3

The Quantum Theory of Light

Chapter Outline

At the beginning of the 20th century, following the lead of Newton and Maxwell, physicists might have rewritten the biblical story of creation as follows:

In the beginning He created the heavens and the earth—

$$F = G\frac{mm'}{r^2} = ma$$

and He said, "Let there be light"—

$$\oint \mathbf{E}\cdot d\mathbf{A} = \frac{Q}{\varepsilon_0} \qquad \oint \mathbf{E}\cdot d\mathbf{s} = -\frac{d\Phi_B}{dt}$$

$$\oint \mathbf{B}\cdot d\mathbf{A} = 0 \qquad \oint \mathbf{B}\cdot d\mathbf{s} = \mu_0 I + \varepsilon_0\mu_0\frac{d\Phi_E}{dt}$$

Actually, in addition to the twin pillars of mechanics and electromagnetism erected by the giants Newton and Maxwell, there was a third sturdy support for physics in 1900—thermodynamics and statistical mechanics. Classical thermodynamics was the work of many men (Carnot, Mayer, Helmholtz, Clausius, Lord Kelvin). It is especially notable because it starts with two simple propositions and gives solid and conclusive results independent of detailed physical mechanisms. Statistical mechanics, founded by Maxwell, Clausius, Boltzmann,[1] and Gibbs, uses the methods of probability theory to calculate averages and fluctuations from the average for systems containing many particles or modes of vibration. It is interesting that quantum physics started not with a breakdown of Maxwell's or Newton's laws applied to the atom, but with a problem of classical statistical mechanics—that of calculating the intensity of radiation at a given wavelength from a heated cavity. The desperate solution to this radiation problem was found by a thoroughly classical thermodynamicist, Max Planck, in 1900. Indeed, it is significant that both Planck and Einstein returned again and again to the simple and general foundation of thermodynamics and statistical mechanics as the only certain bases for the new quantum theory. Although we shall not follow the original thermodynamic arguments completely, we shall see in this chapter how Planck arrived at the correct spectral distribution for cavity radiation by allowing only certain energies for the radiation-emitting oscillators in the cavity walls. We shall also see how Einstein extended this quantization of energy to light itself, thereby brilliantly explaining the photoelectric effect. We conclude our brief history of the quantum theory of light with a discussion of the scattering of light by electrons (Compton effect), which showed conclusively that the light quantum carried momentum as well as energy. Finally, we describe the pull of gravity on light in Section 3.7.

3.1 HERTZ'S EXPERIMENTS—LIGHT AS AN ELECTROMAGNETIC WAVE

It is ironic that the same experimentalist who so carefully confirmed that the "newfangled" waves of Maxwell actually existed and possessed the same properties as light also undermined the electromagnetic wave theory as the complete explanation of light. To understand this irony, let us briefly review the theory of electromagnetism developed by the great Scottish physicist James Clerk Maxwell between 1865 and 1873.

Maxwell was primarily interested in the effects of electric current oscillations in wires. According to his theory, an alternating current would set up fluctuating electric and magnetic fields in the region surrounding the original disturbance. Moreover, these waves were predicted to have a frequency equal to the frequency of the current oscillations. *In addition, and most importantly, Maxwell's theory predicted that the radiated waves would behave in every way like light:* electromagnetic waves would be reflected by metal mirrors, would be refracted by dielectrics like glass, would exhibit polarization and interference, and would travel outward from the wire through a vacuum with a speed of 3.0×10^8 m/s. Naturally this led to the unifying and simplifying postulate that light was also a type of Maxwell wave or electromagnetic disturbance, created by extremely high frequency electric oscillators in matter. At the end of the 19th century the precise nature of these charged submicroscopic oscillators was unknown (Planck called them resonators), but physicists assumed that somehow they were able to emit light waves whose frequency was equal to the oscillator's frequency of motion.

Even at this time, however, it was apparent that this model of light emission was incapable of direct experimental verification, because the highest

[1] On whose tombstone is written $S = k_B \log W$, a basic formula of statistical mechanics attributed to Boltzmann.

electrical frequencies then attainable were about 10^9 Hz and visible light was known to possess a frequency a million times higher. But Heinrich Hertz (Fig. 3.1) did the next best thing. In a series of brilliant and exhaustive experiments, he showed that Maxwell's theory was correct and that an oscillating electric current does indeed radiate electromagnetic waves that possess every characteristic of light except the same wavelength as light. Using a simple spark gap oscillator consisting of two short stubs terminated in small metal spheres separated by an air gap of about half an inch, he applied pulses of high voltage, which caused a spark to jump the gap and produce a high-frequency electric oscillation of about 5×10^8 Hz. This oscillation, or ringing, occurs while the air gap remains conducting, and charge surges back and forth between the spheres until electrical equilibrium is established. Using a simple loop antenna with a small spark gap as the receiver, Hertz very quickly succeeded in detecting the radiation from his spark gap oscillator, even at distances of several hundred meters. Moreover, he found the detected radiation to have a wavelength of about 60 cm, corresponding to the oscillator frequency of 5×10^8 Hz. (Recall that $c = \lambda f$, where λ is the wavelength and f is the frequency.)

Figure 3.1 Heinrich Hertz (1857–1894), an extraordinarily gifted German experimentalist. (©*Bettmann/Corbis*)

In an exhaustive tour de force, Hertz next proceeded to show that these electromagnetic waves could be reflected, refracted, focused, polarized, and made to interfere—in short, he convinced physicists of the period that Hertzian waves and light waves were one and the same. The classical model for light emission was an idea whose time had come. It spread like wildfire. The idea that light was an electromagnetic wave radiated by oscillating submicroscopic electric charges (now known to be atomic electrons) was applied in rapid succession to the transmission of light through solids, to reflection from metal surfaces, and to the newly discovered Zeeman effect. In 1896, Pieter Zeeman, a Dutch physicist, discovered that a strong magnetic field changes the frequency of the light emitted by a glowing gas. In an impressive victory for Maxwell, it was found that Maxwell's equations correctly predicted (in most cases) the change of vibration of the electric oscillators and hence, the change in frequency of the light emitted. (See Problem 1.) Maxwell, with Hertz behind the throne, reigned supreme, for he had united the formerly independent kingdoms of electricity, magnetism, and light! (See Fig. 3.2.)

A terse remark made by Hertz ends our discussion of his confirmation of the electromagnetic wave nature of light. In describing his spark gap transmitter, he emphasizes that "it is essential that the pole surfaces of the spark gap

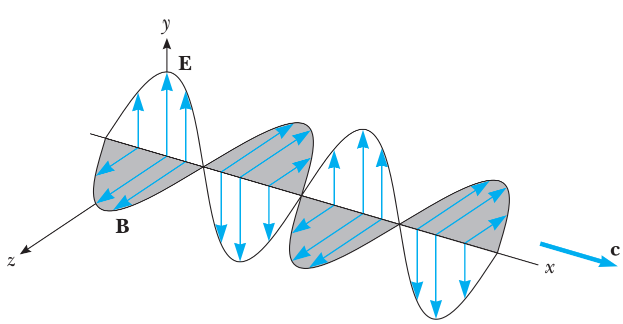

Figure 3.2 A light or radio wave far from the source according to Maxwell and Hertz.

should be frequently repolished" to ensure reliable operation of the spark.[2] Apparently this result was initially quite mysterious to Hertz. In an effort to resolve the mystery, he later investigated this side effect and concluded that it was the ultraviolet light from the initial spark acting on a clean metal surface that caused current to flow more freely between the poles of the spark gap. In the process of verifying the electromagnetic wave theory of light, Hertz had discovered the photoelectric effect, a phenomenon that would undermine the priority of the wave theory of light and establish the particle theory of light on an equal footing.

3.2 BLACKBODY RADIATION

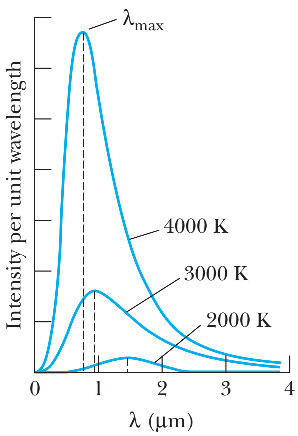

Figure 3.3 Emission from a glowing solid. Note that the amount of radiation emitted (the area under the curve) increases rapidly with increasing temperature.

The tremendous success of Maxwell's theory of light emission immediately led to attempts to apply it to a long-standing puzzle about radiation—the so-called "blackbody" problem. The problem is to predict the radiation intensity at a given wavelength emitted by a hot glowing solid at a specific temperature. Instead of launching immediately into Planck's solution of this problem, let us develop a feeling for its importance to classical physics by a quick review of its history.

Thomas Wedgwood, Charles Darwin's relative and a renowned maker of china, seems to have been the first to note the universal character of all heated objects. In 1792, he observed that all the objects in his ovens, regardless of their chemical nature, size, or shape, became red at the same temperature. This crude observation was sharpened considerably by the advancing state of spectroscopy, so that by the mid-1800s it was known that glowing solids emit continuous spectra rather than the bands or lines emitted by heated gases. (See Fig. 3.3.) In 1859, Gustav Kirchhoff proved a theorem as important as his circuit loop theorem when he showed by arguments based on thermodynamics that for any body in thermal equilibrium with radiation[3] the emitted power is proportional to the power absorbed. More specifically,

$$e_f = J(f, T)A_f \tag{3.1}$$

where e_f is the power emitted per unit area per unit frequency by a particular heated object, A_f is the absorption power (fraction of the incident power absorbed per unit area per unit frequency by the heated object), and $J(f, T)$ is a universal function (the same for all bodies) that depends only on f, the light frequency, and T, the absolute temperature of the body. A *blackbody* is defined as an object that absorbs all the electromagnetic radiation falling on it and consequently appears black. It has $A_f = 1$ for all frequencies and so Kirchhoff's theorem for a blackbody becomes

Blackbody

$$e_f = J(f, T) \tag{3.2}$$

[2] H. Hertz, *Ann. Physik* (Leipzig), 33:983, 1887.

[3] An example of a body in equilibrium with radiation would be an oven with closed walls at a fixed temperature and the radiation within the oven cavity. To say that radiation is in thermal equilibrium with the oven walls means that the radiation has exchanged energy with the walls many times and is homogeneous, isotropic, and unpolarized. In fact, thermal equilibrium of radiation within a cavity can be considered to be quite similar to the thermal equilibrium of a fluid within a container held at constant temperature—both will cause a thermometer in the center of the cavity to achieve a final stationary temperature equal to that of the container.

Equation 3.2 shows that the power emitted per unit area per unit frequency by a blackbody depends only on temperature and light frequency and not on the physical and chemical makeup of the blackbody, in agreement with Wedgwood's early observation.

Because absorption and emission are connected by Kirchhoff's theorem, we see that a blackbody or perfect absorber is also an ideal radiator. In practice, a small opening in any heated cavity, such as a port in an oven, behaves like a blackbody because such an opening traps all incident radiation (Fig. 3.4). If the direction of the radiation is reversed in Figure 3.4, the light emitted by a small opening is in thermal equilibrium with the walls, because it has been absorbed and re-emitted many times.

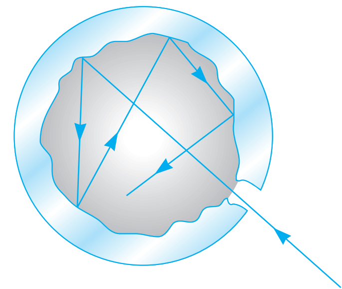

Figure 3.4 The opening to the cavity inside a body is a good approximation of a blackbody. Light entering the small opening strikes the far wall, where some of it is absorbed but some is reflected at a random angle. The light continues to be reflected, and at each reflection a portion of the light is absorbed by the cavity walls. After many reflections essentially all of the incident energy is absorbed.

The next important development in the quest to understand the universal character of the radiation emitted by glowing solids came from the Austrian physicist Josef Stefan (1835–1893) in 1879. He found experimentally that the total power per unit area emitted at all frequencies by a hot solid, e_{total}, was proportional to the fourth power of its absolute temperature. Therefore, Stefan's law may be written as

$$e_{\text{total}} = \int_0^\infty e_f \, df = \sigma T^4 \tag{3.3}$$

where e_{total} is the power per unit area emitted at the surface of the blackbody at all frequencies, e_f is the power per unit area per unit frequency emitted by the blackbody, T is the absolute temperature of the body, and σ is the Stefan–Boltzmann constant, given by $\sigma = 5.67 \times 10^{-8}\ \text{W}\cdot\text{m}^{-2}\cdot\text{K}^{-4}$. A body that is not an ideal radiator will obey the same general law but with a coefficient, a, less than 1:

Stefan's law

$$e_{\text{total}} = a\sigma T^4 \tag{3.4}$$

Only 5 years later another impressive confirmation of Maxwell's electromagnetic theory of light occurred when Boltzmann derived Stefan's law from a combination of thermodynamics and Maxwell's equations.

EXAMPLE 3.1 Stefan's Law Applied to the Sun

Estimate the surface temperature of the Sun from the following information. The Sun's radius is given by $R_s = 7.0 \times 10^8$ m. The average Earth–Sun distance is $R = 1.5 \times 10^{11}$ m. The power per unit area (at all frequencies) from the Sun is measured at the Earth to be 1400 W/m^2. Assume that the Sun is a blackbody.

