Illustrating Probability through Roulette: A Spreadsheet Simulation Model

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Abstract

Teaching probability can be challenging because the mathematical formulas often are too abstract and complex for the students to fully grasp the underlying meaning and effect of the concepts. Games can provide a way to address this issue. For example, the game of roulette can be an exciting application for teaching probability concepts. In this paper, we implement a model of roulette in a spreadsheet that can simulate outcomes of various betting strategies. The simulations can be analyzed to gain better insights into the corresponding probability structures. We use the model to simulate a particular betting strategy known as the bet-doubling, or Martingale, strategy. This strategy is quite popular and is often erroneously perceived as a winning strategy even though the probability analysis shows that such a perception is incorrect. The simulation allows us to present the true implications of such a strategy for a player with a limited betting budget and relate the results to the underlying theoretical probability structure. The overall validation of the model, its use for teaching, including its application to analyze other types of betting strategies are discussed.

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1 Introduction and literature review

The quest for novel teaching and delivery methods is never ending. As professors we are continually challenged to stimulate students' interest by having exciting examples or interesting applications that accomplish the associated pedagogical goals. This is the motivation behind the approach presented in this paper. In our statistics and management science/operations research (MS/OR) classes we use a model familiar to most or all students–the game of roulette–for teaching probability concepts and introducing spreadsheet simulation techniques.

The game of roulette is an interesting and intriguing problem. The game is simple to play and has the lure of easy winning. As a result, many betting strategies have been proposed over the years with claims from sure wins to minimization of the risk of losing in the game. The analysis of the underlying probability structures of these betting strategies can be worthwhile exercises for students in a statistics or MS/OR course to get many interesting insights into the validity of these claims. However, these analyses can be mathematically complex and initially difficult for the students to understand. Simulation of roulette in a spreadsheet allows the student to investigate the outcomes of various strategies for playing the game and can be an excellent means for carrying out the analysis of the underlying probability structures without resorting to complex mathematical models. Such a simulation model can also help educators to enhance teaching probability concepts in a statistics course and simulation techniques in a MS/OR course.

In this paper we show a way to implement a simulation model of the game of roulette in spreadsheets and use it to analyze various betting strategies for the game in order to help students understand the underlying probability concepts. In particular, we use one of the well-known strategies of roulette, the bet-doubling, or Martingale strategy as our example for the simulation model. It is easy for students to believe the Martingale strategy to be a fail-safe system (O'Connor and Dickerson [16]). The simulation model, easily implemented in a spreadsheet, can be used to investigate that strategy and show the inherent fallacy in that logic.

Games can be exciting examples for teaching. They can raise students' interest in the subject, and provide better retention, comprehension and overall learning (Leemkuil, et al., [13]). Simulated games have been shown to be beneficial in training practitioners as well as teaching in traditional classrooms. Dempsey et al., [7] presents the various characteristics of games that can lend themselves for use in an educational setting. Additionally, research shows that simulation can be used for teaching a variety of subjects such as statistics (Doane, [8]), Economics (Craft, [6]), Organizational Development (Ruohomaki, [18]), Business Education (Chang, et al., [5]) and Operations Management/Management Science (Thiriez, [21], Seal, [19], Al-Faraj, et al., [1]). Mills [15], in particular, provides a comprehensive literature review of using computer simulation methods to teach statistics while Yeates [22] provides a very comprehensive bibliography on the use of simulation in business games.

In this paper, we bring the game and simulation together through very familiar software to the students: the spreadsheet. Since Bodily [4], spreadsheets have become a very popular medium for development and implementation of many types of simulation models. Spreadsheets are almost ubiquitous in present day personal computers and are well accepted as an analysis tool by the end-users. They provide an intuitive and easy to understand interface and allow one to build sophisticated simulation models with relative ease. Simulation add-in packages such as Crystal Ball, PopTools, or @RISK, can provide a very powerful simulation environment for large complex problems. The virtues of spreadsheets as a medium for simulation and modeling along with examples of its application in multiple fields, as well as their use for teaching statistics and probability concepts, quantitative modeling and simulation, and operations management, have been mentioned by many researchers (Leon, et al., [14], Bodily, [4], Ragsdale, [17], Kohler, [12], Anderson *et al*, [3], Ammar and Wright, [2], Grossman, [10], Eppen *et al* [9]).

Because of these advantages, we have used the spreadsheet as our implementation medium for the simulation of roulette. An earlier paper by Seal and Przasnyski, [20] introduced the basic mechanics of a spreadsheet simulation model for roulette with simple betting strategies. This paper shows the development of a more advanced model and its application for analysis of a particular betting strategy along with theoretical validation.

In the following section, we begin with a brief description of the game of roulette and the associated rules. This is followed by the description of the implementation of the simulation model of the game in the spreadsheet and its use through the Martingale strategy. We validate the simulation model by comparing its outputs with that of the mathematical model for the strategy. We discuss the results and conclude with a critical analysis of the approach, the contribution, and key points of the paper.

2 Roulette

The game of roulette consists of a spinning wheel with numbered and colored slots, a small ball that can drop into one of the slots, and a betting table. There are two versions of the roulette wheel, the Monte Carlo wheel and the Las Vegas wheel. In the Monte Carlo wheel, the slots are numbered from 0 to 36 while in Las Vegas wheel the numbers are 00, and 0 to 36. In this paper we use the Las Vegas version. The number slots are alternately colored red or black with the exception of the slots with "0" and "00", which are colored green. Color and number combinations offer a variety of betting options. The dealer, a person representing the gambling house and in charge of controlling the game, spins the wheel and the ball. The players place their bets on the betting table on individual numbers, colors, groups of numbers, or on various combinations of numbers and colors. While bets are being placed, the dealer spins the roulette ball in the opposite direction of the spinning wheel. Bets may be placed until the dealer announces "no more bets" and the ball begins to drop. When the ball falls into a numbered slot on the roulette wheel, the dealer places a marker on the winning number on the table layout and pays the winning amount against the bets placed on that number and/or the color. Note that the winning amount for a payoff of $n : 1$ indicates that, upon winning, the player will get n times the original bet amount plus the original bet. For example, a payoff of $35 : 1$ means that if $1 bet is placed, and the player wins, then the player will get a winning amount of $36 ($\$35 + \$1$).

The full list of the possible bets and associated payoffs and the layout of the roulette table is provided in Appendix A.

3 The basic model

The model considers a single player playing roulette against the house in a simplified roulette game where the only allowed betting scenarios (and the corresponding payback for winning) are as listed below.

- A bet on an individual number which pays back 35:1,
- A bet on an even number which pays back 1:1,
- A bet on an odd number which pays back 1:1,
- A bet on the numbers 1-6 which pays back 5:1,
- A bet on the numbers 13-24 which pays back 2:1.

