Chapter 3 Vectors

After reading this chapter the student should be able to:

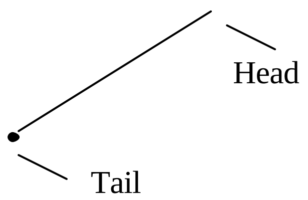
1. Distinguish between **a scalar** and **a vector.**
2. Add vectors **geometrically** and **analytically**.
3. Be familiar with the **unit vector** notation.
4. Know how to perform **vector** (or **cross) product** and **scalar** (or **dot) product**.

3.1 Vectors and Scalars

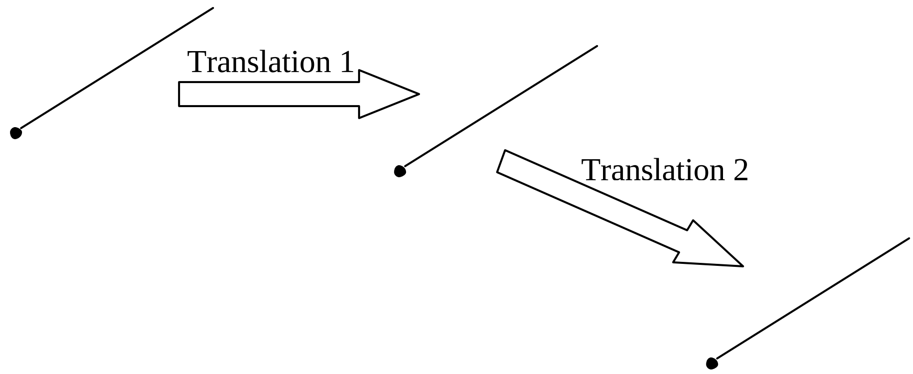
A vector has a **magnitude** and a **direction**. Some physical quantities that are vector quantities are displacement, velocity, acceleration and force.

A scalar on the other hand is given as a **single value** with a sign. There are many scalar quantities in physics such as temperature, pressure, and mass.

A graphical representation of a vector quantity in two dimensions is shown below. Its length represents the magnitude of the physical quantity and the arrow indicates the direction. A vector has a tail and a head.

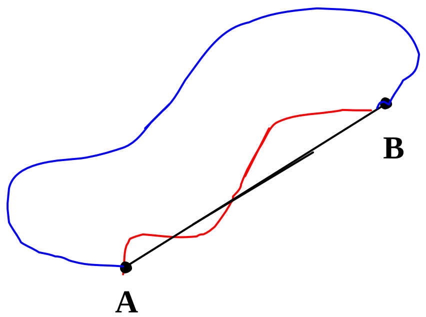


If we translate a vector without changing its magnitude and direction, the vector remains the same.



These three vectors are equivalents, i.e., they represent the same vector.

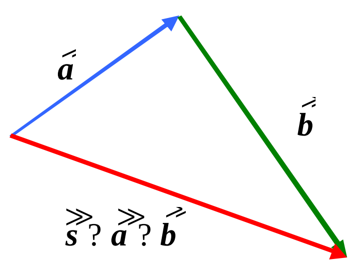
Suppose an object moves along the three paths as shown in the figure below. These three paths have the same displacement vector as they start at end at the same points A and B.



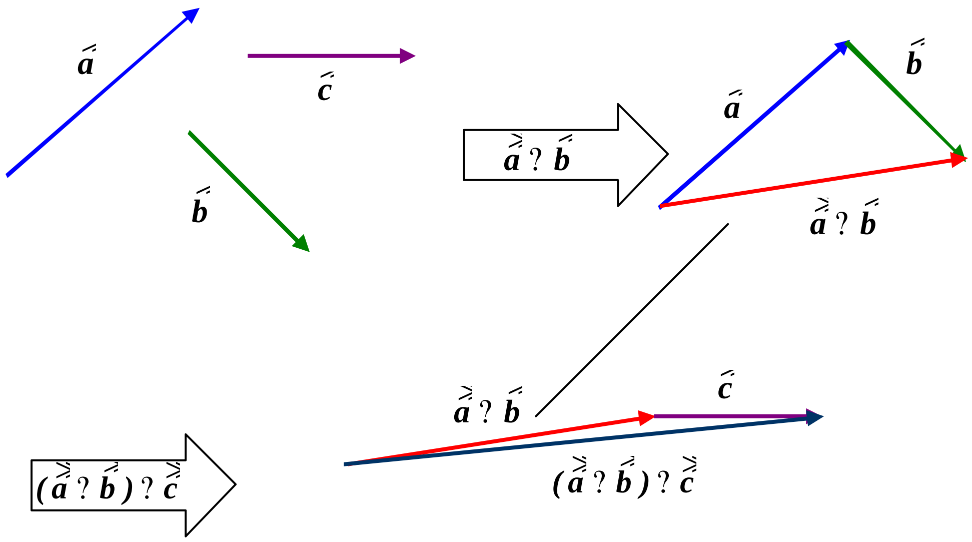
Note that the distance traveled by the object in moving from A to B is different for the three paths. It is the biggest for the blue path.