Solution For a black body, we take $a = 1$, so Equation 3.4 gives

$$e_{\text{total}}(R_s) = \sigma T^4 \tag{3.5}$$

where the notation $e_{\text{total}}(R_s)$ stands for the total power per unit area at the surface of the Sun. Because the problem gives the total power per unit area at the Earth, $e_{\text{total}}(R)$, we need the connection between $e_{\text{total}}(R)$ and $e_{\text{total}}(R_s)$. This comes from the conservation of energy:

$$e_{\text{total}}(R_s)\cdot 4\pi R_s^2 = e_{\text{total}}(R)\cdot 4\pi R^2$$

or

$$e_{\text{total}}(R_s) = e_{\text{total}}(R)\cdot \frac{R^2}{R_s^2}$$

Using Equation 3.5, we have

$$T = \left[\frac{e_{\text{total}}(R)\cdot R^2}{\sigma R_s^2}\right]^{1/4}$$

or

$$T = \left[\frac{(1400\ \text{W/m}^2)(1.5\times 10^{11}\ \text{m})^2}{(5.6\times 10^{-8}\ \text{W/m}^2\cdot\text{K}^4)(7.0\times 10^8\ \text{m})^2}\right]^{1/4}$$
$$= 5800\ \text{K}$$

As can be seen in Figure 3.3, *the wavelength marking the maximum power emission of a blackbody,* λ_{max}, shifts toward shorter wavelengths as the blackbody gets hotter. This agrees with Wedgwood's general observation that objects in his kiln progressed from dull red to orange to white in color as the temperature was raised. This simple effect of $\lambda_{max} \propto T^{-1}$ was not definitely established, however, until about 20 years after Kirchhoff's seminal paper had started the search to find the form of the universal function $J(f, T)$. In 1893, Wilhelm Wien proposed a general form for the blackbody distribution law $J(f, T)$ that gave the correct experimental behavior of λ_{max} with temperature. This law is called *Wien's displacement law* and may be written

$$\lambda_{max} T = 2.898 \times 10^{-3}\ \mathrm{m \cdot K} \tag{3.6}$$

where λ_{max} is the wavelength in meters corresponding to the blackbody's maximum intensity and T is the absolute temperature of the surface of the object emitting the radiation. Assuming that the peak sensitivity of the human eye (which occurs at about 500 nm—blue-green light) coincides with λ_{max} for the Sun (a blackbody), we can check the consistency of Wien's displacement law with Stefan's law by recalculating the Sun's surface temperature:

$$T = \frac{2.898 \times 10^{-3}\ \mathrm{m \cdot K}}{500 \times 10^{-9}\ \mathrm{m}} = 5800\ \mathrm{K}$$

Thus we have good agreement between measurements made at all wavelengths (Example 3.1) and at the maximum-intensity wavelength.

Exercise 1 How convenient that the Sun's emission peak is at the same wavelength as our eyes' sensitivity peak! Can you account for this?

Spectral energy density of a blackbody

So far, the power radiated per unit area per unit frequency by the blackbody, $J(f, T)$ has been discussed. However, it is more convenient to consider the spectral energy density, or *energy per unit volume per unit frequency of the radiation within the blackbody cavity, $u(f, T)$.* For light in equilibrium with the walls, the power emitted per square centimeter of opening is simply proportional to the energy density of the light in the cavity. Because the cavity radiation is isotropic and unpolarized, one can average over direction to show that the constant of proportionality between $J(f, T)$ and $u(f, T)$ is $c/4$, where c is the speed of light. Therefore,

$$J(f, T) = u(f, T)c/4 \tag{3.7}$$

An important guess as to the form of the universal function $u(f, T)$ was made in 1893 by Wien and had the form

$$u(f, T) = Af^3 e^{-\beta f/T} \tag{3.8}$$

where A and β are constants. This result was known as Wien's exponential law; it resembles and was loosely based on Maxwell's velocity distribution for gas molecules. Within a year the great German spectroscopist Friedrich Paschen

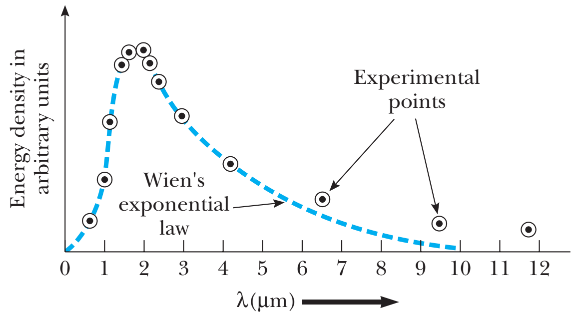

Figure 3.5 Discrepancy between Wien's law and experimental data for a blackbody at 1500 K.

had confirmed Wien's guess by working in the then difficult infrared range of 1 to 4 μm and at temperatures of 400 to 1600 K.[4]

As can be seen in Figure 3.5, Paschen had made most of his measurements in the maximum energy region of a body heated to 1500 K and had found good agreement with Wien's exponential law. In 1900, however, Lummer and Pringsheim extended the measurements to 18 μm, and Rubens and Kurlbaum went even farther—to 60 μm. Both teams concluded that Wien's law failed in this region (see Fig. 3.5). The experimental setup used by Rubens and Kurlbaum is shown in Figure 3.6. It is interesting to note that these historic

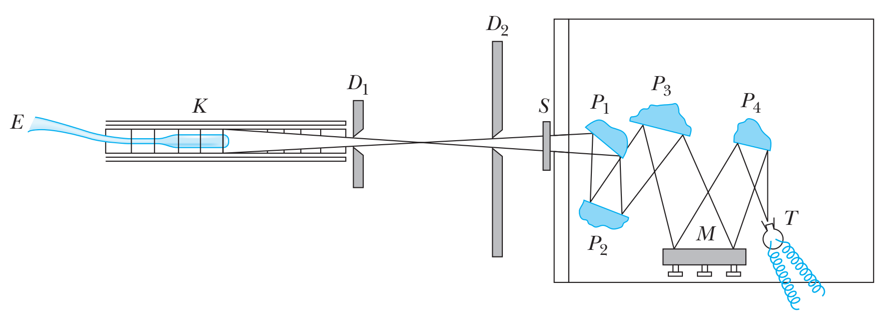

Figure 3.6 Apparatus for measuring blackbody radiation at a single wavelength in the far infrared region. The experimental technique that disproved Wien's law and was so crucial to the discovery of the quantum theory was the method of residual rays (*Restrahlen*). In this technique, one isolates a narrow band of far infrared radiation by causing white light to undergo multiple reflections from alkalide halide crystals (P_1–P_4). Because each alkali halide has a maximum reflection at a characteristic wavelength, quite pure bands of far infrared radiation may be obtained with repeated reflections. These pure bands can then be directed onto a thermopile (T) to measure intensity. E is a thermocouple used to measure the temperature of the blackbody oven, K.

[4]We should point out the great difficulty in making blackbody radiation measurements and the singular advances made by German spectroscopists in the crucial areas of blackbody sources, sensitive detectors, and techniques for operating far into the infrared region. In fact, it is dubious whether Planck would have found the correct blackbody law as quickly without his close association with the experimentalists at the Physikalisch Technische Reichsanstalt of Berlin (a sort of German National Bureau of Standards)—Otto Lummer, Ernst Pringsheim, Heinrich Rubens, and Ferdinand Kurlbaum.

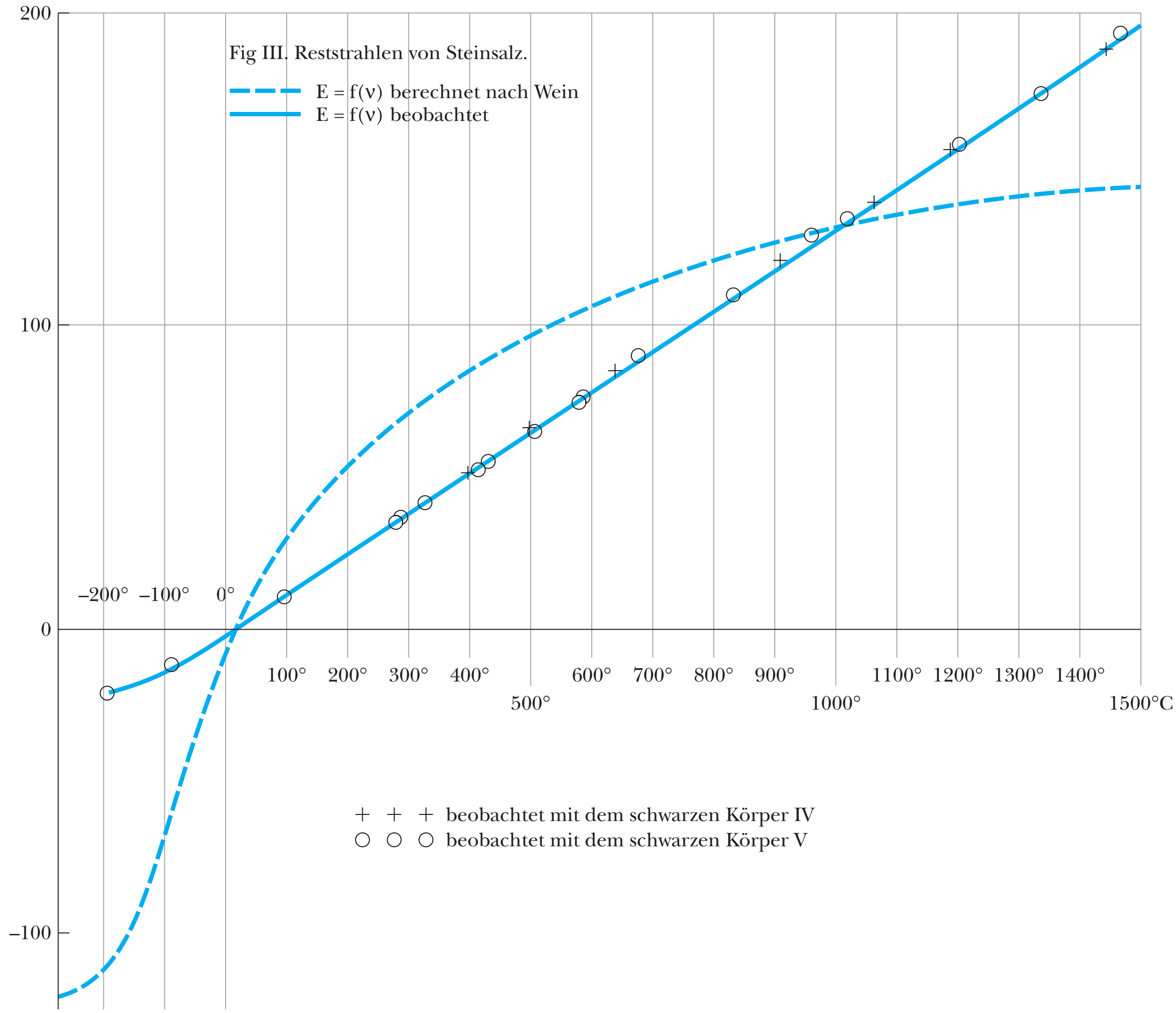

Figure 3.7 Comparison of theoretical and experimental blackbody emission curves at 51.2 μm and over the temperature range of $-188°$ to 1500°C. The title of this modified figure is "Residual Rays from Rocksalt." *Berechnet nach* means "calculated according to," and *beobachtet* means "observed." The vertical axis is emission intensity in arbitrary units. (*From H. Rubens and S. Kurlbaum,* Ann. Physik, *4:649, 1901.*)

experiments involved the measurement of blackbody radiation intensity at a fixed wavelength and variable temperature. Typical results measured at $\lambda = 51.2$ μm and over the temperature range of $-200°$ to $+1500°$C are shown in Figure 3.7, from the paper by Rubens and Kurlbaum.

Enter Planck

On a Sunday evening early in October of 1900, Max Planck discovered the famous blackbody formula, which truly ushered in the quantum theory. Planck's proximity to the Reichsanstalt experimentalists was extremely important for his discovery—earlier in the day he had heard from Rubens that his latest measurements showed that $u(f, T)$, the spectral energy density, was proportional to T for long wavelengths or low frequency. Planck knew that Wien's law agreed well with the data at high frequency and indeed had been working hard for several years to derive Wien's exponential law from the principles of statistical mechanics and Maxwell's laws. Interpolating between the two limiting forms (Wien's exponential law and an energy density proportional to temperature), he immediately found a general formula, which he sent to Rubens, on a postcard, the same evening. His formula was[5]

$$u(f, T) = \frac{8\pi h f^3}{c^3}\left(\frac{1}{e^{hf/k_B T} - 1}\right) \tag{3.9}$$

where h is Planck's constant $= 6.626 \times 10^{-34}\ \mathrm{J\cdot s}$, and k_B is Boltzmann's constant $= 1.380 \times 10^{-23}\ \mathrm{J/K}$. We can see that Equation 3.9 has the correct limiting behavior at high and low frequencies with the help of a few approximations. At high frequencies, where $hf/k_B T \gg 1$,

$$\frac{1}{e^{hf/k_B T} - 1} \approx e^{-hf/k_B T}$$

so that

$$u(f, T) = \frac{8\pi h f^3}{c^3}\left(\frac{1}{e^{hf/k_B T} - 1}\right) \approx \frac{8\pi h f^3}{c^3}\, e^{-hf/k_B T}$$

and we recover Wien's exponential law, Equation 3.8. At low frequencies, where $hf/k_B T \ll 1$,

$$\frac{1}{e^{hf/k_B T} - 1} = \frac{1}{1 + \dfrac{hf}{k_B T} + \cdots - 1} \approx \frac{k_B T}{hf}$$

and

$$u(f, T) = \frac{8\pi h f^3}{c^3}\left(\frac{1}{e^{hf/k_B T} - 1}\right) \approx \frac{8\pi f^2}{c^3}\, k_B T$$

This result shows that the spectral energy density is proportional to T in the low-frequency or so-called classical region, as Rubens had found.