In all cases the amount of the bet is returned to the player if he/she wins.

Other betting scenarios, described in Appendix A, could be implemented as well but many of them are equivalent to those above (e.g., betting on 7-12 is equivalent to betting on 1-6, betting on red is equivalent to betting on even numbers) and others can be incorporated as easy extensions to the model.

The model can be used to investigate various betting strategies, both pure and mixed. In a pure strategy, only one type of bet is placed by a player, e.g.,

- Player bets a dollar amount on a single number each play,
- Player bets a dollar amount on even or odd each play,
- Player bets a dollar amount on the number 1-6 each play,
- Player bets a dollar amount on the number 13-24 each play,
- Player bets double the previous bet amount on even or odd if the previous play was a loss and bets on the same type of number again (even or odd), the Martingale strategy.

In mixed strategies, multiple bets are placed by the player, e.g.,

- Player bets a dollar amount on even and odd each play,
- Player bets a dollar amount on even and odd each play and an individual number.

The basic spreadsheet model is presented in Figure 1. This shows an input section `A2:E9` and the main model for 240 spins of the roulette wheel `A12:I254`. For the purpose of illustration, we selected 30 spins to represent about two and half hours of play by a player, and 240 spins to represent 20 hours of play by a player. We are assuming that one spin, including the placement of bets, spinning the wheel, and distribution of the winnings and preparing the table for the next spin, takes approximately 5 minutes. We further assume that an average gambler will not spend more than two and a half hours on a roulette table and 20 hours represents an average day since some of the night time hours may be slow. The point of the "day's play" is to let the user get a sense of the long term pay-off from the game. Clearly, one can extend the spreadsheet to any number of spins by simply copying the rows.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	**BASIC MODEL**									**1-trial measures of performance**					Expected
2			**Inputs**								Player's final	**Amount**	Max	Min	
3	Player's initial amount:				$1,000						after 50 spins	**$1,094**	$1,094	$977	$961
4	Lucky number chosen to bet on:				17		Odds	Payoff			after 100 spins	**$1,034**	$1,138	$922	$921
5	Amount to bet on this number:				$1		2.63%	$36			after 200 spins	**$1,206**	$1,206	$892	$842
6	Amount to bet on even:				$2		47.37%	$4			after 240 spins	**$1,156**	$1,203	$934	$822
7	Amount to bet on odd:				$3		47.37%	$6							
8	Amount to bet on 1-6:				$4		15.79%	$24							
9	Amount to bet on 13-24:				$5		31.58%	$15							
10	Player's Total Bet:				$15		Expected total paid per spin:	$14.21							
11							Expected net return:	-$0.79							
12					Amount paid to player										
13	Spin Number	Ball	if number comes up	if even comes up	if odd comes up	if 1-6 comes up	if 13-24 comes up	Total paid to player	Player's cumulative $						
14									$1,000						
15	1	27			$6			$6	$991						
16	2	1			$6	$24		$30	$1,006						
17	3	23			$6		$15	$21	$1,012						
18	4	3			$6	$24		$30	$1,027						
19	5	14		$4			$15	$19	$1,031						
20	6	17	$36		$6		$15	$57	$1,073						
21	7	37							$1,058						
22	8	14		$4			$15	$19	$1,062						
23	9	3			$6	$24		$30	$1,077						
24	10	32		$4				$4	$1,066						
25	11	24		$4			$15	$19	$1,070						
26	12	38							$1,055						
246	.	.	.	.	.	.	.	.	.						
247	.	.	.	.	.	.	.	.	.						
248	.	.	.	.	.	.	.	.	.						
249	235	2		$4		$24		$28	$1,148						
250	236	19			$6		$15	$21	$1,154						
251	237	29			$6			$6	$1,145						
252	238	17	$36		$6		$15	$57	$1,187						
253	239	16		$4			$15	$19	$1,191						
254	240	1			$6	$24		$30	$1,206						

Figure 1: The basic spreadsheet model.

In the main part of the spreadsheet model, values in `E5:E9` record the amounts bet by the player for the various strategies identified in the corresponding row labels. The dropping of the ball on a particular slot is simulated by generating a random number in column `B` using the built-in `RANDBETWEEN` function in Microsoft Excel. Columns `C` through `G` record the amounts paid to the player for the corresponding bets, which are then totaled in column `H`. Finally, column `I` tracks the player's cumulative amount after each spin. The values of the measures of performance for this particular trial of the simulation are shown in `J1:N6`, and the corresponding theoretical expected amounts are displayed in column `O`.

Figure 2 explains the formulas used in detail. Formulas are entered once in row 15 and then copied down for as many spins of the roulette wheel as are to be simulated. Note that in order for the `RANDBETWEEN` function to work in Excel, one needs to have Analysis ToolPak installed (Analysis ToolPak can be installed by clicking on Tools - Add-Ins and checking on Analysis ToolPak and Analysis ToolPak-VBA).

For compactness of display in Figure 1 only two strategies, one that returns 6 to 1 (i.e., bet on a value in the ranges 1 – 6, 7 – 12, 13 – 18, 19 – 24, 25 – 30 and 31 – 36), and another that returns 3 to 1 are included. A full implementation would simply have the extra columns with corresponding formulas. Similarly, for reasons of compactness of the displayed model the strategy involving betting on a color is omitted. The underlying logic (i.e., Excel formulas) is the same as for the odd-even strategies. Indeed, we have used the model in a teaching environment and found that given the model shown in Figure 1, implementing various strategies described in this section is a useful exercise for advanced undergraduate and MBA students to hone their spreadsheet and modeling skills.

We used the following assumptions and conventions in building the model:

- In order to facilitate comparison of strategies it is assumed that the same strategy will be followed by the player for all plays (240 in the illustrative example).
- 0 and 00 are represented by the random numbers 37 and 38, when using the `RANDBETWEEN` function. This is purely to make the resulting formulas simpler to read because it is then possible to test for values > 36 with one test to represent the generated 0 or 00. Another way to implement this would be to use -1, and 0 to represent 00 and 0, respectively.

4 Using the model

After the basic model is built, it is necessary to identify the appropriate measures of performance or outputs such as long-term payoffs for the player, the maximum bet amount, the duration of plays for a given amount of initial money or others as appropriate.