3.2 Adding Vectors Geometrically

To add two vectors, say displacement vectors, means to evaluate the net displacement we can use what is called the **graphical method**. The **net** (or **resultant**) displacement of two displacement vectors $\vec{a}$ and $\vec{b}$ is given by the vector equation $\vec{s}$? $\vec{a}$? $\vec{b}$. The resultant is also a vector. The procedure to add the vectors geometrically is to bring the tail of $\vec{b}$ at the head of $\vec{a}$ keeping the orientation of the two vectors unchanged. The vector sum $\vec{s}$ extends from the tail of vector $\vec{a}$ to the head of vector $\vec{b}$.



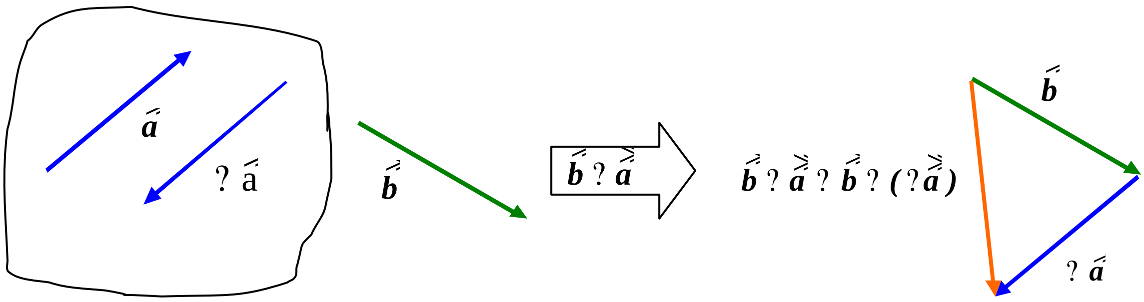
Next, let us add three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$. The graphical way to do it is to perform the sum of two vectors say $\vec{a}+\vec{b}$ and then add the third one. The vector equation is $(\vec{a}+\vec{b})+\vec{c}$. This sum is illustrated in the following diagram.



It is to be noted that $(\vec{a} ? \vec{b}) ? \vec{c} ? \vec{a} ? (\vec{b} ? \vec{c})$. The law is associative.

The vector $?\vec{a}$ has the same magnitude as $\vec{a}$, but points in the opposite direction.

So $\vec{b} ? \vec{a} ? \vec{b} ? (?\vec{a})$.



This is the rule of vector subtraction.

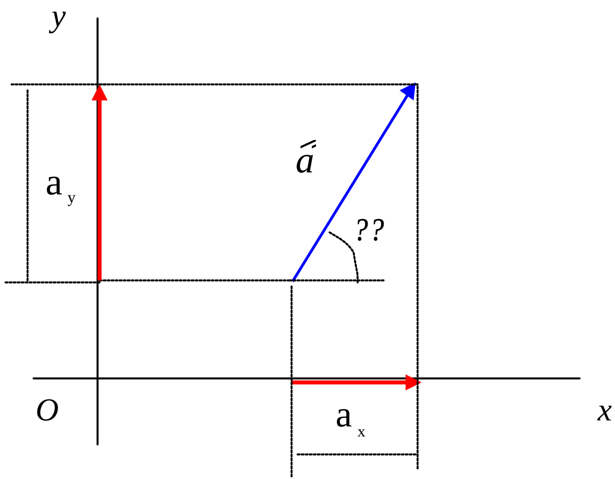
If a vector is moved from one side of an equation to the other, a change is sign is needed similar to rules of algebra.

It is to be noted that only vectors of **the same kind** can be added or subtracted. We cannot add a displacement vector to a velocity vector!

3.3 Components of Vectors

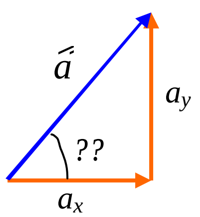
Adding vectors geometrically does not tell us how to do the math with vector addition. We need to learn how to calculate vector quantities mathematically. We will do this just in two dimensions, but it can be extended to three dimensions.