We should emphasize that Planck's work entailed much more than clever mathematical manipulation. For more than six years Planck (Fig. 3.8) labored to find a rigorous derivation of the blackbody distribution curve. He was driven, in his own words, by the fact that the emission problem "represents something absolute, and since I had always regarded the search for the absolute as the loftiest goal of all scientific activity, I eagerly set to work." This work was to occupy most of his life as he strove to give his formula an ever deeper physical interpretation and to reconcile discrete quantum energies with classical theory.

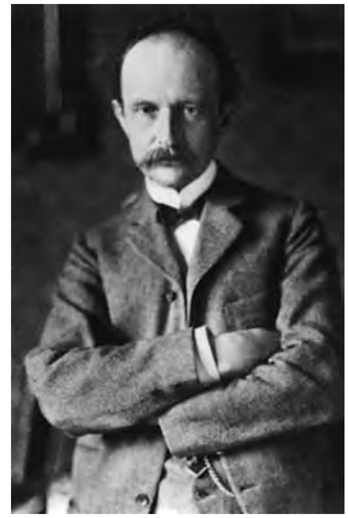

Figure 3.8 Max Planck (1858–1947). The work leading to the "lucky" blackbody radiation formula was described by Planck in his Nobel prize acceptance speech (1920): "But even if the radiation formula proved to be perfectly correct, it would after all have been only an interpolation formula found by lucky guess-work and thus, would have left us rather unsatisfied. I therefore strived from the day of its discovery, to give it a real physical interpretation and this led me to consider the relations between entropy and probability according to Boltzmann's ideas. After some weeks of the most intense work of my life, light began to appear to me and unexpected views revealed themselves in the distance." (*AIP Niels Bohr Library, W. F. Meggers Collection*)

[5] Planck originally published his formula as $u(\lambda, T) = \dfrac{C_1}{\lambda^5}\left(\dfrac{1}{e^{C_2/\lambda T} - 1}\right)$, where $C_1 = 8\pi ch$ and $C_2 = hc/k_B$. He then found best-fit values to the experimental data for C_1 and C_2 and evaluated $h = 6.55 \times 10^{-34}\ \mathrm{J\cdot s}$ and $k_B = N_A/R = 1.345 \times 10^{-23}\ \mathrm{J/K}$. As R, the universal gas constant, was fairly well known at the time, this technique also resulted in another method for finding N_A, Avogadro's number.

The Quantum of Energy

Planck's original theoretical justification of Equation 3.9 is rather abstract because it involves arguments based on entropy, statistical mechanics, and several theorems proved earlier by Planck concerning matter and radiation in equilibrium.[6] We shall give arguments that are easier to visualize physically yet attempt to convey the spirit and revolutionary impact of Planck's original work.

Planck was convinced that blackbody radiation was produced by vibrating submicroscopic electric charges, which he called resonators. He assumed that the walls of a glowing cavity were composed of literally billions of these resonators (whose exact nature was unknown at the time), all vibrating at different frequencies. Hence, according to Maxwell, each oscillator should emit radiation with a frequency corresponding to its vibration frequency. **Also according to classical Maxwellian theory, an oscillator of frequency f could have any value of energy and could change its amplitude continuously as it radiated any fraction of its energy.** This is where Planck made his revolutionary proposal. To secure agreement with experiment, **Planck had to assume that the total energy of a resonator with mechanical frequency f could only be an integral multiple of hf or**

$$E_{\text{resonator}} = nhf \qquad n = 1, 2, 3, \ldots \tag{3.10}$$

where h is a fundamental constant of quantum physics, $h = 6.626 \times 10^{-34}\ \text{J}\cdot\text{s}$, known as Planck's constant. In addition, he concluded that emission of radiation of frequency f occurred when a resonator dropped to the next lowest energy state. *Thus the resonator can change its energy only by the difference* ΔE *according to*

$$\Delta E = hf \tag{3.11}$$

That is, it cannot lose just any amount of its total energy, but only a finite amount, hf, the so-called quantum of energy. Figure 3.9 shows the quantized energy levels and allowed transitions proposed by Planck.

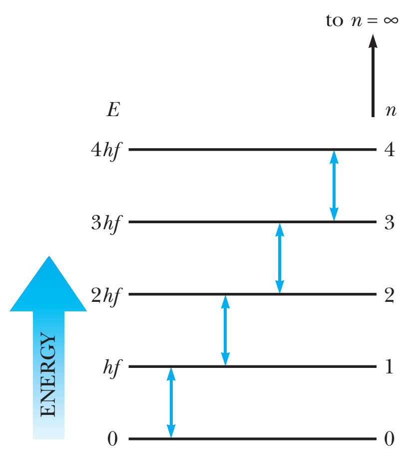

Figure 3.9 Allowed energy levels according to Planck's original hypothesis for an oscillator with frequency f. Allowed transitions are indicated by the double-headed arrows.

[6] M. Planck, *Ann. Physik*, 4:553, 1901.

EXAMPLE 3.2 A Quantum Oscillator versus a Classical Oscillator

Consider the implications of Planck's conjecture that *all* oscillating systems of natural frequency f have discrete allowed energies $E = nhf$ and that the smallest change in energy of the system is given by $\Delta E = hf$.

(a) First compare an atomic oscillator sending out 540-nm light (green) to one sending out 700-nm light (red) by calculating the minimum energy change of each. For the green quantum,

$$\Delta E_{\text{green}} = hf = \frac{hc}{\lambda}$$
$$= \frac{(6.63 \times 10^{-34}\ \text{J}\cdot\text{s})(3.00 \times 10^{8}\ \text{m/s})}{540 \times 10^{-9}\ \text{m}}$$
$$= 3.68 \times 10^{-9}\ \text{J}$$

Actually, the joule is much too large a unit of energy for describing atomic processes; a more appropriate unit of energy is the electron volt (eV). The electron volt takes the charge on the electron as its unit of charge. By definition, an electron accelerated through a potential difference of 1 volt has an energy of 1 eV. An electron volt may be converted to joules by noting that

$$E = V\cdot q = 1\ \text{eV} = (1.602 \times 10^{-19}\ \text{C})(1\ \text{J/C})$$
$$= 1.602 \times 10^{-19}\ \text{J}$$

It is also useful to have expressions for h and hc in terms of electron volts. These are

$$h = 4.136 \times 10^{-15}\ \text{eV}\cdot\text{s}$$

$$hc = 1.240 \times 10^{-6}\ \text{eV}\cdot\text{m} = 1240\ \text{eV}\cdot\text{nm}$$

Returning to our example, we see that the minimum energy change of an atomic oscillator sending out green light is

$$\Delta E_{\text{green}} = \frac{3.68 \times 10^{-19}\ \text{J}}{1.602 \times 10^{-19}\ \text{J/eV}} = 2.30\ \text{eV}$$

For the red quantum the minimum energy change is

$$\Delta E_{\text{red}} = \frac{hc}{\lambda} = \frac{(6.63 \times 10^{-34}\ \text{J}\cdot\text{s})(3.00 \times 10^{8}\ \text{m/s})}{700 \times 10^{-9}\ \text{m}}$$
$$= 2.84 \times 10^{-19}\ \text{J} = 1.77\ \text{eV}$$

Note that the minimum allowed amount or "quantum" of energy is not uniform under all conditions as is the quantum of charge—the quantum of energy is proportional to the natural frequency of the oscillator. Note, too, that the high frequency of atomic oscillators produces a measurable quantum of energy of several electron volts.

(b) Now consider a pendulum undergoing small oscillations with length $\ell = 1$ m. According to classical theory, if air friction is present, the amplitude of swing and consequently the energy decrease *continuously* with time, as shown in Figure 3.10a. Actually, *all* systems vibrating with frequency f are quantized (according to Equation 3.10) and lose energy in discrete packets or quanta, hf. This would lead to a decrease of the pendulum's energy in a stepwise manner, as shown in Figure 3.10b. We shall show that there is no contradiction between quantum theory and the observed behavior of laboratory pendulums and springs.

An energy change of one quantum corresponds to

$$\Delta E = hf$$

where the pendulum frequency f is

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{\ell}} = 0.50\ \text{Hz}$$

Thus,

$$\Delta E = (6.63 \times 10^{-34}\ \text{J}\cdot\text{s})(0.50\ \text{s}^{-1})$$
$$= 3.3 \times 10^{-34}\ \text{J}$$
$$= 2.1 \times 10^{-15}\ \text{eV}$$

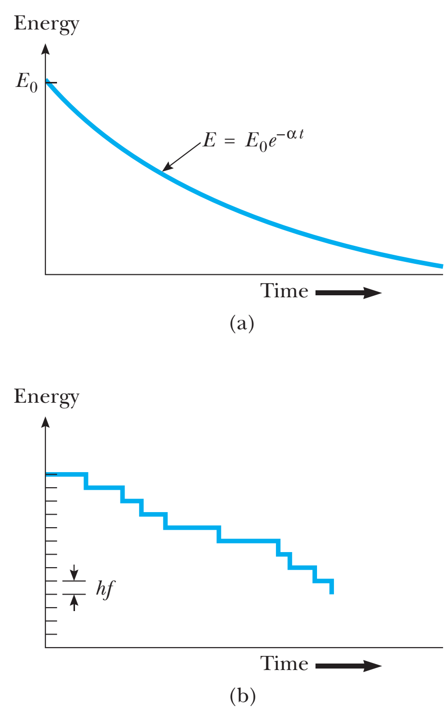

Figure 3.10 (Example 3.2) (a) Observed classical behavior of a pendulum. (b) Predicted quantum behavior of a pendulum.

Because the total energy of a pendulum of mass m and length ℓ displaced through an angle θ is

$$E = mg\ell(1 - \cos\theta)$$

we have for a typical pendulum with $m = 100$ g, $\ell = 1.0$ m, and $\theta = 10°$,

$$E = (0.10 \text{ kg})(9.8 \text{ m/s}^2)(1.0 \text{ m})(1 - \cos 10°) = 0.015 \text{ J}$$

Therefore, the fractional change in energy, $\Delta E/E$, is unobservably small:

$$\frac{\Delta E}{E} = \frac{3.3 \times 10^{-34} \text{ J}}{1.5 \times 10^{-2} \text{ J}} = 2.2 \times 10^{-32}$$

Note that the energy quantization of large vibrating systems is unobservable because of their low frequencies compared to the high frequencies of atomic oscillators. Hence there is no contradiction between Planck's quantum postulate and the behavior of macroscopic oscillators.

Exercise 2 Calculate the quantum number, n, for this pendulum with $E = 1.5 \times 10^{-2}$ J.

Answer 4.6×10^{31}

Exercise 3 An object of mass m on a spring of stiffness k oscillates with an amplitude A about its equilibrium position. Suppose that $m = 300$ g, $k = 10$ N/m, and $A = 10$ cm. (a) Find the total energy. (b) Find the mechanical frequency of vibration of the mass. (c) Calculate the change in amplitude when the system loses one quantum of energy.

Answer (a) $E_{\text{total}} = 0.050$ J; (b) $f = 0.92$ Hz; (c) $\Delta E_{\text{quantum}} = 6.1 \times 10^{-34}$ J, so

$$\Delta A \approx -\frac{\Delta E}{\sqrt{2Ek}} = -6.1 \times 10^{-34} \text{ m}$$

Until now we have been concentrating on the remarkable quantum properties of single oscillators of frequency f. Planck explained the continuous spectrum of the blackbody by assuming that the heated walls contained resonators vibrating at many different frequencies, each emitting light at the same frequency as its vibration frequency. By considering the conditions leading to equilibrium between the wall resonators and the radiation in the blackbody cavity, he was able to show that the spectral energy density $u(f, T)$ could be expressed as the product of the number of oscillators having frequency between f and $f + df$, denoted by $N(f)\,df$, and the average energy emitted per oscillator, $\overline{E}$. Thus we have the important result

$$u(f, T)\,df = \overline{E}N(f)\,df \qquad (3.12)$$

Furthermore, Planck showed that the number of oscillators with frequency between f and $f + df$ was proportional to f^2 or

$$N(f)\,df = \frac{8\pi f^2}{c^3}\,df \qquad (3.13)$$

(See Appendix 1 on our book Web site at http://info.brookscole.com/mp3e for details.)

Substituting Equation 3.13 into Equation 3.12 gives

$$u(f, T)\,df = \overline{E}\,\frac{8\pi f^2}{c^3}\,df \qquad (3.14)$$

This result shows that the spectral energy density is proportional to the product of the frequency squared and the average oscillator energy. Also, since $u(f, T)$ approaches zero at high frequencies (see Fig. 3.5), $\overline{E}$ must tend to zero at high frequencies faster than $1/f^2$. The fact that the mean oscillator energy must become extremely small when the frequency becomes high guided Planck in the development of his theory. In the next section we shall see that the failure of $\overline{E}$ to become small at high frequencies in the classical Rayleigh–Jeans theory led to the "ultraviolet catastrophe"—the prediction of an infinite spectral energy density at high frequencies in the ultraviolet region.

3.3 THE RAYLEIGH–JEANS LAW AND PLANCK'S LAW

O P T I O N A L

Rayleigh–Jeans Law

Both Planck's law and the Rayleigh–Jeans law (the classical theory of blackbody radiation formulated by Lord Rayleigh, John William Strutt, 1842–1919, English physicist, and James Jeans, 1887–1946, English astronomer and physicist) may be derived using the idea that the blackbody radiation energy per unit volume with frequency between f and $f + df$ can be expressed as the product of the number of oscillators per unit volume in this frequency range and the average energy per oscillator:

$$u(f, T)\,df = \overline{E}\,N(f)\,df \tag{3.12}$$

It is instructive to perform both the Rayleigh–Jeans and Planck calculations to see the effect on $u(f, T)$ of calculating $\overline{E}$ from a *continuous* distribution of classical oscillator energies (Rayleigh–Jeans) as opposed to a *discrete* set of quantum oscillator energies (Planck). We discuss Lord Rayleigh's derivation first because it is a more direct classical calculation.