Once the outputs are identified, the actual simulation begins by running the model multiple times and noting the output measures. It is important to repeat the runs a large number of times to obtain the steady state or the long-term average behavior of the system. The multiple runs and collection of the output measures for each run can be done in various ways. The simplest way is to use an add-in package such as Crystal Ball from Decisioneering Inc., @Risk from Pallisades Corporation, or the free PopTools developed by Greg Hood at Pest Animal Control Co-operative Research Centre in Australia. Alternatively, one can manually execute the runs by simply using the recalculate feature (F9) of the spreadsheet and keeping track of the outputs systematically in an Excel table for eventual calculation of the statistical parameters of the outputs. However, manual runs can be tedious. We used Crystal Ball which ran the simulation many times with a few clicks of the mouse, produced various summary statistics, and allowed us to see the behavior of the system in multiple formats. The software is easy to use with a point-and-click interface and works seamlessly in a spreadsheet. In Appendix B, an illustrative example shows the Crystal Ball results for a player's total purse using the mixed strategy of "bet the same dollar amount on odd and even", after 30 and 240 spins when the player started with an initial purse of $1,000.

The model also lends itself to more challenging extensions and improvements. For example, a reasonably routine extension would be the strategy of betting on $1 - 18$, $19 - 36$ or on four values. Another, slightly more advanced extension, is the so-called

Input Section		
Cell	**Formula**	**Explanation**
E10	=SUM(E5:E9)	Player's Total Bet.
G5	=1/38	Odds of winning for bet on individual number.
G6	=18/38	Odds of winning for bet on even number.
G7	=18/38	Odds of winning for bet on odd number.
G8	=6/38	Odds of winning for bet on 1-6.
G9	=12/38	Odds of winning for bet on 13-24.
H5	=36*E5	Payoff if an indiviual number comes up.
H6	=2*E6	Payoff if the number that comes up is even.
H7	=2*E7	Payoff if the number that comes up is odd.
H8	=6*E8	Payoff if the number that comes up is 1-6.
H9	=3*E9	Payoff if the number that comes up is 13-24.
H10	=SUMPRODUCT(G5:G9,H5:H9)	Expected win per spin.
H11	=H10-E10	Expected net return is the difference between the expected total paid per win [H10] and the total bet [E10]
Main Model		
Cell	**Formula**	**Explanation**
B15	=RANDBETWEEN(1,38)	Simulation of the final value of the ball on this play, i.e., a random number in the range 1-38, where 37 and 38 represent 0 and 00 respectively.
C15	=IF(B15>36,0,IF(E4=B15,36*E5,0))	If the ball comes up >36 (i.e., 0 or 00) the amount paid to the player is zero. If the ball comes up with the lucky number [E4] the amount paid to the player is 36 times the bet [E5] on this number.
D15	=IF(B15>36,0,IF(ISEVEN(B15)=TRUE,E6*2,0))	If the ball comes up >36 (i.e., 0 or 00) the amount paid to the player is zero. If the ball comes up with an even value the amount paid to the player is twice the bet made [E6].
E15	=IF(B15>36,0,IF(ISODD(B15)=TRUE,E7*2,0))	If the ball comes up >36 (i.e., 0 or 00) the amount paid to the player is zero. If the ball comes up with an odd value the amount paid to the player is twice the bet made [E7].
F15	=IF(B15>36,0,IF(B15>6,0,E8*6))	If the ball comes up >36 (i.e., 0 or 00) the amount paid to the player is zero. If the ball comes up with a value >6 the player is paid zero, otherwise the player is paid 6 times the bet [E8].
G15	=IF(B15>36,0,IF(B15<13,0,IF(B15>24,0,E9*3)))	If the ball comes up >36 (i.e., 0 or 00) the amount paid to the player is zero. If the ball comes up with a value <13 or >24 the player is paid zero, otherwise the player is paid 3 times the bet [E9].
H15	=SUM(C15:G15)	Total amount paid to player, as a result of all bets placed.
I15	=I14-E10+H15	Player's cumulative amount after this spin = previous cumulative [I14] - total amount bet [E10] + total paid to player on this spin [H15].
Results Section		
Cell	**Formula**	**Explanation**
L3	=I64	Player's cumulative amount after 50 spins.
L4	=I114	Player's cumulative amount after 100 spins.
L5	=I214	Player's cumulative amount after 200 spins.
M3	=MAX(I15:I64)	The maximum of the player's cumulative amount over 50 spins.
M4	=MAX(I15:I114)	The maximum of the player's cumulative amount over 100 spins.
M5	=MAX(I15:I214)	The maximum of the player's cumulative amount over 200 spins.
M6	=MAX(I15:I254)	The maximum of the player's cumulative amount over 240 spins.
N3	=MIN(I15:I64)	The minimum of the player's cumulative amount over 50 spins.
N4	=MIN(I15:I114)	The minimum of the player's cumulative amount over 100 spins.
N5	=MIN(I15:I214)	The minimum of the player's cumulative amount over 200 spins.
N6	=MIN(I15:I254)	The minimum of the player's cumulative amount over 240 spins.
O3	=E$3+A64*(H$10-E$10)	Expected amount player should have after 50 spins = initial amount [E3] + 50 (expected win [H10] - amount bet [E10]) on each spin.
O4	=E$3+A114*(H$10-E$10)	Expected amount player should have after 100 spins = initial amount [E3] + 100 (expected win [H10] - amount bet [E10]) on each spin.
O5	=E$3+A214*(H$10-E$10)	Expected amount player should have after 200 spins = initial amount [E3] + 200 (expected win [H10] - amount bet [E10]) on each spin.
O6	=E$3+A254*(H$10-E$10)	Expected amount player should have after 240 spins = initial amount [E3] + 240 (expected win [H10] - amount bet [E10]) on each spin.

Figure 2: Formulas for the basic model presented in Figure 1.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
1	**DOUBLE ON EVEN MODEL**											**1-trial measures of performance**					Player's
2				**Inputs**			Bet							**Amount**	max	min	Max Bet
3	Player's initial amount (I):					$1,000	Multiplying					Player's cumulative		$1,106	$1,113	$922	$128
4	Lucky number chosen to bet on:						Factor (r)	Odds	Payoff								
5	Amount to bet on this number:							2.63%									
6	***Initial*** amount to bet on even (a):					$1	2	47.37%	$2								
7	Amount to bet on odd:							47.37%									
8	Amount to bet on 1-6:							15.79%									
9	Amount to bet on 13-24:							31.58%									
10	Player's Total Bet:					$1		Expected total paid per spin:	$0.95								
11								Expected net return:	-$0.05						Total number of games (G)	Total number of spins played (T)	Average spins per game (S)
12				Amount paid to player											113	240	2.1239
13	Spin Number	Amount Bet on Even	**Ball**	if number comes up	if even comes up	if odd comes up	if 1-6 comes up	if 13-24 comes up	Total paid to player	Player's amount putting winnings aside	Player's cumulative amount				Indicator for end of a game, i.e., a win	Number of spins till a game ends	Spin number where the player goes bust, else 1000
14										$1,000	$1,000				1		
15	1	$1	**29**							$999	$999					1	10000
16	2	$2	**37**							$997	$997					2	10000
17	3	$4	**12**		$8				**$8**	$1,000	$1,001				1	3	10000
18	4	$1	**9**							$999	$1,000					1	10000
19	5	$2	**1**							$997	$998					2	10000
20	6	$4	**14**		$8				**$8**	$1,000	$1,002				1	3	10000
21	7	$1	**33**							$999	$1,001					1	10000
22	8	$2	**18**		$4				**$4**	$1,000	$1,003				1	2	10000
23	9	$1	**7**							$999	$1,002					1	10000
24	10	$2	**33**							$997	$1,000					2	10000
25	11	$4	**12**		$8				**$8**	$1,000	$1,004				1	3	10000
26	12	$1	**8**		$2				**$2**	$1,000	$1,005				1	1	10000
246	.	.	.		.				.	.	.				.	.	.
247	.	.	.		.				.	.	.				.	.	.
248	.	.	.		.				.	.	.				.	.	.
249	235	$4	**4**		$8				**$8**	$1,000	$1,111				1	3	10000
250	236	$1	**14**		$2				**$2**	$1,000	$1,112				1	1	10000
251	237	$1	**12**		$2				**$2**	$1,000	$1,113				1	1	10000
252	238	$1	**17**							$999	$1,112					1	10000
253	239	$2	**17**							$997	$1,110					2	10000
254	240	$4	**33**							$993	$1,106					3	10000