A **component** of a vector is the projection of the vector on one of the axes of a *set of cartesian coordinates system* as shown in the figure below.



$\boldsymbol{a_x}$ is the component of vector $\vec{a}$ along the **x-axis** (or ***x-component***) and $\boldsymbol{a_y}$ is the component of vector $\vec{a}$ along the **y-axis** (or ***y-component***). We say that the **vector is resolved** into its components. ? is the angle the vector $\vec{a}$ makes with the positive *x*-axis. In this figure both components of vector $\vec{a}$ are positive.

Note that $\vec{a}$? $\vec{a}_x$? $\vec{a}_y$, graphically



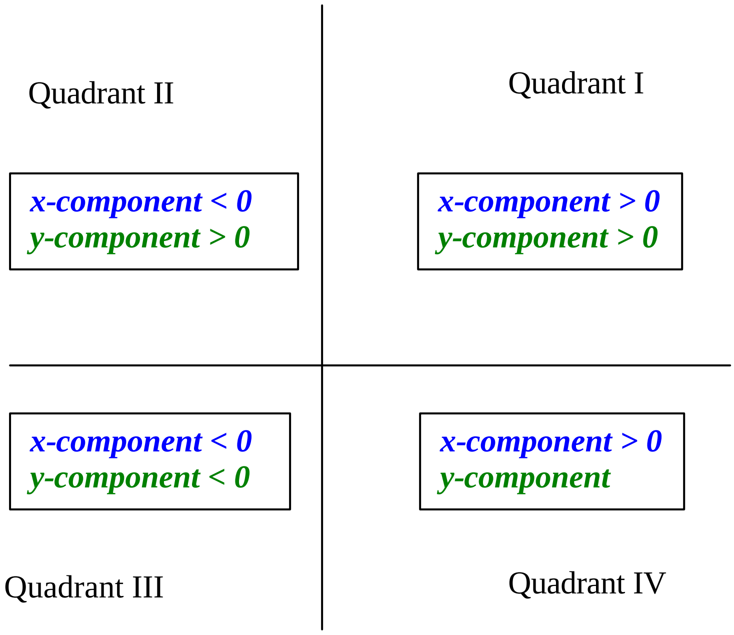
The values of the components a_x and a_y can be found if the angle ? and the magnitude of the vector $\vec{a}$ are known:

a_x ? $a\cos$?	a_y ? $a\sin$?

On the other hand, if the components a_x and a_y of the vector $\vec{a}$ are known, we can find its magnitude and direction:

$a \;?\; \sqrt{a_x^{\,2} \;?\; a_y^{\,2}}$ (using Pythagoras theorem) and $tan? \;?\; \dfrac{a_y}{a_x}$

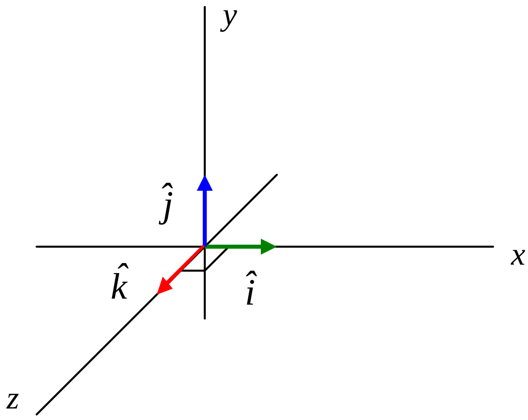
General rule for the **sign** of the vector components:



3.4 Unit Vectors

A **unit vector** is one that has a magnitude of 1 and points in a particular direction. It is often indicated by putting a "hat" of top of the vector symbol, for example **unit vector** $= \hat{a}$ and $|\hat{a}| = 1$.

We will label the unit vectors in the positive directions of the x, y , and z-axes as $\hat{i}, \hat{j}, and\ \hat{k}$.



A two dimensional vector $\vec{a}$ can be expressed in terms of unit vectors as $\vec{a} ? a_x\hat{i} ? a_y\hat{j}$. The quantities $a_x\hat{i}$ and $a_y\hat{j}$ are vectors and called the **vector components** of $\vec{a}$ while a_x and a_y are scalars and called the **scalar components** of $\vec{a}$.