While Planck concentrated on the thermal equilibrium of cavity radiation with oscillating electric charges in the cavity walls, Rayleigh concentrated directly on the electromagnetic waves in the cavity. Rayleigh and Jeans reasoned that the standing electromagnetic waves in the cavity could be considered to have a temperature T, because they constantly exchanged energy with the walls and caused a thermometer within the cavity to reach the same temperature as the walls. Further, they considered a standing polarized electromagnetic wave to be equivalent to a one-dimensional oscillator (Fig. 3.11). Using the same general idea as Planck, they expressed the energy density as a product of the number of standing waves (oscillators) and the average energy per oscillator. They found the average oscillator energy $\overline{E}$ to be independent of frequency and equal to k_BT from the Maxwell-Boltzmann distribution law (see Chapter 10). According to this distribution law, the probability P of finding an individual system (such as a molecule or an atomic oscillator) with energy E above some minimum energy, E_0, in a large group of systems at temperature T is

$$P(E) = P_0 e^{-(E-E_0)/k_BT} \tag{3.15}$$

where P_0 is the probability that a system has the minimum energy. In the case of a *discrete* set of allowed energies, the average energy, $\overline{E}$, is given by

$$\overline{E} = \frac{\Sigma E \cdot P(E)}{\Sigma P(E)} \tag{3.16}$$

where division by the sum in the denominator serves to normalize the total probability to 1. *In the classical case considered by Rayleigh,* an oscillator could have any

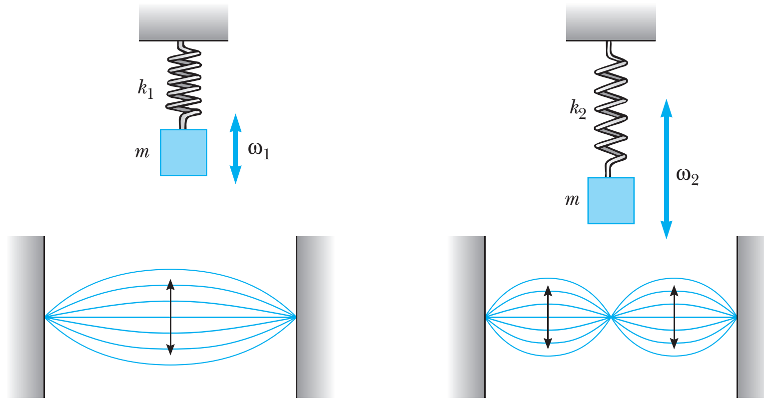

Figure 3.11 A one-dimensional harmonic oscillator is equivalent to a plane-polarized electromagnetic standing wave.

energy E in a continuous range from 0 to ∞. Thus the sums in Equation 3.16 must be replaced with integrals, and the expression for $\overline{E}$ becomes

$$\overline{E} = \frac{\int_0^\infty E e^{-E/k_B T} dE}{\int_0^\infty e^{-E/k_B T}\, dE} = k_B T$$

The calculation of $N(f)$ is a bit more complicated but is of importance here as well as in the free electron model of metals. Appendix 1 on our Web site gives the derivation of the density of modes, $N(f)\, df$. One finds

$$N(f)\, df = \frac{8\pi f^2}{c^3}\, df \tag{3.45}$$

or in terms of wavelength,

$$N(\lambda)\, d\lambda = \frac{8\pi}{\lambda^4}\, d\lambda \tag{3.46}$$

The spectral energy density is simply the density of modes multiplied by $k_B T$, or

Density of standing waves in a cavity

$$u(f, T)\, df = \frac{8\pi f^2}{c^3}\, k_B T\, df \tag{3.17}$$

In terms of wavelength,

Rayleigh–Jeans blackbody law

$$u(\lambda, T)\, d\lambda = \frac{8\pi}{\lambda^4}\, k_B T\, d\lambda \tag{3.18}$$

However, as one can see from Figure 3.12, this classical expression, known as the Rayleigh–Jeans law, does not agree with the experimental results in the short wavelength region. Equation 3.18 diverges as $\lambda \rightarrow 0$, predicting unlimited energy emission in the ultraviolet region, which was dubbed the "ultraviolet catastrophe." One is forced to conclude that classical theory fails miserably to explain blackbody radiation.

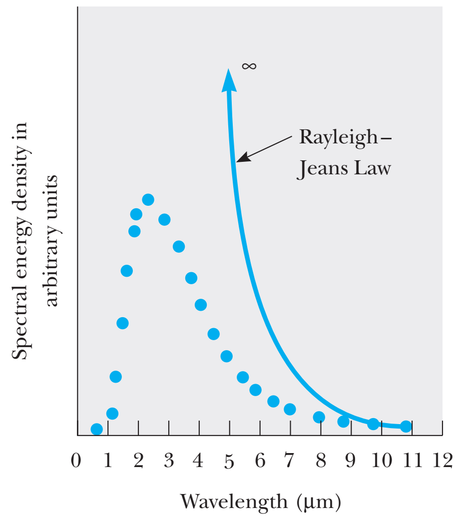

Figure 3.12 The failure of the classical Rayleigh–Jeans law (Equation 3.18) to fit the observed spectrum of a blackbody heated to 1000 K.

Planck's Law

As mentioned earlier, Planck concentrated on the energy states of resonators in the cavity walls and used the condition that the resonators and cavity radiation were in-equilibrium to determine the spectral quality of the radiation. By thermodynamic reasoning (and apparently unaware of Rayleigh's derivation), he arrived at the same expression for $N(f)$ as Rayleigh. However, Planck arrived at a different form for $\overline{E}$ by allowing only discrete values of energy for his resonators. He found, using the Maxwell-Boltzmann distribution law,

$$\overline{E} = \frac{hf}{e^{hf/k_B T} - 1} \tag{3.19}$$

(See the book Web site at http://info.brookscole.com/mp3e for Planck's derivation of $\overline{E}$.)

Multiplying $\overline{E}$ by $N(f)$ gives the Planck distribution formula:

Planck blackbody law

$$u(f, T)\,df = \frac{8\pi f^2}{c^3}\left(\frac{hf}{e^{hf/k_B T} - 1}\right) df \tag{3.9}$$

or in terms of wavelength, λ,

$$u(\lambda, T)\,d\lambda = \frac{8\pi hc\,d\lambda}{\lambda^5(e^{hc/\lambda k_B T} - 1)} \tag{3.20}$$

Equation 3.9 shows that the ultraviolet catastrophe is avoided because the $\overline{E}$ term dominates the f^2 term at high frequencies. One can qualitatively understand why $\overline{E}$ tends to zero at high frequencies by noting that the first allowed oscillator level (hf) is so large for large f compared to the average thermal energy available ($k_B T$) that Boltzmann's law predicts almost zero probability that the first excited state is occupied.

In summary, Planck arrived at his blackbody formula by making two startling assumptions: (1) the energy of a charged oscillator of frequency f is limited to discrete values nhf and (2) during emission or absorption of light, the change in energy of an oscillator is hf. But Planck was every bit the "unwilling revolutionary." From most of Planck's early correspondence one gets the impression that his concept of energy quantization was really a desperate calculational device, and moreover a device that applied only in the case of blackbody radiation. It remained for the great Albert Einstein, the popular icon of physics in the 20th century, to elevate quantization to the level of a universal phenomenon by showing that light itself was quantized.

EXAMPLE 3.3 Derivation of Stefan's Law from the Planck Distribution

In this example, we show that the Planck spectral distribution formula leads to the experimentally observed Stefan law for the total radiation emitted by a blackbody at all wavelengths,

$$e_{\text{total}} = 5.67 \times 10^{-8}\, T^4\ \text{W}\cdot\text{m}^{-2}\cdot\text{K}^{-4}$$

Solution Since Stefan's law is an expression for the total power per unit area radiated at all wavelengths, we must integrate the expression for $u(\lambda, T)\,d\lambda$ given by Equation 3.20 over λ and use Equation 3.7 for the connection between the energy density inside the blackbody cavity and the power emitted per unit area of blackbody surface. We find

$$e_{\text{total}} = \frac{c}{4}\int_0^\infty u(\lambda, T)\,d\lambda = \int_0^\infty \frac{2\pi hc^2}{\lambda^5(e^{hc/\lambda k_{\mathrm{B}}T} - 1)}\,d\lambda$$

If we make the change of variable $x = hc/\lambda k_{\mathrm{B}}T$, the integral assumes a form commonly found in tables:

$$e_{\text{total}} = \frac{2\pi k_{\mathrm{B}}^4 T^4}{c^2 h^3}\int_0^\infty \frac{x^3}{(e^x - 1)}\,dx$$

Using

$$\int_0^\infty \frac{x^3}{(e^x - 1)}\,dx = \frac{\pi^4}{15}$$

we find

$$e_{\text{total}} = \frac{2\pi^5 k_{\mathrm{B}}^4}{15c^2h^3}\,T^4 = \sigma T^4$$

Finally, substituting for k_{B}, c, and h, we have

$$\sigma = \frac{(2)(3.141)^5(1.381 \times 10^{-23}\ \text{J/K})^4}{(15)(2.998 \times 10^8\ \text{m/s})^2(6.626 \times 10^{-34}\ \text{J}\cdot\text{s})^3}$$
$$= 5.67 \times 10^{-8}\ \text{W}\cdot\text{m}^{-2}\cdot\text{K}^{-4}$$

Exercise 4 Show that

$$\int_0^\infty \frac{2\pi hc^2}{\lambda^5(e^{hc/\lambda k_{\mathrm{B}}T} - 1)}\,d\lambda = \frac{2\pi k_{\mathrm{B}}^4 T^4}{h^3c^2}\int_{x=0}^\infty \frac{x^3}{(e^x - 1)}\,dx$$

3.4 LIGHT QUANTIZATION AND THE PHOTOELECTRIC EFFECT

We now turn to the year 1905, in which the next major development in quantum theory took place. The year 1905 was an incredible one for the "willing revolutionary" Albert Einstein (Fig. 3.13). In this year Einstein produced three immortal papers on three different topics, each revolutionary and each worthy of a Nobel prize. All three papers contained balanced, symmetric, and unifying new results achieved by spare and clean logic and simple mathematics. The first work, entitled "A Heuristic[7] Point of View

[7]A heuristic argument is one that is plausible and enlightening but not rigorously justified.

About the Generation and Transformation of Light," formulated the theory of light quanta and explained the photoelectric effect.[8] The second paper was entitled "On the Motion of Particles Suspended in Liquids as Required by the Molecular-Kinetic Theory of Heat." It explained Brownian motion and provided strong proof of the reality of atoms.[9] The third paper, which is perhaps his most famous, contained the invention of the theory of special relativity[10] and was entitled "On the Electrodynamics of Moving Bodies." It is interesting to note that when Einstein was awarded the Nobel prize in 1922, the Swedish Academy judged his greatest contribution to physics to have been the theory of the photoelectric effect. No mention was made at all of his theory of relativity!

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Let us turn now to the paper concerning the light quantum, in which Einstein crossed swords with Maxwell and challenged the unqualified successes of the classical wave theory of light. Einstein recognized an inconsistency between Planck's quantization of oscillators in the walls of the blackbody and Planck's insistence that the cavity radiation consisted of classical electromagnetic waves. By showing that the change in entropy of blackbody radiation was like the change in entropy of an ideal gas consisting of independent particles, Einstein reached the conclusion that light itself is composed of "grains," irreducible finite amounts, or quanta of energy.[11] Furthermore, he asserted that light interacting with matter also consists of quanta, and he worked out the implications for photoelectric and photochemical processes. His explanation of the photoelectric effect offers such convincing proof that light consists of energy packets that we shall describe it in more detail. First, however, we need to consider the main experimental features of the photoelectric effect and the failure of classical theory to explain this effect.

As noted earlier, Hertz first established that clean metal surfaces emit charges when exposed to ultraviolet light. In 1888 Hallwachs discovered that the emitted charges were negative, and in 1899 J. J. Thomson showed that the emitted charges were electrons, now called photoelectrons. He did this by measuring the charge-to-mass ratio of the particles produced by ultraviolet light and even succeeded in measuring e separately by a cloud chamber technique (see Chapter 4).

The last crucial discovery before Einstein's explanation was made in 1902 by Philip Lenard, who was studying the photoelectric effect with intense carbon arc light sources. He found that electrons are emitted from the metal with a range of velocities and that the maximum kinetic energy of photoelectrons, K_{max}, *does not* depend on the *intensity* of the exciting light. Although he

[8] A. Einstein, *Ann. Physik*, 17:132, 1905 (March).

[9] A. Einstein, *Ann. Physik*, 17:549, 1905 (May).

[10] A. Einstein, *Ann. Physik*, 17:891, 1905 (June).

[11] Einstein, as Planck before him, fell back on the unquestionable solidity of thermodynamics and statistical mechanics to derive his revolutionary results. At the time it was well known that the probability, W, for n independent gas atoms to be in a partial volume V of a larger volume V_0 is $(V/V_0)^n$. Einstein showed that light of frequency f and total energy E enclosed in a cavity obeys an identical law, where in this case W is the probability that all the radiation is in the partial volume and $n = E/hf$.