Figure 3: Variation of the basic model of Figure 1, to illustrate the Martingale strategy

bet doubling-strategy or Martingale strategy. In this strategy one continuously bets on a particular outcome with a 50/50 chance (e.g., either odd or even) and whenever the previous bet is a losing bet the player doubles the bet in an attempt to recoup his/her loses on the next spin. Basic probability texts show that such a strategy is not sustainable without access to an infinite amount of money, but do not provide more complex results such as the number of average wins or average spins played by a player with limited amount of betting money. We show that our spreadsheet-based model can simulate these types of scenarios and produce the corresponding outputs very easily without resorting to complex mathematical analyses.

The spreadsheet model for the Martingale strategy is shown in Figure 3 with the corresponding formulas in Figure 4. This is essentially the same as the basic model except for the new extra columns B and J. In this betting scenario the player starts with a fixed bet amount on an even number. If the player loses on a particular spin, he/she multiplies the bet by two and bets on even again for the next spin. The player continues with this strategy for every loss till he/she runs out of money (i.e., goes bust). If the player, on the other hand, wins on a spin, he/she puts the winnings aside and then starts a new game with the same initial purse and fixed bet amount on even again. We define a game as an event that results in either a win for the player, or the loss of his/her entire purse (not including the winnings). The model is set up with a bet-multiplying factor parameter (cell `G6`) so that variants of the doubling strategy, such as multiplying the previous losing bet by 1.5 or 3, can easily be investigated.

Cells `B16`, `J15`, `K15` and `O15:Q15` are copied down for the required number of plays. Two alternatives for the formula in Cell `B16` are shown. Either can be used depending upon one's preference. The notation used in section 5.1 for the theoretical model is indicated in parentheses in the appropriate cells shown in Figure 3.

The formulas in the range `O11:Q254` allow us to calculate the total number of games, the total number of spins played and the average number of spins per game. We need to compute these quantities as part of the validation of the model described in section 5.1 below.

We ran the simulation model for the Martingale strategy under various input assumptions on the initial bet value and the starting purse amount of the player. We simulated the player's (i) cumulative amount and (ii) maximum amount, (iii) the number of wins, i.e., games, (iv) the number of spins played before going bust or reaching the maximum number of spins allowed in the simulation (i.e., 30, and 240) and (v) the average number of spins per game. We carried out the simulations for combinations of initial bet values of \$1, \$2, \$5, and \$10 and initial purse amounts of \$50, \$100, \$200, \$250, \$500, \$1,000, and \$2,000. Appendix C displays the mean values of items (i) through (v) of these simulations provided by Crystal Ball. We include the coefficient of variation for each of (i) through (iv) as an indicator of the variation associated with the value.

Based on the results presented in Appendix C we make the following observations:

1. The average number of spins per game is roughly constant at two, irrespective of the initial bet, the player's initial purse amount or the number of spins played.

2. The longer the player plays (i.e., the more spins of the wheel) the lower will be their (average) total winnings whereas their maximum possible winnings will be higher (irrespective of the initial bet and the players initial purse amount).

3. The higher the player's initial purse amount and the lower the initial bet the longer it takes for the player to go bust (i.e., the higher the number of spins played by the player).

4. The higher the player's initial purse amount, the less variability there is in all of the above results.

5. Even with a relatively small initial purse, a player can play for a reasonably long time on a roulette table. For example, a player with an initial purse of \$100 and an initial bet of \$2 can expect to play about 24 spins, which is roughly equivalent to about two hours of play at the table.

Cell	Formula	Explanation
B15	=F6	Original bet amount (on even).
B16	=IF(E15>0,F6,IF(J15>=B15*G6,B15*G6,0))	If the preceding bet produced a win, then bet the *initial* amount on this spin [F6]. Otherwise, if the amount the player has left (the player's cumulative [J15]) is ≥ than twice (the bet multiplying factor [G6]) times the bet amount on the preceding spin [B15], then bet twice (the bet multiplying factor [G6]) times the bet amount on the preceding spin [B15], else no bet can be made.
B16 alter-native	=IF(E15>0,F6,MIN(J15,B15*G6))	If the preceding bet produced a win, then bet the *initial* amount on this spin [F6]. Otherwise bet the minimum of how much the player has left (the player's cumulative [J15]) and twice (the bet multiplying factor [G6]) times the bet amount on the preceding spin [B15].
J15	=IF(I15>0,F3,J14-B15)	If the player wins this spin, she/he puts aside the winnings and has the initial purse amount [F3]. Otherwise, the player has what they had on the previous spin [J14] - amount bet (on even) this spin [B15]
K15	=K14-B15+I15	Player's cumulative amount after this spin = previous cumulative [K14] - amount bet on this spin [B15] + total paid to player on this spin [I15].
N3	=K254	The last value for the player's cumulative amount.
O3	=MAX(K15:K254)	The maximum value of the player's cumulative amount.
P3	=MIN(K15:K254)	The minimum value of the player's cumulative amount.
Q3	=MAX(B15:B254)	The maximum bet made (on even) after 240 spins.
O15	=IF(I15>0,1,0)	If the total paid to the player >0, the player wins and this game ends, i.e., indicator is set to 1. Otherwise indicator is set to 0.
P15	=IF(O14,1,P14+1)	If there was a win [if O14=1] on the previous spin, then reset the number of spins in current game to 1, else the number of spins in current game is what it was [P14] + 1.
Q15	=IF(AND(B15=0,SUM(B14:B15)>0),A15-1,10000)	If the amount bet on this spin [B15] = 0 (i.e., player cannot bet on this spin) and the player betted on the previous spin [sum(B14:B15)>0] the player went bust on the previous spin [A14]. Otherwise, generate an arbitrarily large number [e.g., 10000] significantly larger than the maximum possible number of spins [240 in this example].
O12	=IF(P12=A254,SUM(O15:O254),SUM(O15:O254)+1)	If the total number of spins played = the maximum number of spins [A254], the total number of games played = the number of wins [SUM(O15:O254)]. Otherwise, the total number of games played = the number of wins [SUM(O15:O254)] + 1, since the player must have bust on the last game
P12	=MIN((SUMPRODUCT(O15:O254,P15:P254)+ INT((LOG(F3/F6,2)*(G6-1)))),A254)	Total number of spins played = minimum of {number of games played [O15:O254] times the number of spins in the corresponding game [P15:P254] + integer part of (log base 2 of the player's initial amount [F3] / fixed initial bet amount [F6] * (bet multiplying factor[G6]-1)} and {the maximum number of spins [A254]}.
Q12	=IF(O12<>0,P12/O12,0)	If the total number of games played <> 0, then the average number of spins per game = total number of spins played [P12] / total number of games played [O12].