3.5 Adding Vectors by Components

We have seen in section 3.2 how to add vectors graphically. The resolution of a vector into its components can be used to add and subtract vectors arithmetically. To illustrate this let us take an example. What is the sum of the following three vectors using the components method?

$$\vec{a} ? a_x\hat{i} ? a_y\hat{j} ? a_z\hat{k}$$

$$\vec{b} ? b_x\hat{i} ? b_y\hat{j} ? b_z\hat{k}$$

$$\vec{c} ? c_x\hat{i} ? c_y\hat{j} ? c_z\hat{k}$$

The vector sum is

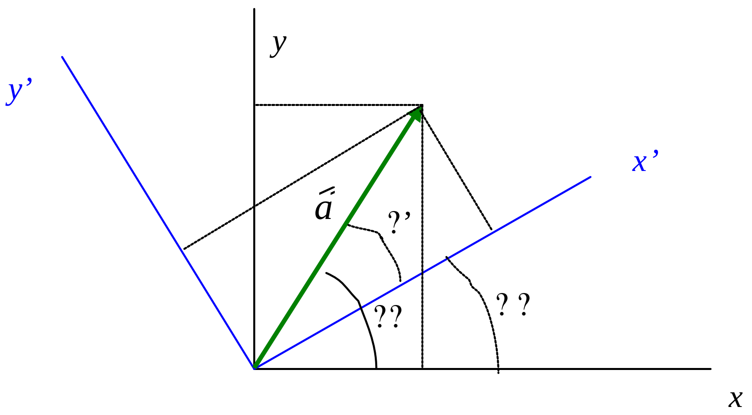
$$\vec{s} ? \vec{a} ? \vec{b} ? \vec{c} ? (a_x ? b_x ? c_x)\hat{i} ? (a_y ? b_y ? c_y)\hat{j} ? (a_z ? b_z ? c_z)\hat{k}$$

The components of the vector $\vec{s}$ are:

$$s_x ? a_x ? b_x ? c_x$$
$$s_y ? a_y ? b_y ? c_y$$
$$s_z ? a_z ? b_z ? c_z$$

3.6 Vectors and the Laws of Physics

So far we have used a convenient set of coordinate system that aligns with the edges of your notebook. However, we can also choose another set of coordinates as long as we are interested in the physical vector quantity such as displacement, acceleration, force, etc. Let us take a graphical example.

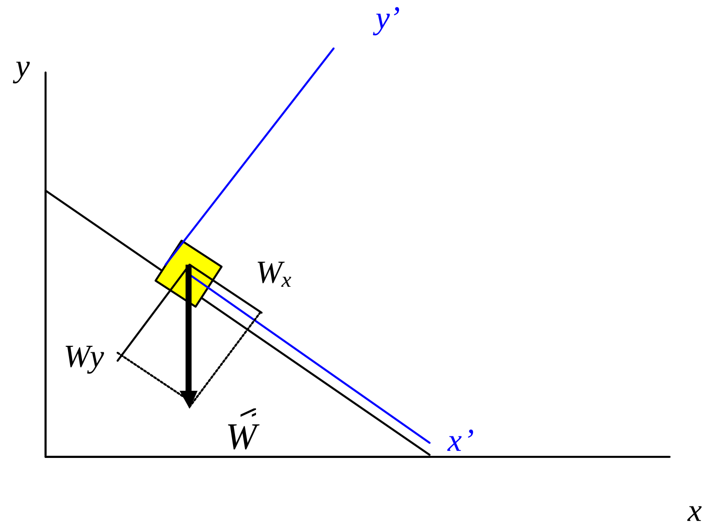


The components of vector $\vec{a}$ have are different when we change the coordinates system from (*x*,*y*) to (*x'*,*y'*), however, the magnitude and direction of the vector itself have remained the same or **invariant** under **rotation**.

$$a ? \sqrt{a_x{}^2 ? a_y{}^2} ? \sqrt{a_{x'}{}^2 ? a_{y'}{}^2} \quad \text{and} \quad ? \; ? ?' ? ?$$

The new coodinate system (*x'*,*y'*)has been formed by by rotating the old coordinate system (*x*,*y*) by an angle ? .

The Physics is not affected by the choice of the coordinates system. You should always look for the most convenient system to use in solving your problem. For example, in the case of the inclined plane:



It is convenient to use the (*x'*,*y'*) coordinate system rather than using the (*x*,*y*) one. $\vec{W}$ represents the weight of the object. The object will move in the direction of the *x'*- axis. $\vec{W}$ is resolved in the (*x'*,*y'*) coordinate system as seen in te figure.