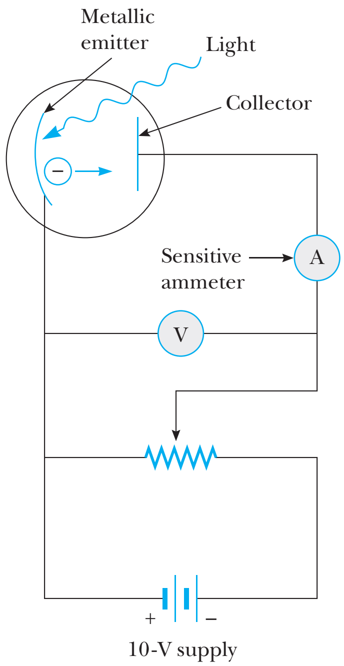


Figure 3.14 Photoelectric effect apparatus.

was unable to establish the precise relationship, Lenard also indicated that K_{max} increases with light *frequency*. A typical apparatus used to measure the maximum kinetic energy of photoelectrons is shown in Figure 3.14. K_{max} is easily measured by applying a retarding voltage and gradually increasing it until the most energetic electrons are stopped and the photocurrent becomes zero. At this point,

$$K_{max} = \frac{1}{2} m_e v_{max}^2 = eV_s \qquad (3.21)$$

where m_e is the mass of the electron, v_{max} is the maximum electron speed, e is the electronic charge, and V_s is the stopping voltage. A plot of the type found by Lenard is shown in Figure 3.15a; it illustrates that K_{max} or V_s is independent of light intensity I. The increase in current (or number of electrons per second) with increasing light intensity shown in Figure 3.15a was expected and could be explained classically. However, the result that K_{max} *does not depend on the intensity* was completely unexpected.

Two other experimental results were completely unexpected classically as well. One was *the linear dependence of* K_{max} *on light frequency*, shown in Figure 3.15b. Note that Figure 3.15b also shows the existence of a threshold frequency, f_0, below which no photoelectrons are emitted. (Actually, a threshold energy called the *work function*, ϕ, is associated with the binding energy of an electron in a metal and is expected classically. That there is an energy barrier holding electrons in a solid is evident from the fact that electrons are not spontaneously emitted from a metal in a vacuum, but require high temperatures or incident light to provide an energy of ϕ and cause emission.) The other interesting result impossible to explain classically is that *there is no time lag between the start of illumination and the start of the photocurrent.* Measurements have shown that if there is a time lag, it is less than 10^{-9} s. In summary, as shown in detail in the following example, classical electromagnetic theory has major difficulties explaining the independence of K_{max} and light intensity, the linear dependence of K_{max} on light frequency, and the instantaneous response of the photocurrent.

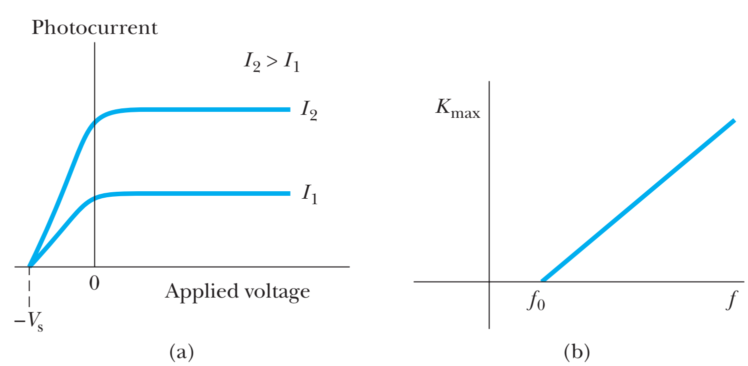


Figure 3.15 (a) A plot of photocurrent versus applied voltage. The graph shows that K_{max} is independent of light intensity I for light of fixed frequency. (b) A graph showing the dependence of K_{max} on light frequency.

EXAMPLE 3.4 Maxwell Takes a Licking

For a typical case of photoemission from sodium, show that *classical theory* predicts that (a) $K_{\max}$ depends on the incident light intensity, I, (b) $K_{\max}$ does not depend on the frequency of the incident light, and (c) there is a long time lag between the start of illumination and the beginning of the photocurrent. The work function for sodium is $\phi = 2.28$ eV and an absorbed power per unit area of 1.00×10^{-7} mW/cm^2 produces a measurable photocurrent in sodium.

Solution (a) *According to classical theory, the energy in a light wave is spread out uniformly and continuously over the wavefront.* Assuming that all absorption of light occurs in the top atomic layer of the metal, that each atom absorbs an equal amount of energy proportional to its cross-sectional area, A, and that each atom somehow funnels this energy into one of its electrons, we find that each electron absorbs an energy K in time t given by

$$K = CIAt$$

where C is a fraction accounting for less than 100% light absorption. Because the most energetic electrons are held in the metal by a surface energy barrier or work function of ϕ, these electrons will be emitted with $K_{\max}$ once they have absorbed enough energy to overcome the barrier ϕ. We can express this as

$$K_{\max} = CIAt - \phi \qquad (3.22)$$

Thus, classical theory predicts that for a fixed absorption period, t, at low light intensities when $CIAt < \phi$, no electrons ought to be emitted. At higher intensities, when $CIAt > \phi$, electrons should be emitted with higher kinetic energies the higher the light intensity. Therefore, classical predictions contradict experiment at both very low and very high light intensities.

(b) According to classical theory, *the intensity of a light wave is proportional to the square of the amplitude of the electric field*, E_0^2, and it is this electric field amplitude that increases with increasing intensity and imparts an increasing acceleration and kinetic energy to an electron. Replacing I with a quantity proportional to E_0^2 in Equation 3.22 shows that $K_{\max}$ should not depend at all on the frequency of the classical light wave, again contradicting the experimental results.

(c) To estimate the time lag between the start of illumination and the emission of electrons, we assume that an electron must accumulate just enough light energy to overcome the work function. Setting $K_{\max} = 0$ in Equation 3.22 gives

$$0 = CIAt - \phi$$

or

$$t = \frac{\phi}{CIA} = \frac{\phi}{IA}$$

assuming that I is the actual absorbed intensity. Because ϕ and I are given, we need A, the cross-sectional area of an atom, to calculate the time. As an estimate of A we simply use $A = \pi r^2$, where r is a typical atomic radius. Taking $r = 1.0 \times 10^{-8}$ cm, we find $A = \pi \times 10^{-16}$ cm^2. Finally, substituting this value into the expression for t, we obtain

$$t = \frac{2.28 \text{ eV} \times 1.60 \times 10^{-16} \text{ mJ/eV}}{(10^{-7} \text{ mJ/s}\cdot\text{cm}^2)(\pi \times 10^{-16} \text{ cm}^2)}$$
$$= 1.2 \times 10^7 \text{ s} \approx 130 \text{ days}$$

Thus we see that the classical calculation of the time lag for photoemission does not agree with the experimental result, disagreeing by a factor of 10^{16}!

Exercise 5 Why do the I–V curves in Figure 3.15a rise gradually between $-V_s$ and 0, that is, why do they not rise abruptly upward at $-V_s$? What statistical information about the conduction electrons inside the metal is contained in the slope of the I–V curve?

Einstein's explanation of the puzzling photoelectric effect was as brilliant for what it focused on as for what it omitted. For example, he stressed that Maxwell's classical theory had been immensely successful in describing the progress of light through space *over long time intervals* but that a different theory might be needed to describe *momentary interactions* of light and matter, as in light emission by oscillators or the transformation of light energy to kinetic energy of the electron in the photoelectric effect. He also focused only on the energy aspect of the light and avoided models or mechanisms concerning the conversion of the quantum of light energy to kinetic energy

Table 3.1 Work Functions of Selected Metals

Metal	Work Function, ϕ, (in eV)
Na	2.28
Al	4.08
Cu	4.70
Zn	4.31
Ag	4.73
Pt	6.35
Pb	4.14
Fe	4.50

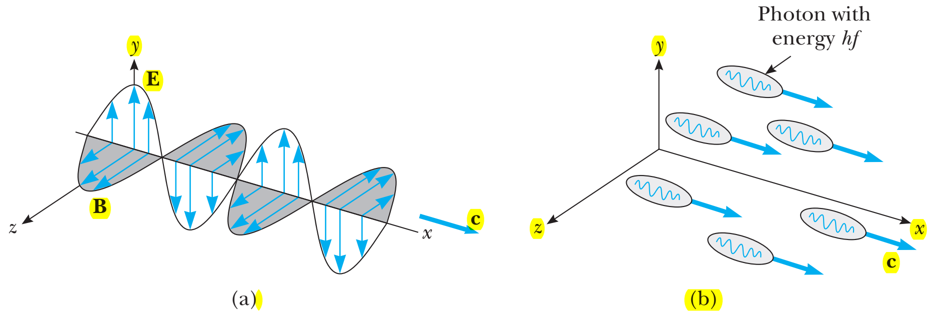


Figure 3.16 (a) A classical view of a traveling light wave. (b) Einstein's photon picture of "a traveling light wave."

of the electron. In short, he introduced only those ideas necessary to explain the photoelectric effect. **He maintained that the energy of light is not distributed evenly over the classical wavefront, but is concentrated in discrete regions (or in "bundles"), called quanta, each containing energy, hf.** A suggestive image, not to be taken too literally, is shown in Figure 3.16b. Einstein's picture was that a light quantum was so localized that it gave all its energy, hf, directly to a single electron in the metal. Therefore, according to Einstein, the maximum kinetic energy for emitted electrons is

Einstein's theory of the photoelectric effect

$$K_{\max} = hf - \phi \tag{3.23}$$

where ϕ is the work function of the metal, which corresponds to the minimum energy with which an electron is bound in the metal. Table 3.1 lists values of work functions measured for different metals.

Equation 3.23 beautifully explained the puzzling independence of $K_{\max}$ and intensity found by Lenard. For a fixed light frequency f, an increase in light intensity means more photons and more photoelectrons per second, although $K_{\max}$ remains unchanged according to Equation 3.23. In addition, Equation 3.23 explained the phenomenon of threshold frequency. Light of threshold frequency f_0, which has just enough energy to knock an electron out of the metal surface, causes the electron to be released with zero kinetic energy. Setting $K_{\max} = 0$ in Equation 3.23 gives

$$f_0 = \frac{\phi}{h} \tag{3.24}$$

Thus the variation in threshold frequency for different metals is produced by the variation in work function. Note that light with $f < f_0$ has insufficient energy to free an electron. Consequently, the photocurrent is zero for $f < f_0$.

With any theory, one looks not only for explanations of previously observed results but also for new predictions. This was indeed the case here, as Equation 3.23 predicted the result (new in 1905) that $K_{\max}$ should vary linearly with f for any material and that the slope of the $K_{\max}$ versus f plot should yield

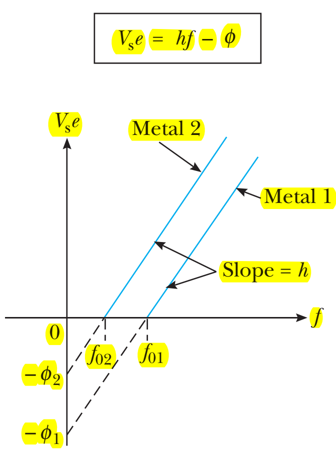


Figure 3.17 Universal characteristics of all metals undergoing the photoelectric effect.

the universal constant h (see Fig. 3.17). In 1916, the American physicist Robert Millikan (1868–1953) reported photoelectric measurement data, from which he substantiated the linear relation between K_{max} and f and determined h with a precision of about 0.5%.[12]

EXAMPLE 3.5 The Photoelectric Effect in Zinc

Philip Lenard determined that photoelectrons released from zinc by ultraviolet light were stopped by a voltage of 4.3 V. Find K_{max} and v_{max} for these electrons.

Solution

$$K_{max} = eV_s = (1.6 \times 10^{-19}\ \mathrm{C})(4.3\ \mathrm{V}) = 6.9 \times 10^{-19}\ \mathrm{J}$$

To find v_{max}, we set the work done by the electric field equal to the change in the electron's kinetic energy, to obtain

$$\tfrac{1}{2} m_e v_{max}^2 = eV_s$$

or

$$v_{max} = \sqrt{\frac{2eV_s}{m_e}} = \sqrt{\frac{2(6.9 \times 10^{-19}\ \mathrm{J})}{9.11 \times 10^{-31}\ \mathrm{kg}}} = 1.2 \times 10^6\ \mathrm{m/s}$$

Therefore, a 4.3-eV electron is rather energetic and moves with a speed of about a million meters per second. Note, however, that this is still only about 0.4% of the speed of light, so relativistic effects are negligible in this case.

EXAMPLE 3.6 The Photoelectric Effect for Iron

Suppose that light of total intensity 1.0 $\mu\mathrm{W/cm^2}$ falls on a clean iron sample 1.0 $\mathrm{cm^2}$ in area. Assume that the iron sample reflects 96% of the light and that only 3.0% of the absorbed energy lies in the violet region of the spectrum above the threshold frequency.

(a) What intensity is actually available for the photoelectric effect?

Because only 4.0% of the incident energy is absorbed, and only 3.0% of this energy is able to produce photoelectrons, the intensity available is

$$I = (0.030)(0.040)I_0 = (0.030)(0.040)(1.0\ \mu\mathrm{W/cm^2}) = 1.2\mathrm{nW/cm^2}$$

(b) Assuming that all the photons in the violet region have an effective wavelength of 250 nm, how many electrons will be emitted per second?