Figure 4: New formulas for the bet doubling model presented in Figure 3. Others are as shown in Figure 2.

All of the above observations conform to the results that can be derived from probability analyses thereby providing the students with an understanding of the underlying structure when they carry out the probability analyses using mathematics. Additionally, the simulation results in Appendix C can be used as a rough guide to form decision rules for playing, bearing in mind that different individuals will have different tolerances for risk and expectations of winning or losing. For example, by looking at the values in the player's maximum winnings column which, incidentally have a low coefficient of variation, a player could decide when it would be prudent to leave the table based upon their current purse amount.

The model can also be easily used for investigating the outcomes of mixed betting strategies such as betting on a specific number in addition to odd or even or any other combinations of the various betting strategies described in section 3.0. Users of the model only need to type the values for the desired bets in the input cells of the model.

5 Validation of the model

All simulations must be validated to show that they simulate the intended scenarios correctly. The validation can be done in a variety of ways. If one has pre-specified data with known outputs then it can be fed to the simulation system to see if the intended output is indeed obtained. Another way to validate a simulation is through a theoretical analysis of the scenario and then comparing the theoretical results with the outputs obtained from the simulation. In the following section we provide a theoretical model of the Martingale strategy in roulette and compare the theoretical results with outputs from our simulation model for 5000 spins as a means of model validation. We also show why 5000 spins were needed to validate the implemented model.

5.1 Theoretical model for the expected number of games and spins

Symbols used are defined in Table 1.

Table 1: Symbols used.

I	initial purse amount
a	the fixed bet amount at the start of a game
r	the bet multiplying factor
S	number of spins in a game
G	total number of games played by the player
T	total number of spins played by a player
p	probability of winning on any one spin of the wheel
n	the maximum number of spins that can be played within a game

We observe that S and G are random variables. S can take a value of $1, 2, \ldots, n$ while G can be $1, 2, \ldots \infty$. We need to find the probability distribution of S and G and the corresponding expected values.

It is easy to see that

$$n = \left\lfloor \log_r \left(\left(\frac{I}{a} \right) (r-1) \right) \right\rfloor \tag{1}$$

Furthermore,

$$\Pr(S = k) = \begin{cases} (1-p)^{k-1}p & \text{for} \quad k = 1, 2, ..., n-1 \\ (1-p)^{n-1} & \text{for} \quad k = n \end{cases} \tag{2}$$

The expression of eq (2) follows from the fact that S is a geometric random variable for all values of $k < n$. The event that S will take a value n can happen in two ways. The player can win at the n^{th} spin or can lose for all n spins and thus lose the entire purse and cannot continue the game any more.

We know that the probability that one wins on the n^{th} spin is $(1-p)^{n-1}p$, and the probability that one loses on all n spins is $(1-p)^n$. Hence,

$$\begin{aligned} \Pr(S = n) &= (1-p)^n + (1-p)^{n-1}p \\ &= (1-p)^{n-1}(p + 1 - p) \\ &= (1-p)^{n-1} \end{aligned} \tag{3}$$

The expected value of S, is given by

$$\begin{aligned} E(S) &= \sum_{k=1}^{n-1} k \Pr(S = k) + n(1-p)^{n-1} \\ &= \sum_{k=1}^{n-1} k(1-p)^{k-1}p + n(1-p)^{n-1} \\ &= \frac{p}{(1-p)} \sum_{k=1}^{n-1} k(1-p)^k + n(1-p)^{n-1} \\ &= \frac{p}{(1-p)} \left[\frac{(1-p) - (1-p)^n}{p^2} - \frac{(n-1)(1-p)^n}{p} \right] + n(1-p)^{n-1} \\ &= \frac{1}{p} - \frac{1}{p}(1-p)^{n-1} - (n-1)(1-p)^{n-1} + n(1-p)^{n-1} \\ &= \frac{1}{p} - (1-p)^{n-1} \left(\frac{1}{p} - 1 \right) \\ &= \frac{1}{p} - \frac{(1-p)^n}{p} \\ &= \frac{1-f}{p} \end{aligned}$$

where $f = (1-p)^n$ is the probability that the player loses on all n spins in a game and thus stops by losing the entire purse.

The random variable G, the total number of games played by a player is a geometric random variable because it indicates the number of spins needed for a player to lose the entire purse for the first time (clearly the player cannot continue after that). It is to be noted that theoretically, G can be any positive integer up to infinity as it is conceivable that a player will never lose the entire purse in any one game and thus would continue forever by winning each game.

Clearly, the probability that G will take a value of j is the same as the probability that the player does not go bust in the previous $j-1$ games, and goes bust on the j^{th} game. Therefore,

$$\Pr(G=j) = (1-f)^{(j-1)} f \qquad \text{for} \qquad j = 1, 2, \ldots, \infty,$$

and

$$E(G) = \frac{1}{f}$$

The total number of spins played by a player is given by

$$T = \sum_{i=1}^{G-1} S_i + n$$

In this equation, S_i is the number of spins played in the i^{th} game, and $G = 1, 2, \ldots, \infty$.