3.7 Multiplying Vectors

The multiplication of a vector $\vec{a}$ by a scalar *m* gives a new vector.

$$\vec{a} \cdot m ? m \cdot \vec{a} ? m\vec{a}$$

If *m* > 0, the new vector will have the same direction as the vector $\vec{a}$.

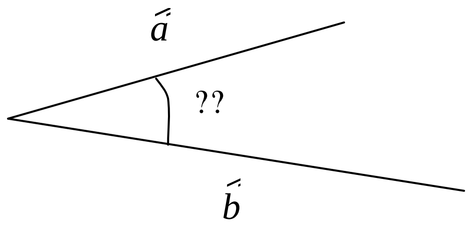
If *m* < 0, the new vector will have apposite direction to the vector $\vec{a}$.

To divide $\vec{a}$ by m, we multiply $\vec{a}$ by $1/m$.

There are two types of vector multiplication, namely, the **scalar** or **dot product** of two vectors, which results in a scalar, and the **vector** or **cross product** of two vectors, which results in a vector.

The **scalar product** of two vectors $\vec{a}$ and $\vec{b}$, denoted $\vec{a} ? \vec{b}$, is defined to be

$$\boxed{\vec{a} ? \vec{b} ? |a||b| \cos ?}$$



$|a|$ and $|b|$ are the magnitudes of vector $\vec{a}$ and vector $\vec{b}$, respectively.

And in particular we have $\hat{i} ? \hat{i} ? \hat{j} ? \hat{j} ? \hat{k} ? \hat{k} ? 1$, since the angle between a vector and itself is 0 and the cosine of 0 is 1.

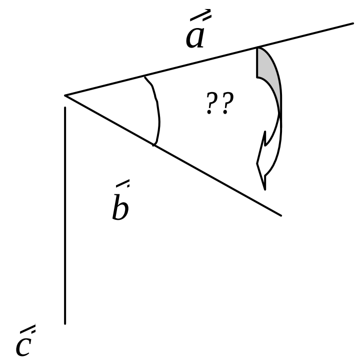
Alternatively, we have $\hat{i} ? \hat{j} ? \hat{i} ? \hat{k} ? \hat{j} ? \hat{k} ? 0$, since the angle between $\hat{i}$ and $\hat{j}$, $\hat{i}$ and $\hat{k}$, and $\hat{j}$ and $\hat{k}$ is 90º and the cosine of 90º is 0.

The laws for scalar products are given in the following:

$$\vec{a} ? \vec{b} ? \vec{b} ? \vec{a}$$
$$\vec{a} ?(\vec{b} ? \vec{c}) ? \vec{a} ? \vec{b} ? \vec{a} ? \vec{c}$$
$$m(\vec{a} ? \vec{b}) ? (m\vec{a}) ? \vec{b}$$
$$? \vec{a} ?(m\vec{b})$$
$$? \vec{a} ?(m\vec{b})$$
$$? m(\vec{a} ? \vec{b})$$

The **vector product** of two vectors $\vec{a}$ and $\vec{b}$, denoted $\vec{a} ? \vec{b}$, produces a third vector $\vec{c}$ whose magnitude is

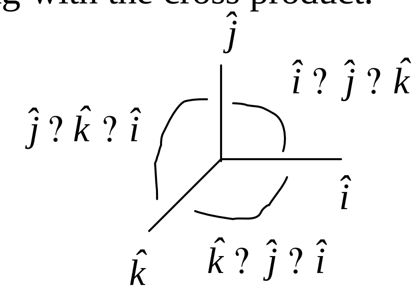
$$c ? |a||b| \sin ?$$



The direction of $\vec{c}$ is perpendicular to the plane that contains the two vectors $\vec{a}$ and $\vec{b}$ as shown in the figure. The direction is found by what is called the right hand rule as seen in the following demo.

In particular we have $\hat{i} ?\hat{i} ? \hat{j} ?\hat{j} ? \hat{k} ?\hat{k} ? 0$, since the angle between a vector and itself is 0 and the sine of 0 is 1.

Note that the angle between $\hat{i}$ and $\hat{j}$, $\hat{i}$ and $\hat{k}$, and $\hat{j}$ and $\hat{k}$ is 90^{o} and the sine of 90^{o} is 1. Therefore we use the following diagram to help us solve problems dealing with the cross product.



However, if the rotation in the figure is <u>clockwise</u> such as $\hat{j} ? \hat{i} ? ?\hat{k}$ etc…