For an efficiency of 100%, one photon of energy, hf, will produce one electron, so

Number of electrons/s

$$= \frac{1.2 \times 10^{-9}\ \mathrm{W}}{hf} = \frac{\lambda(1.2 \times 10^{-9})}{hc} = \frac{(250 \times 10^{-9}\ \mathrm{m})(1.2 \times 10^{-9}\ \mathrm{J/s})}{(6.6 \times 10^{-34}\ \mathrm{J \cdot s})(3.0 \times 10^8\ \mathrm{m/s})} = 1.5 \times 10^9$$

(c) Calculate the current in the phototube in amperes.

$$i = (1.6 \times 10^{-19}\ \mathrm{C})(1.5 \times 10^9\ \mathrm{electrons/s}) = 2.4 \times 10^{-10}\ \mathrm{A}$$

A sensitive electrometer is needed to detect this small current.

(d) If the cutoff frequency is $f_0 = 1.1 \times 10^{15}$ Hz, find the work function, ϕ, for iron.

From Equation 3.24, we have

$$\phi = hf_0 = (4.14 \times 10^{-15}\ \mathrm{eV \cdot s})(1.1 \times 10^{15}\ \mathrm{s^{-1}}) = 4.5\ \mathrm{eV}$$

(e) Find the stopping voltage for iron if photoelectrons are produced by light with $\lambda = 250$ nm.

From the photoelectric equation,

$$eV_s = hf - \phi = \frac{hc}{\lambda} - \phi = \frac{(4.14 \times 10^{-15}\ \mathrm{eV \cdot s})(3.0 \times 10^8\ \mathrm{m/s})}{250 \times 10^{-9}\ \mathrm{m}} - 4.5\ \mathrm{eV} = 0.46\ \mathrm{eV}$$

Thus the stopping voltage is 0.46 V.

[12] R. A. Millikan, *Phys. Rev.*, 7:355, 1916. Some of the experimental difficulties in the photoelectric effect were the lack of strong monochromatic uv sources, small photocurrents, and large effects of rough and impure metal surfaces on f_0 and K_{max}. Millikan cleverly circumvented these difficulties by using alkali metal cathodes, which are sensitive in the visible to about 600 nm (thus making it possible to use the strong visible lines of the mercury arc), and machining fresh alkali surfaces while the metal sample was held under high vacuum. Also when the phototube emitter and collector are composed of different metals, the work function ϕ determined from plots of V_s vs. f is actually that of the collector. See J. Rudnick and D. S. Tannhauser, AJP 44, 796, 1976.

3.5 THE COMPTON EFFECT AND X-RAYS

Although Einstein introduced the concept that light consists of pointlike quanta of *energy* in 1905, he did not directly treat the *momentum* carried by light until 1906. In that year, in a paper treating a molecular gas in thermal equilibrium with electromagnetic radiation (statistical mechanics again!), Einstein concluded that a light quantum of energy E travels in a single direction (unlike a spherical wave) and carries a momentum directed along its line of motion of E/c, or hf/c. In his own words, "If a bundle of radiation causes a molecule to emit or absorb an energy packet hf, then momentum of quantity hf/c is transferred to the molecule, directed along the line of motion of the bundle for absorption and opposite the bundle for emission."

After developing the first theoretical justification for photon momentum, and treating the photoelectric effect much earlier, it is curious that Einstein carried the treatment of photon momentum no further. The theoretical treatment of photon–particle collisions had to await the insight of Peter Debye (1884–1966, Dutch physical chemist), and Arthur Holly Compton (1892–1962, American physicist). In 1923, both men independently realized that the scattering of x-ray photons from electrons could be explained by treating photons as pointlike particles with energy hf and momentum hf/c and by conserving relativistic energy and momentum of the photon–electron pair in a collision.[13,14] This remarkable development completed the particle picture of light by showing that photons, in addition to carrying energy, hf, carry momentum, hf/c, and scatter like particles. Before treating this in detail, a brief introduction to the important topic of x-rays will be given.

X-Rays

X-rays were discovered in 1895 by the German physicist Wilhelm Roentgen. He found that a beam of high-speed electrons striking a metal target produced a new and extremely penetrating type of radiation (Fig. 3.18). Within months of Roentgen's discovery the first medical x-ray pictures were taken, and within several years it became evident that x-rays were electromagnetic vibrations similar to light but with extremely short wavelengths and great penetrating power (see Fig. 3.19). Rough estimates obtained from the diffraction of x rays by a narrow slit showed x-ray wavelengths to be about 10^{-10} m, which is of the same order of magnitude as the atomic spacing in crystals. Because the best artificially ruled gratings of the time had spacings of 10^{-7} m, Max von Laue in Germany and William Henry Bragg and William Lawrence Bragg (a father and son team) in England suggested using single crystals such as calcite as natural three-dimensional gratings, the periodic atomic arrangement in the crystals constituting the grating rulings.

A particularly simple method of analyzing the scattering of x-rays from parallel crystal planes was proposed by W. L. Bragg in 1912. Consider two successive planes of atoms as shown in Figure 3.20. Note that adjacent atoms *in a single plane,* A, will scatter constructively if the angle of incidence, θ_i,

[13] P. Debye, *Phys. Zeitschr.*, 24:161, 1923. In this paper, Debye acknowledges Einstein's pioneering work on the quantum nature of light.

[14] A. H. Compton, *Phys. Rev.*, 21:484, 1923.

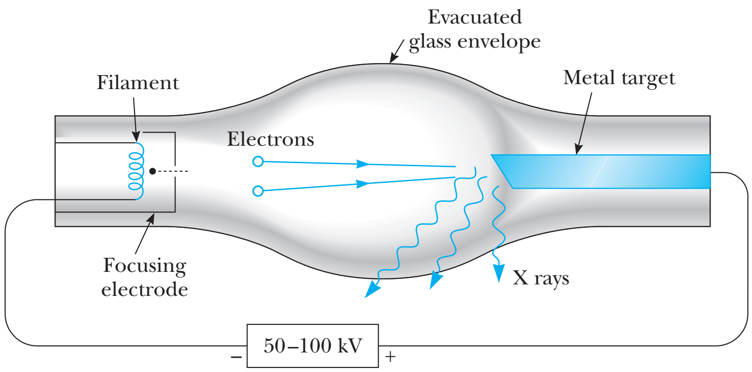


Figure 3.18 X-rays are produced by bombarding a metal target (copper, tungsten, and molybdenum are common) with energetic electrons having energies of 50 to 100 keV.

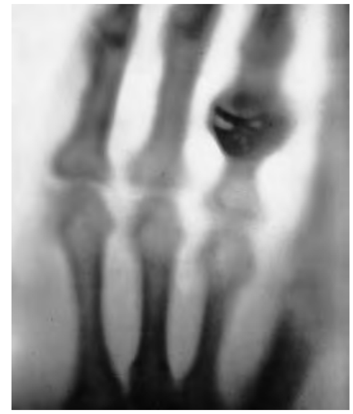


Figure 3.19 One of the first images made by Roentgen using x-rays (December 22, 1895).

equals the angle of reflection, θ_r. Atoms in *successive planes* (A and B) will scatter constructively at an angle θ if the path length difference for rays (1) and (2) is a whole number of wavelengths, $n\lambda$. From the diagram, constructive interference will occur when

$$AB + BC = n\lambda \qquad n = 1, 2, 3, \ldots$$

and because $AB = BC = d\sin\theta$, it follows that

Bragg equation

$$n\lambda = 2d\sin\theta \qquad n = 1, 2, 3, \ldots \tag{3.25a}$$

where n is the order of the intensity maximum, λ is the x-ray wavelength, d is the spacing between planes, and θ is the angle of the intensity maximum measured from plane A. Note that there are several maxima at different angles for a fixed d and λ corresponding to $n = 1, 2, 3, \ldots$. Equation 3.25a is known as the Bragg equation; it was used with great success by the Braggs to determine atomic positions in crystals. A diagram of a Bragg x-ray spectrometer is shown in Figure 3.21a. The crystal is slowly rotated until a strong reflection is

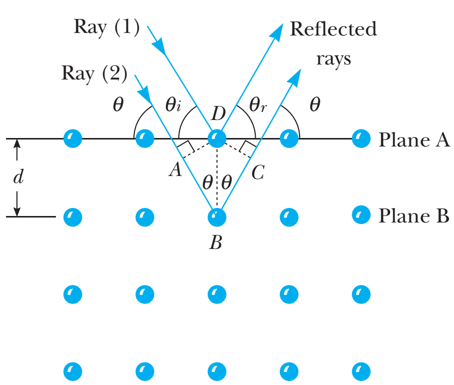
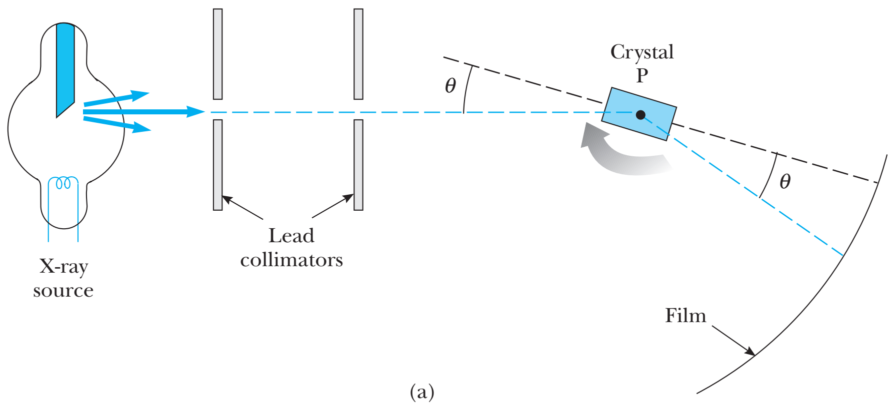
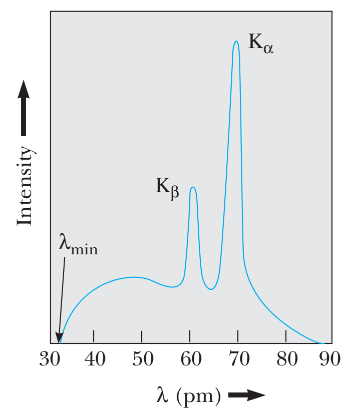


Figure 3.20 Bragg scattering of x-rays from successive planes of atoms. Constructive interference occurs for ABC equal to an integral number of wavelengths.



(a)



(b)

Figure 3.21 (a) A Bragg crystal x-ray spectrometer. The crystal is rotated about an axis through *P*. (b) The x-ray spectrum of a metal target consists of a broad, continuous spectrum plus a number of sharp lines, which are due to the characteristic x-rays. Those shown were obtained when 35-keV electrons bombarded a molybdenum target. Note that 1 pm $= 10^{-12}$ m $= 10^{-3}$ nm.

observed, which means that Equation 3.25a holds. If λ is known, d can be calculated and, from the series of d values found, the crystal structure may be determined. (See Problem 38.) If measurements are made with a crystal with known d, the x-ray intensity vs. wavelength may be determined and the x-ray emission spectrum examined.

The actual x-ray emission spectrum produced by a metal target bombarded by electrons is interesting in itself and is shown in Figure 3.21b. Although the broad, continuous spectrum is well explained by classical electromagnetic theory, a feature of Figure 3.21b, λ_{min}, shows proof of the photon theory. The broad continuous x-ray spectrum shown in Figure 3.21b results from glancing or indirect scattering of electrons from metal atoms. In such collisions only part of the electron's energy is converted to electromagnetic radiation. This radiation is called *bremsstrahlung* (German for braking radiation), which refers to the radiation given off by any charged particle when it is decelerated. The minimum continuous x-ray wavelength, $\lambda_{\min}$, is found to be independent of target composition and depends only on the tube voltage, V. It may be explained by attributing it to the case of a head-on electron–atom collision in which all of the incident electron's kinetic energy is converted to electromagnetic energy in the form of a single x-ray photon. For this case we have

$$eV = hf = \frac{hc}{\lambda_{\min}}$$

or

$$\lambda_{\min} = \frac{hc}{eV} \tag{3.26}$$

where V is the x-ray tube voltage.

Superimposed on the continuous spectrum are sharp x-ray lines labeled K_α and K_β, which are like sharp lines emitted in the visible light spectrum. The sharp lines depend on target composition and provide evidence for discrete atomic energy levels separated by thousands of electron volts, as explained in Chapter 9.

The Compton Effect

Let us now turn to the year 1922 and the experimental confirmation by Arthur Holly Compton that x-ray photons behave like particles with momentum hf/c. For some time prior to 1922, Compton and his coworkers had been accumulating evidence that showed that classical wave theory failed to explain the scattering of x-rays from free electrons. In particular, classical theory predicted that incident radiation of frequency f_0 should accelerate an electron in the direction of propagation of the incident radiation, and that it should cause forced oscillations of the electron and reradiation at frequency f', where $f' \leq f_0$ (see Fig. 3.22a).[15] Also, according to classical theory, the frequency or wavelength of the scattered radiation should depend on the length of time the electron was exposed to the incident radiation as well as on the intensity of the incident radiation.

Imagine the surprise when Compton showed experimentally that the wavelength shift of x-rays scattered at a given angle is absolutely independent of the intensity of radiation and the length of exposure, and depends only on the scattering angle. Figure 3.22b shows the quantum model of the transfer of momentum and energy between an individual x-ray photon and an electron. Note that the quantum model easily explains the lower scattered frequency f', because the incident photon gives some of its original energy hf to the recoiling electron.