Expected value of T or the expected number of total spins played by a player is given by

$$E(T) = \sum_{j=1}^{\infty} \left(E\left[\sum_{i=1}^{j-1} S_i + n \right] \Pr(G=j) \right)$$

At $j=1$, $E(T) = n \Pr(G=1)$
At $j=2$, $E(T) = (E(S_1)+n) \Pr(G=2)$
At $j=3$, $E(T) = (E(S_1+S_2)+n) \Pr(G=3) = [E(S_1)+E(S_2)+n] \Pr(G=3)$
thus,
At $j=m$ $E(T) = [E(S_1)+E(S_2)+\cdots+E(S_{m-1})+n] \Pr(G=m)$

Since the S_i are independent and identically distributed random variables, $E(S_i) = E(S_j)$ for all integers i and j. Let $\mu = E(S_i)$ for all i. Substituting for $E(S_i)$ in $E(T)$ we obtain an expression for the expected number of total spins.

$$\begin{aligned}
E(T) &= n \sum_{j=1}^{\infty} \Pr(G=j) + \mu \left[f(1-f) + 2f(1-f)^2 + \cdots + mf(1-f)^m + \cdots \right] \\
&= n + \mu(1-f) \left[f + 2f(1-f) + \cdots + mf(1-f)^{m-1} + \cdots \right] \\
&= n + \mu(1-f) \frac{1}{f} \\
&= n + \left(\frac{1-f}{p} \right) (1-f) \frac{1}{f}, \text{ since } \mu = \left(\frac{1-f}{p} \right) \\
&= \frac{(1-f)^2}{pf} + n
\end{aligned}$$

Total Spins (T)	Number of games (m)	Approximate value of the Reduction Term
100	49	347.8349
500	239	209.9039
1000	477	81.1669
1500	716	27.0572
2000	954	8.4108
2500	1192	2.4980
3000	1430	0.7194
3500	1669	0.2015
4000	1907	0.0558
4500	2145	0.0152
5000	2383	0.0041

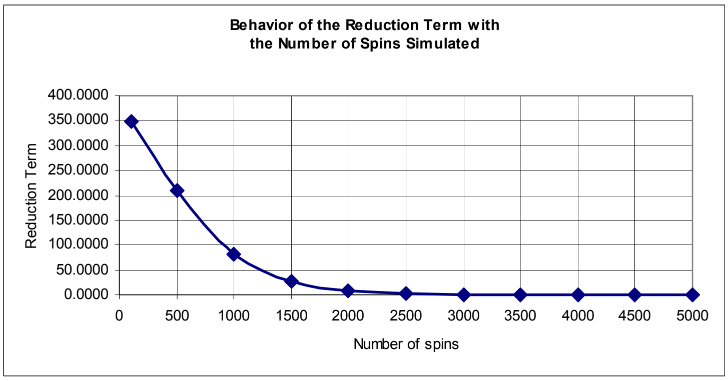

Figure 5: The behavior of the reduction term with the number of spins simulated.

The middle step follows because the expression in the square brackets is the expected value of a geometric distribution with parameter f and thus evaluates to $1/f$.

The formula also makes intuitive sense as we can see that the expected number of total spins played is the product of the expected number of games and the expected number of spins per game given that the game continues. When the game stops then the expected number of spins is n. The probability that a game continues is same as the probability that one does not go bust in the current game and that probability is $(1 - f)$. Hence, the expected number of total spins is given by eq (4).

$$\begin{aligned} E(T) &= (1-f) \times E(S) \times E(G) + n \\ &= \frac{(1-f)^2}{pf} + n \end{aligned} \tag{4}$$

5.2 Reason for choosing 5000 spins in the simulation for validating the model

The total number of spins played by a player is dependent on the number of games played and the number of games, as we have seen, theoretically can be unlimited. Clearly, we cannot create a simulation model that can be extended to an infinite number of rows. We thus have to investigate the total number of spins needed to be simulated so that the simulation is close to the theoretical model and thus can be validated. We note that the partial sum for $E(T)_m$ with m games is

$$\begin{aligned} E(T)_m &= n - n(1-f)^m + \frac{(1-f)^2}{pf} - \frac{(1-f)^{m+2}}{f} - m(1-f)^{m+1} \\ &= n + \frac{(1-f)^2}{pf} - \left[n(1-f)^m + \frac{(1-f)^{m+2}}{f} + m(1-f)^{m+1} \right] \\ &= E(T) - \left[n(1-f)^m + \frac{(1-f)^{m+2}}{f} + m(1-f)^{m+1} \right] \end{aligned}$$

Since $0 < f < 1$, when $m \to \infty$, the reduction term (the terms inside the square bracket above) reduces to zero. We created a spreadsheet to evaluate the reduction term and experimented with various values of m to see when the reduction term becomes very close to zero and thus insignificant. The result, for a value of $n = 8$ (a maximum bet of \$256.00) is shown in Figure 5. Similar results are obtained for other values of n.

	Average Number of Games				Average Number of Spins/Game				Average Number of Total Spins			
	Simulated Value (with 95% C.I.)				Simulated Value (with 95% C.I.)				Simulated Value (with 95% C.I.)			
Initial Purse Amount	Lower Limit	Mean	Upper Limit	Theoretical Value	Lower Limit	Mean	Upper Limit	Theoretical Value	Lower Limit	Mean	Upper Limit	Theoretical Value
\$ 2.00	1.85	1.93	2.01	1.90	0.98	1.00	1.02	1.00	1.85	1.93	2.00	1.90
\$ 4.00	3.49	3.64	3.79	3.61	1.50	1.52	1.54	1.53	5.28	5.55	5.78	5.98
\$ 8.00	6.45	6.85	7.25	6.86	1.77	1.80	1.83	1.80	11.68	12.35	13.02	13.57
\$ 16.00	12.07	12.86	13.65	13.03	1.91	1.94	1.97	1.95	23.53	24.96	26.39	27.45
\$ 32.00	22.77	24.24	25.71	24.76	2.00	2.03	2.06	2.03	46.30	49.09	51.88	53.14
\$ 64.00	46.79	49.69	52.59	47.05	2.02	2.05	2.08	2.07	96.06	102.05	108.04	101.14
\$ 128.00	81.10	86.19	91.28	89.39	2.06	2.08	2.10	2.09	168.60	179.37	190.14	191.51
\$ 256.00	157.92	167.23	176.54	169.84	2.09	2.10	2.11	2.10	331.68	351.82	371.96	362.33

Initial Purse Amount	n	p	f	E(G)	E(S)	E(T)
\$ 2.00	1	0.4737	0.5263	1.90	1.00	1.90
\$ 4.00	2	0.4737	0.2770	3.61	1.53	5.98
\$ 8.00	3	0.4737	0.1458	6.86	1.80	13.57
\$ 16.00	4	0.4737	0.0767	13.03	1.95	27.45
\$ 32.00	5	0.4737	0.0404	24.76	2.03	53.14
\$ 64.00	6	0.4737	0.0213	47.05	2.07	101.14
\$ 128.00	7	0.4737	0.0112	89.39	2.09	191.51
\$ 256.00	8	0.4737	0.0059	169.84	2.10	362.33
\$ 512.00	9	0.4737	0.0031	322.69	2.10	686.01
\$1,024.00	10	0.4737	0.0016	613.11	2.11	1300.12
\$2,048.00	11	0.4737	0.0009	1164.90	2.11	2466.02
\$4,096.00	12	0.4737	0.0005	2213.31	2.11	4680.33
\$8,192.00	13	0.4737	0.0002	4205.30	2.11	8886.63

Figure 6: Comparison of theoretical and simulated results.

