_0})^{3/2} \frac{r}{a_0\sqrt{3}} e^{-r/2a_0}$ where a_0 is Bohr radius.

(a) Find the most probable distance of the electron from the proton when the electron is in this state.

$$P_{2p} = r^2 R_{2p}^2 = \frac{1}{(2a_0)^3} \frac{r^4}{a_0^2(3)} e^{-r/a_0}$$

most probable distance $\Rightarrow \frac{dP_{2p}}{dr} = 0$ (P_{2p} is maximum)

$$\frac{d}{dr}\left(r^4 e^{-r/a_0}\right) = 4r^3 e^{-r/a_0} - \frac{r^4}{a_0} e^{-r/a_0} = 0$$

$$\left(4 - \frac{r}{a_0}\right) r^3 e^{-r/a_0} = 0 \Rightarrow r = 0 \text{ (impossible)}, \quad \boxed{r = 4a_0}$$

(b) Calculate the average value of *r* when the electron is in this state.

$$\langle r \rangle = \int_0^{+\infty} r P_{2p}(r)\, dr = \frac{1}{(2a_0)^3} \frac{1}{3a_0^2} \int_0^{+\infty} r^5 e^{-r/a_0} dr$$

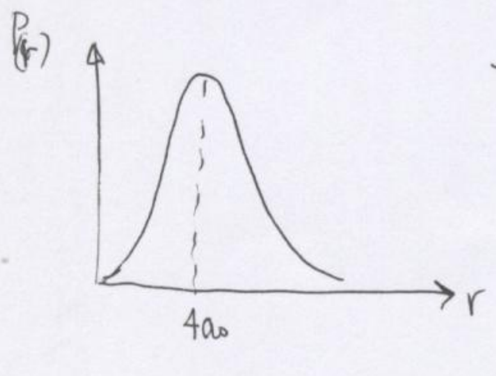
set $z = \frac{r}{a_0} \Rightarrow r^5 = a_0^5 z^5 \quad dr = a_0\, dz$

$$\Rightarrow \langle r \rangle = \frac{1}{(2a_0)^3} \frac{1}{3a_0^2} \int_0^{+\infty} a_0^6 z^5 e^{-z} dz = \frac{a_0}{24} \underbrace{\int_0^{+\infty} z^5 e^{-z} dz}_{=5!}$$

$$\langle r \rangle = \frac{120}{24} a_0 = 5a_0$$

$$\boxed{\langle r \rangle = 5a_0}$$

(c) Explain briefly why is the most probable distance is different from the average value?



The function $P_{2p}(r)$ is not symmetric as seen as seen from the figure.