A schematic diagram of the apparatus used by Compton is shown in Figure 3.23a. In the original experiment, Compton measured the dependence of scattered x-ray intensity on wavelength at three different scattering angles

[15]This decrease in frequency of the reradiated wave is caused by a double Doppler shift, first because the electron is receding from the incident radiation, and second because the electron is a moving radiator as viewed from the fixed lab frame. See D. Bohm, *Quantum Theory*, Upper Saddle River, NJ, Prentice-Hall, 1961, p. 35.

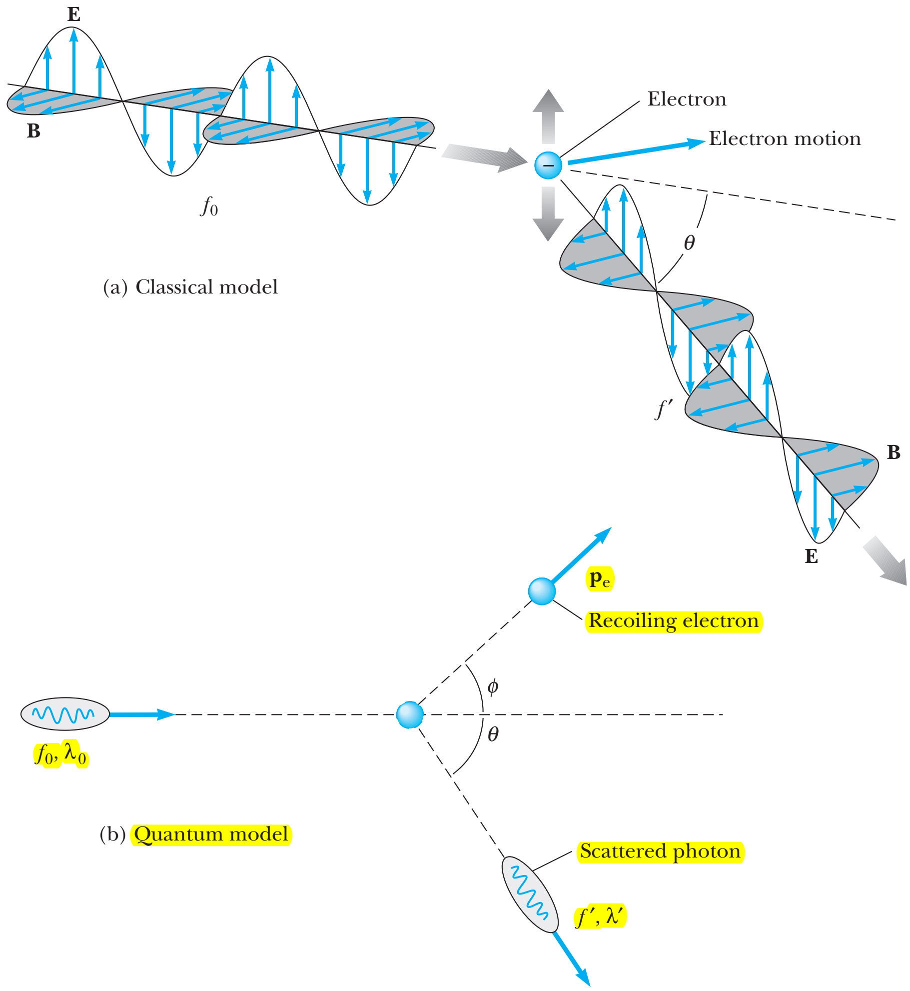


Figure 3.22 X-ray scattering from an electron: (a) the classical model, (b) the quantum model.

of 45°, 90°, and 135°. The wavelength was measured with a rotating crystal spectrometer, and the intensity was determined by an ionization chamber that generated a current proportional to the x-ray intensity. Monochromatic x-rays of wavelength $\lambda_0 = 0.71$ Å constituted the incident beam. A carbon target with a low atomic number, $Z = 12$, was used because atoms with small Z have a higher percentage of loosely bound electrons. The experimental intensity versus wavelength plots observed by Compton for scattering angles of 0°, 45°, 90°, and 135° are shown in Figure 3.23b. They show two peaks, one at λ_0 and a shifted peak at a longer wavelength λ'. The shifted peak at λ' is caused by the scattering of x-rays from nearly free electrons. Assuming that x-rays behave like particles, λ' was predicted by Compton to depend on scattering angle as

Compton effect

$$\lambda' - \lambda_0 = \frac{h}{m_e c}(1 - \cos\theta) \tag{3.27}$$

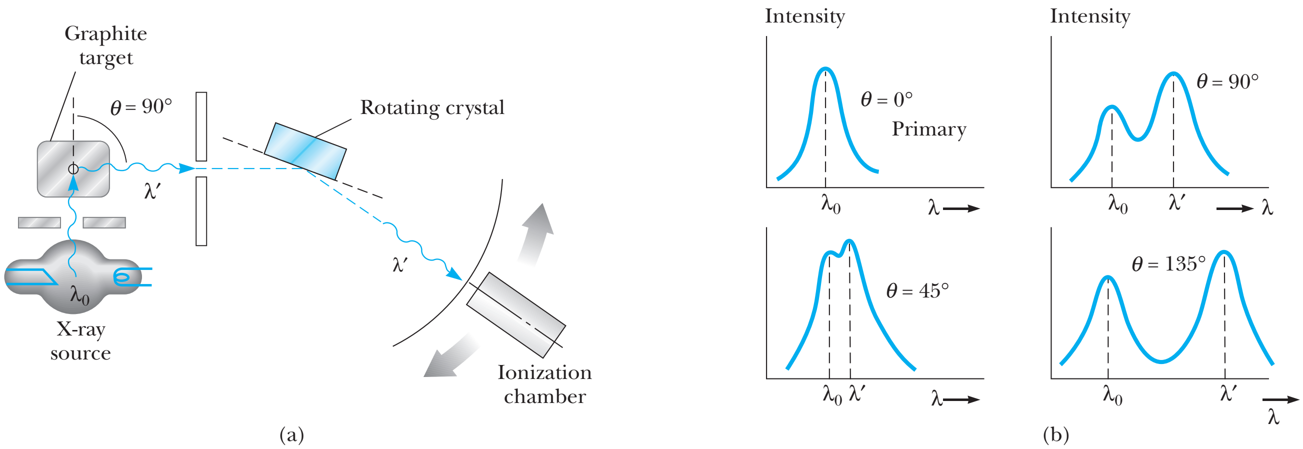


Figure 3.23 (a) Schematic diagram of Compton's apparatus. The wavelength was measured with a rotating crystal spectrometer using graphite (carbon) as the target. The intensity was determined by a movable ionization chamber that generated a current proportional to the x-ray intensity. (b) Scattered x-ray intensity versus wavelength of Compton scattering at $\theta = 0°$, 45°, 90°, and 135°.

where m_e = electron mass; the combination of constants h/m_ec is called the Compton wavelength of the electron and has a currently accepted value of

$$\frac{h}{m_ec} = 0.0243\ \text{Å} = 0.00243\ \text{nm}$$

Compton's careful measurements completely confirmed the dependence of λ' on scattering angle θ and determined the Compton wavelength of the electron to be 0.0242 Å, in excellent agreement with the currently accepted value. It is fair to say that these results were the first to really convince most American physicists of the basic validity of the quantum theory!

The unshifted peak at λ_0 in Figure 3.23 is caused by x-rays scattered from electrons tightly bound to carbon atoms. This unshifted peak is actually predicted by Equation 3.27 if the electron mass is replaced by the mass of a carbon atom, which is about 23,000 times the mass of an electron.

Let us now turn to the derivation of Equation 3.27 assuming that the photon exhibits particle-like behavior and collides elastically like a billiard ball with a free electron initially at rest. Figure 3.24 shows the photon–electron collision for which energy and momentum are conserved. Because the electron typically recoils at high speed, we treat the collision relativistically. The expression for conservation of energy gives

$$E + m_ec^2 = E' + E_e \tag{3.28}$$

Energy conservation

where E is the energy of the incident photon, E' is the energy of the scattered photon, m_ec^2 is the rest energy of the electron, and E_e is the total relativistic energy of the electron after the collision. Likewise, from momentum conservation we have

$$p = p' \cos\theta + p_e \cos\phi \tag{3.29}$$

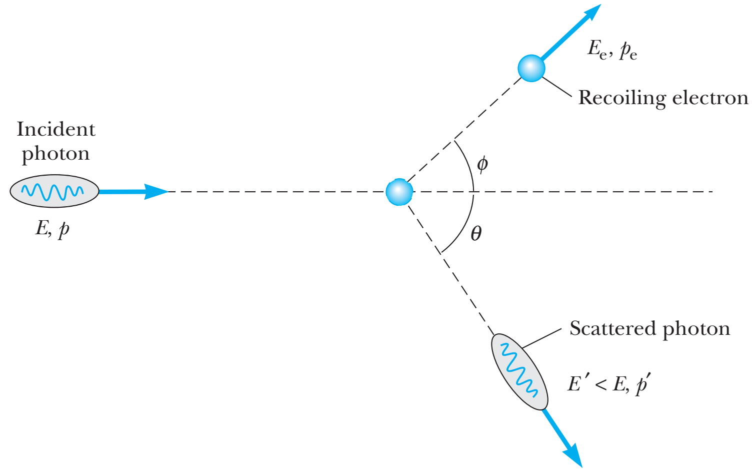


Figure 3.24 Diagram representing Compton scattering of a photon by an electron. The scattered photon has less energy (or longer wavelength) than the incident photon.

$$p' \sin\theta = p_e \sin\phi \tag{3.30}$$

where p is the momentum of the incident photon, p' is the momentum of the scattered photon, and p_e is the recoil momentum of the electron. Equations 3.29 and 3.30 may be solved simultaneously to eliminate ϕ, the electron scattering angle, to give the following expression for p_e^2:

$$p_e^2 = (p')^2 + p^2 - 2pp' \cos\theta \tag{3.31}$$

At this point it is necessary, paradoxically, to use the wave nature of light to explain the particle-like behavior of photons. We have already seen that the energy of a photon and the frequency of the associated light wave are related by $E = hf$. If we assume that a photon obeys the relativistic expression $E^2 = p^2c^2 + m^2c^4$ and that a photon has a mass of zero, we have

$$p_{\text{photon}} = \frac{E}{c} = \frac{hf}{c} = \frac{h}{\lambda} \tag{3.32}$$

Here again we have a paradoxical situation; a particle property, the photon momentum, is given in terms of a wave property, λ, of an associated light wave. If the relations $E = hf$ and $p = hf/c$ are substituted into Equations 3.28 and 3.31, these become respectively

$$E_e = hf - hf' + m_e c^2 \tag{3.33}$$

and

$$p_e^2 = \left(\frac{hf'}{c}\right)^2 + \left(\frac{hf}{c}\right)^2 - \frac{2h^2ff'}{c^2}\cos\theta \tag{3.34}$$

Because the Compton measurements do not concern the total energy and momentum of the electron, we eliminate E_e and p_e by substituting Equations 3.33 and 3.34 into the expression for the electron's relativistic energy,

$$E_e^2 = p_e^2c^2 + m_e^2c^4$$

After some algebra (see Problem 33), one obtains Compton's result for the increase in a photon's wavelength when it is scattered through an angle θ:

$$\lambda' - \lambda_0 = \frac{h}{m_e c}(1 - \cos\theta) \tag{3.27}$$

EXAMPLE 3.7 The Compton Shift for Carbon

X-rays of wavelength $\lambda = 0.200$ nm are aimed at a block of carbon. The scattered x-rays are observed at an angle of 45.0° to the incident beam. Calculate the increased wavelength of the scattered x-rays at this angle.

Solution The shift in wavelength of the scattered x-rays is given by Equation 3.27. Taking $\theta = 45.0°$, we find

$$\Delta\lambda = \frac{h}{m_e c}(1 - \cos\theta)$$

$$= \frac{6.63 \times 10^{-34}\ \text{J}\cdot\text{s}}{(9.11 \times 10^{-31}\ \text{kg})(3.00 \times 10^8\ \text{m/s})}(1 - \cos 45.0°)$$

$$= 7.11 \times 10^{-13}\ \text{m} = 0.00071\ \text{nm}$$

Hence, the wavelength of the scattered x-ray at this angle is

$$\lambda = \Delta\lambda + \lambda_0 = 0.200711\ \text{nm}$$

Exercise 6 Find the fraction of energy lost by the photon in this collision.

Answer Fraction $= \Delta E/E = 0.00355$.

EXAMPLE 3.8 X-ray Photons versus Visible Photons

(a) Why are x-ray photons used in the Compton experiment, rather than visible-light photons? To answer this question, we shall first calculate the Compton shift for scattering at 90° from graphite for the following cases: (1) very high energy γ-rays from cobalt, $\lambda = 0.0106$ Å; (2) x-rays from molybdenum, $\lambda = 0.712$ Å; and (3) green light from a mercury lamp, $\lambda = 5461$ Å.

Solution In all cases, the Compton shift formula gives $\Delta\lambda = \lambda' - \lambda_0 = (0.0243\ \text{Å})(1 - \cos 90°) = 0.0243\ \text{Å} = 0.00243$ nm. That is, regardless of the incident wavelength, the same small shift is observed. However, the fractional change in wavelength, $\Delta\lambda/\lambda_0$, is quite different in each case:

γ-rays from cobalt:

$$\frac{\Delta\lambda}{\lambda_0} = \frac{0.0243\ \text{Å}}{0.0106\ \text{Å}} = 2.29$$

X-rays from molybdenum:

$$\frac{\Delta\lambda}{\lambda_0} = \frac{0.0243\ \text{Å}}{0.712\ \text{Å}} = 0.0341$$

Visible light from mercury:

$$\frac{\Delta\lambda}{\lambda_0} = \frac{0.0243\ \text{Å}}{5461\ \text{Å}} = 4.45 \times 10^{-6}$$

Because both incident and scattered wavelengths are simultaneously present in the beam, they can be easily resolved only if $\Delta\lambda/\lambda_0$ is a few percent or if $\lambda_0 \le 1$ Å.