5.3 Comparison of outputs from the theoretical and simulation models

Figure 6 shows the theoretical values for the total number of spins and the total number of games along with the corresponding values obtained from the simulation and their 95% confidence intervals. The simulation results are in excellent agreement with the theoretical results and thus show the validity of the model.

6 Classroom experience and learning objectives

Our model is quite adaptable for various levels of students. Depending upon the students' experience with spreadsheets the model can be implemented either from scratch or by having a template or pre-built model provided to the students. In a statistics course, students can be asked to calculate the long-term effects of different betting strategies using probability theory and then can verify their theoretical results against the simulation outputs. Thus they can gain a deeper understanding of the nature of the probabilities and expected values through experiential learning with the model as opposed to (or in addition to) the traditional abstract formulas. In a MS/OR course the focus would be more on model development and the simulation process to help evaluate the consequences of different strategies. We have used the roulette model described in this paper in a classroom situation in various courses and the learning objectives differ slightly based on the course and level.

In an undergraduate operations course the learning objectives are to provide an illustration of the simulation process in an interesting and perhaps non-standard application. Students are first introduced to the mechanics of simulation with some typical textbook examples prior to the basic model shown in Figure 1 being developed in class. All runs are made manually since Crystal Ball is beyond the scope of the course. Students then work through some homework problems for different initial values to determine long term or steady state values for straightforward strategies, such as betting a dollar amount on a single number each play, (pure strategy) and betting a dollar amount on even and a single number each play (mixed strategy).

At the MBA level in a decision support course the learning objectives are to provide an illustration of the simulation process through an exciting example and to show how the model can be used in a decision making context to evaluate alternative strategies. An additional objective is to improve the students' spreadsheet modeling skills, since the course heavily relies on building decision support models. Students are first introduced to the mechanics of simulation with some typical textbook examples prior to the basic model shown in Figure 1 being developed in class and Crystal Ball is used for the runs. Some time is spent on discussing the Crystal Ball output results to address the evaluation aspects of the strategies. Students are then required as practice homework examples themselves to extend the model for strategies such as betting a dollar amount on the numbers 1-6, 13-24 and the Martingale strategy. We have found that this is a useful exercise for MBA students to hone their spreadsheet and modeling skills. Furthermore, the theoretical results can be used to teach the students the process and need for validation of a simulation model. The derivation of the theoretical model is beyond the scope of the MBA class, and thus the students must be given the formulas to calculate the theoretical results.

Finally, the model can be used for advanced Statistics class where the students can be asked to investigate the effect of the Martingale strategies in a scenario where the player is allowed to use the entire purse, which includes the previous winnings, at the start of a new game. Clearly, the player will be able to play longer and will have an increased maximum amount of winning. While the theoretical probability model for this can be quite complex, students can easily build the model in spreadsheets and verify this intuitive conclusion, thereby appreciating the power of spreadsheet simulation.

7 Conclusion

This paper shows a way to implement a simulation of the game of roulette in a spreadsheet environment. The simplicity of the implementation and ready availability of the spreadsheet make this model an attractive tool for teaching probability concepts and simulation techniques in statistics and MS/OR courses. The model can be used for analysis of various betting strategies without resorting to complex mathematics. The model is robust and can be adapted easily to various levels of complexity, sophistication and focus. The spreadsheet medium makes the implementation rather straightforward for anyone with an intermediate level of knowledge of spreadsheets.

The model presented in this paper is useful because it allows the student to look at the probability structure of the Martingale strategy from multiple perspectives. In the traditional analysis of the bet doubling strategy, as presented in text books, only the expected gains are shown assuming an infinite number of plays and betting funds. The specific implementation of our model allows investigation of the more realistic scenarios where player's funds or playing time is limited. The probabilistic analyses of these scenarios are complex and may not be immediately meaningful to students if presented abstractly through mathematical formulas. Simulation shows how these results are obtained and explains and provides insights into the mathematical derivation of the probabilities. The approach also shows the power of simulation to provide insights into complex problems where the analytical derivations can be mathematically intricate.

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A Roulette table and possible bets and payoffs

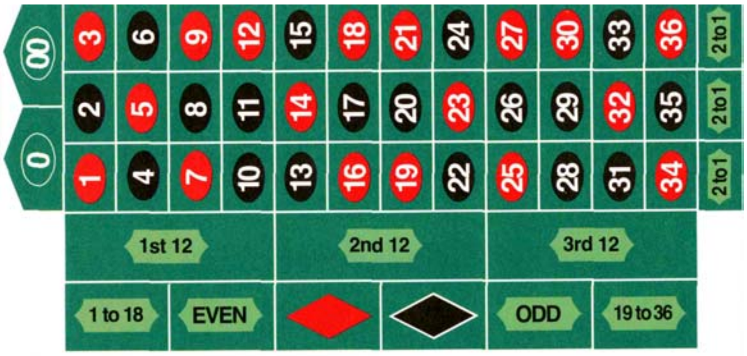


Figure 7: Roulette table layout.

Table 2: Table of payoffs. In all cases the amount of the bet is returned to the player.

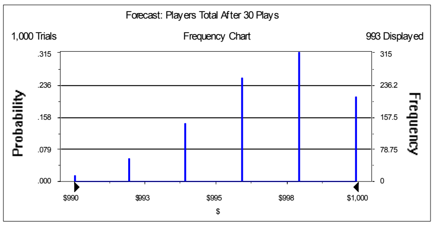
Possible Bet on	Payoff
one number	35 to 1
two numbers	17 to 1
three numbers	11 to 1
four numbers	8 to 1
five numbers	6 to 1
six numbers	5 to 1
twelve number sections	2 to 1
twelve number columns	2 to 1
1 - 18	1 to 1
19 - 36	1 to 1
Red or Black	1 to 1
Even or Odd	1 to 1

B Sample results for the basic model

Crystal Ball Report
Player's initial amount: $1,000
Strategy: Bet same amount on Even and Odd

Forecast: Players Total After 30 Plays **Cell: P43**

Statistics:	Value
Trials	1000
Mean	$997
Median	$998
Mode	$998
Standard Deviation	$3
Variance	$7
Skewness	-0.75
Kurtosis	3.42
Coeff. of Variability	0.00
Range Minimum	$986
Range Maximum	$1,000
Range Width	$14
Mean Std. Error	$0.08

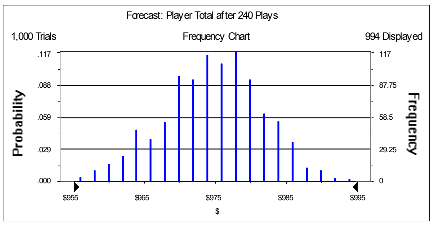


Player's total after 30 spins.