(b) The so-called free electrons in carbon are actually electrons with a binding energy of about 4 eV. Why may this binding energy be ignored for x-rays with $\lambda_0 = 0.712$ Å?

Solution The energy of a photon with this wavelength is

$$E = hf = \frac{hc}{\lambda} = \frac{12\,400\ \text{eV}\cdot\text{Å}}{0.712\ \text{Å}} = 17\,400\ \text{eV}$$

Therefore, the electron binding energy of 4 eV is negligible in comparison with the incident x-ray energy.

3.6 PARTICLE–WAVE COMPLEMENTARITY

As we have seen, the Compton effect offers ironclad evidence that when light interacts with matter it behaves as if it were composed of particles with energy hf and momentum h/λ. Yet the very success of Compton's theory raises many questions. If the photon is a particle, what can be the meaning of the "frequency" and "wavelength" of the particle, which determine its energy and momentum? Is light in some sense simultaneously a wave and a particle? Although photons have zero mass, is there a simple expression for an effective gravitational photon mass that determines a photon's gravitational attraction? What is the spatial extent of a photon, and how does an electron absorb or scatter a photon?

Although answers to some of these questions are possible, it is well to be aware that some demand a view of atomic processes that is too pictorial and literal. Many of these questions issue from the viewpoint of classical mechanics, in which all matter and energy are seen in the context of colliding billiard balls or water waves breaking on a shore. Quantum theory gives light a more flexible nature by implying that different experimental conditions elicit either the wave properties or particle properties of light. In fact, *both views are necessary and complementary*. Neither model can be used exclusively to describe electromagnetic radiation adequately. A complete understanding is obtained only if the two models are combined in a complementary manner.

The physicist Max Born, an important contributor to the foundations of quantum theory, had this to say about the particle–wave dilemma:

> The ultimate origin of the difficulty lies in the fact (or philosophical principle) that we are compelled to use the words of common language when we wish to describe a phenomenon, not by logical or mathematical analysis, but by a picture appealing to the imagination. Common language has grown by everyday experience and can never surpass these limits. Classical physics has restricted itself to the use of concepts of this kind; by analyzing visible motions it has developed two ways of representing them by elementary processes: moving particles and waves. There is no other way of giving a pictorial description of motions—we have to apply it even in the region of atomic processes, where classical physics breaks down.
>
> Every process can be interpreted either in terms of corpuscles or in terms of waves, but on the other hand it is beyond our power to produce proof that it is actually corpuscles or waves with which we are dealing, for we cannot simultaneously determine all the other properties which are distinctive of a corpuscle or of a wave, as the case may be. We can therefore say that the wave and corpuscular descriptions are only to be regarded as complementary ways of viewing one and the same objective process, a process which only in definite limiting cases admits of complete pictorial interpretation.[16]

Thus we are left with an uneasy compromise between wave and particle concepts and must accept, at this point, that both are necessary to explain the observed behavior of light. Further considerations of the dual nature of light and indeed of all matter will be taken up again in Chapters 4 and 5.

[16] M. Born, *Atomic Physics*, fourth edition, New York, Hafner Publishing Co., 1946, p. 92.

3.7 DOES GRAVITY AFFECT LIGHT?

OPTIONAL

It is interesting to speculate on how far the particle model of light may be carried. Encouraged by the successful particle explanation of the photoelectric and Compton effects, one may ask whether the photon possesses an effective gravitational-mass, and whether photons will be attracted gravitationally by large masses, such as those of the Sun or Earth, and experience an observable change in energy.

To investigate these questions, recall that the photon has zero mass, but its effective inertial mass, m_i, may reasonably be taken to be the mass equivalent of the photon energy, E, or

$$m_i = \frac{E}{c^2} = \frac{hf}{c^2} \tag{3.35}$$

The same result is obtained if we divide the photon momentum by the photon speed c:

$$m_i = \frac{p}{c} = \frac{hf}{c^2}$$

Recall that the *effective inertial mass* determines how the photon responds to an applied force such as that exerted on it during a collision with an electron. The *gravitational mass* of an object determines the force of gravitational attraction of that object to another, such as the Earth. Although it is a remarkable unexplained fact in Newtonian mechanics that the inertial mass of all material bodies is equal to the gravitational mass to within one part in 10^{12}, Einstein's Equivalence Principle of general relativity requires this result as mentioned in Chapter 2.

Let us assume that the photon, like other objects, also has a gravitational mass equal to its inertial mass. In this case a photon falling from a height H should increase in energy by mgH and therefore increase in frequency, although its speed cannot increase and remains at c. In fact, experiments have been carried out that show this increase in frequency and confirm that the photon indeed has an effective gravitational mass of hf/c^2. Figure 3.25 shows a schematic representation of the experiment. An expression for f' in terms of f may be derived by applying conservation of energy to the photon at points A and B.

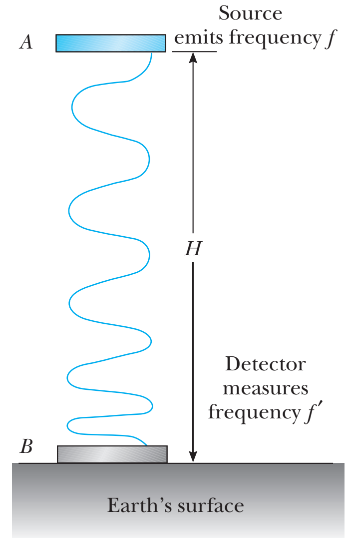


Figure 3.25 Schematic diagram of the falling-photon experiment.

$$KE_B + PE_B = KE_A + PE_A$$

Because the photon's kinetic energy is $E = pc = hf$ and its potential energy is mgH, where $m = hf/c^2$, we have

$$hf' + 0 = hf + \left(\frac{hf}{c^2}\right) gH$$

or

$$f' = f\left(1 + \frac{gH}{c^2}\right) \tag{3.36}$$

The fractional change in frequency, $\Delta f/f$, is given by

$$\frac{\Delta f}{f} = \frac{f' - f}{f} = \frac{gH}{c^2} \tag{3.37}$$

For $H = 50$ m, we find

$$\frac{\Delta f}{f} = \frac{(9.8 \text{ m/s}^2)(50 \text{ m})}{(3.0 \times 10^8 \text{ m/s})^2} = 5.4 \times 10^{-15}$$

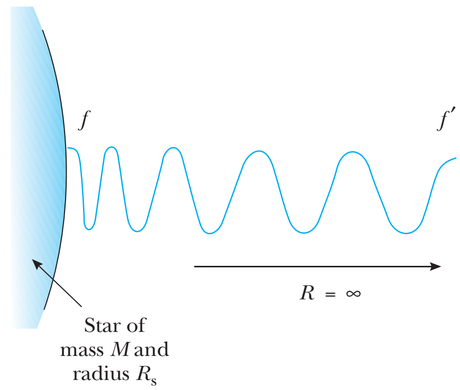


Figure 3.26 Gravitational redshift from a high-density star.

This incredibly small increase in frequency has actually been measured (with difficulty)![17] The shift amounts to only about 1/250 of the line width of the monochromatic γ-ray photons used in the falling-photon experiment.

The increase in frequency for a photon falling inward suggests a decrease in frequency for a photon that escapes outward to infinity against the gravitational pull of a star (see Fig. 3.26). This effect, known as "gravitational redshift," would cause an emitted photon to be shifted in frequency toward the red end of the spectrum. An expression for the redshift may be derived once again by conserving photon energy:

$$[KE + PE]_{R=\infty} = [KE + PE]_{R=R_s}$$

Using hf for the photon's kinetic energy and $-GMm/R$ for its potential energy, with m equal to hf/c^2 and R_s equal to the star's radius, yields

$$hf' - 0 = hf - \frac{GM}{R_s}\left(\frac{hf}{c^2}\right) \tag{3.38}$$

or

$$f' = f\left(1 - \frac{GM}{R_s c^2}\right) \tag{3.39}$$

EXAMPLE 3.9 The Gravitational Redshift for a White Dwarf

White dwarf stars are extremely massive, compact stars that have a mass on the order of the Sun's mass concentrated in a volume similar to that of the Earth. Calculate the gravitational redshift for 300-nm light emitted from such a star.

Solution We can write Equation 3.39 in the alternate form

$$\frac{f' - f}{f} = \frac{\Delta f}{f} = \frac{GM}{R_s c^2}$$

Using the values

$$M = \text{mass of Sun} = 1.99 \times 10^{30}\ \text{kg}$$

[17] R. V. Pound and G. A. Rebka, Jr., *Phys. Rev. Lett.*, 4:337, 1960.

$$R_s = \text{radius of Earth} = 6.37 \times 10^6 \text{ m}$$
$$G = 6.67 \times 10^{-11} \text{ N}\cdot\text{m}^2/\text{kg}^2$$

we find

$$\frac{\Delta f}{f} = \frac{(6.67 \times 10^{-11} \text{ N}\cdot\text{m}^2/\text{kg}^2)(1.99 \times 10^{30} \text{ kg})}{(6.37 \times 10^6 \text{ m})(3.00 \times 10^8 \text{ m/s})^2}$$
$$= 2.31 \times 10^{-4}$$

Because $\Delta f/f \approx df/f$, and $df = -(c/\lambda^2)\ d\lambda$ (from $f = c/\lambda$), we find $|df/f| = d\lambda/\lambda$. Therefore, the shift in wavelength is

$$\Delta\lambda = (300 \text{ nm})(2.31 \times 10^{-4}) = 0.0695 \text{ nm} \approx 0.7 \text{ Å}$$

Note that this is a redshift, so the observed wavelength would be 300.07 nm.

One more observation about Equation 3.39 is irresistible. Is it possible for a very massive star in the course of its life cycle to become so dense that the term $GM/R_s c^2$ becomes greater than 1? In that case Equation 3.38 suggests that the photon cannot escape from the star, because escape requires more energy than the photon initially possesses. Such a star is called a *black hole* because it emits no light and acts like a celestial vacuum cleaner for all nearby matter and radiation. Even though the black hole itself is not luminous, it may be possible to observe it indirectly in two ways. One way is through the gravitational attraction the black hole would exert on a normal luminous star if the two constituted a binary star system. In this case the normal star would orbit the center of mass of the black hole/normal star pair, and the orbital motion might be detectable. A second indirect technique for "viewing" a black hole would be to search for x-rays produced by inrushing matter attracted to the black hole. Although the black hole itself would not emit x-rays, an x-ray-emitting region of roughly stellar diameter should be observable, as shown in Figure 3.27. X-rays are produced by the

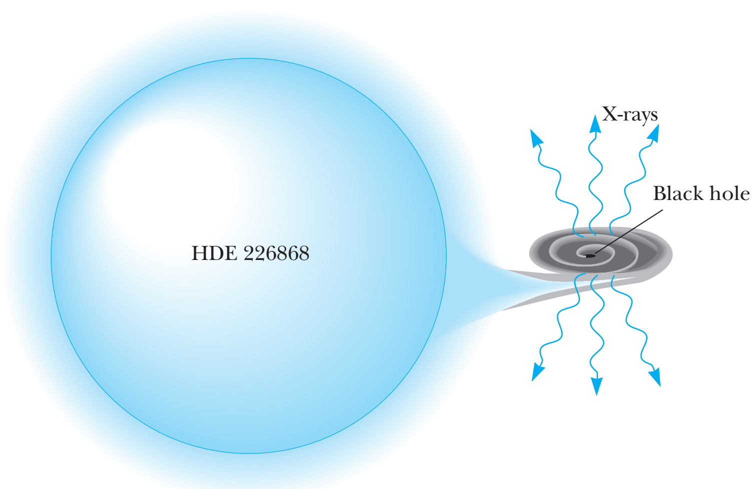


Figure 3.27 The Cygnus X-1 black hole. The stellar wind from HDE 226868 pours matter onto a huge disk around its black hole companion. The infalling gases are heated to enormous temperatures as they spiral toward the black hole. The gases are so hot that they emit vast quantities of x-rays.

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heating of the infalling matter as it circulates, is compressed, and eventually falls into the black hole. Such an intense nonluminous point source of x-rays has been detected in the constellation of the Swan. This source, designated Cygnus X-1, is believed by most astronomers to be a black hole; it possesses a luminosity, or power output, of 10^{30} W in the 2- to 10-keV x-ray range.

Recently, even more convincing evidence of a black hole has been obtained from radio telescope measurements of a dust torus rotating rapidly around a huge central mass at the center of galaxy NGC 4258. (See Figure 3.28.) These observations pinpoint a mass of 39 million solar masses within a radius of 4.0×10^{15} m, a density 10,000 times greater than any known cluster of stars and almost certainly high enough to produce a black hole. The central gravitational mass of 39 million solar masses was calculated from the observed speed of rotation of the dust torus, which is about 1 million m/s. And we needn't even go so far away as NGC 4258. Evidence of a black hole at the center of our own galaxy has been rapidly accumulating, indicating that a black hole of about 3 million solar masses, concealed by dust, is located in the constellation Sagittarius.[18]

SUMMARY

The work of Maxwell and Hertz in the late 1800s conclusively showed that light, heat radiation, and radio waves were all electromagnetic waves differing only in frequency and wavelength. Thus it astonished scientists to find that the

[18] See the interesting book *The Black Hole at the Center of our Galaxy*, by Fulvio Melia, Princeton University Press, 2003.