Crystal Ball Report
Player's initial amount: $1,000
Strategy: Bet same amount on Even and Odd

Forecast: Player Total after 240 Plays **Cell: P253**

Statistics:	Value
Trials	1000
Mean	$975
Median	$974
Mode	$978
Standard Deviation	$7
Variance	$51
Skewness	-0.27
Kurtosis	2.99
Coeff. of Variability	0.01
Range Minimum	$950
Range Maximum	$994
Range Width	$44
Mean Std. Error	$0.23



Player's total after 240 spins.

Note: The coefficient of variability is Crystal Ball's terminology for the coefficient of variation, a relative dispersion measure.

C Sample results for the Martingale strategy

30 spins

Initial Bet	Player's Initial Purse Amount	(i) Player's Cumulative Amount	CoV	(ii) Player's Maximum Amount	CoV	(iii) Number of Wins, i.e., Games	CoV	(iv) Number of Spins Played	CoV	(v) Average Number of Spins/Game
$1	$50	$44	0.57	$75	0.33	25.92	0.94	52.89	0.88	2.04
	$100	$86	0.56	$144	0.25	44.78	0.81	92.91	0.76	2.07
	$200	$168	0.54	$263	0.16	64.04	0.65	134.61	0.62	2.10
	$250	$222	0.41	$315	0.14	66.14	0.64	138.34	0.61	2.09
	$500	$455	0.35	$583	0.07	83.81	0.47	177.36	0.44	2.12
	$1,000	$953	0.27	$1,098	0.03	98.40	0.33	206.60	0.31	2.10
	$2,000	$1,925	0.21	$2,103	0.01	103.03	0.27	218.11	0.25	2.12

Initial Bet	Player's Initial Purse Amount	(i) Player's Cumulative Amount	CoV	(ii) Player's Maximum Amount	CoV	(iii) Number of Wins, i.e., Games	CoV	(v) Number of Spins Played	CoV	(iv) Average Number of Spins/Game
$2	$50	$47	0.50	$68	0.18	9.87	0.58	19.78	0.48	2.00
	$100	$95	0.41	$123	0.09	11.82	0.45	24.27	0.35	2.05
	$200	$191	0.32	$225	0.04	12.88	0.36	26.88	0.24	2.09
	$250	$242	0.25	$275	0.04	12.98	0.36	27.06	0.24	2.08
	$500	$491	0.18	$527	0.02	13.78	0.27	28.63	0.15	2.08
	$1,000	$985	0.14	$1,027	0.01	13.80	0.24	29.32	0.11	2.12
	$2,000	$1,984	0.10	$2,028	0.00	14.01	0.22	29.64	0.08	2.12

Initial Bet	Player's Initial Purse Amount	(i) Player's Cumulative Amount	CoV	(ii) Player's Maximum Amount	CoV	(iii) Number of Wins, i.e., Games	CoV	(v) Number of Spins Played	CoV	(iv) Average Number of Spins/Game
$5	$50	$44	0.71	$76	0.35	6.30	0.80	11.75	0.68	1.87
	$100	$91	0.66	$145	0.22	9.64	0.60	19.09	0.50	1.98
	$200	$189	0.51	$258	0.11	12.07	0.43	24.65	0.33	2.04
	$250	$240	0.41	$308	0.09	12.04	0.44	24.50	0.33	2.04
	$500	$478	0.32	$563	0.04	12.90	0.35	27.03	0.23	2.09
	$1,000	$977	0.23	$1,068	0.02	13.64	0.28	28.49	0.17	2.09
	$2,000	$1,975	0.16	$2,069	0.01	13.91	0.24	29.29	0.11	2.11

Initial Bet	Player's Initial Purse Amount	(i) Player's Cumulative Amount	CoV	(ii) Player's Maximum Amount	CoV	(iii) Number of Wins, i.e., Games	CoV	(v) Number of Spins Played	CoV	(iv) Average Number of Spins/Game
$10	$50	$47	0.67	$75	0.46	3.73	0.85	5.98	0.72	1.60
	$100	$92	0.71	$156	0.36	6.69	0.78	12.32	0.67	1.84
	$200	$181	0.66	$288	0.22	9.62	0.61	19.03	0.51	1.98
	$250	$226	0.52	$336	0.19	9.38	0.62	18.89	0.50	2.01
	$500	$480	0.41	$615	0.09	11.94	0.44	24.57	0.33	2.06
	$1,000	$971	0.31	$1,129	0.04	13.13	0.34	27.15	0.23	2.07
	$2,000	$1,942	0.24	$2,134	0.02	13.58	0.29	28.37	0.17	2.09

Figure 8: Martingale results for 30 spins.

240 spins

Initial Bet	Player's Initial Purse Amount	(i) Player's Cumulative Amount	CoV	(ii) Player's Maximum Amount	CoV	(iii) Number of Wins, i.e., Games	CoV	(iv) Number of Spins Played	CoV	(v) Average Number of Spins/Game
$1	$50	$43	0.56	$74	0.32	24.91	0.94	50.94	0.88	2.04
	$100	$82	0.56	$141	0.25	41.95	0.84	87.55	0.79	2.09
	$200	$170	0.53	$265	0.16	65.28	0.64	136.77	0.61	2.10
	$250	$223	0.41	$315	0.13	65.92	0.64	137.88	0.60	2.09
	$500	$458	0.35	$584	0.07	84.37	0.47	178.26	0.45	2.11
	$1,000	$944	0.28	$1,096	0.03	96.28	0.35	203.15	0.33	2.11
	$2,000	$1,916	0.22	$2,104	0.01	104.31	0.25	219.92	0.23	2.11

Initial Bet	Player's Initial Purse Amount	(i) Player's Cumulative Amount	CoV	(ii) Player's Maximum Amount	CoV	(iii) Number of Wins, i.e., Games	CoV	(v) Number of Spins Played	CoV	(iv) Average Number of Spins/Game
$2	$50	$44	0.57	$74	0.34	12.94	0.96	25.85	0.87	2.00
	$100	$85	0.54	$146	0.31	24.15	0.92	49.58	0.85	2.05
	$200	$166	0.58	$282	0.25	42.13	0.85	87.75	0.80	2.08
	$250	$219	0.44	$335	0.22	43.51	0.83	91.06	0.78	2.09
	$500	$442	0.42	$629	0.14	65.20	0.66	136.39	0.63	2.09
	$1,000	$921	0.35	$1,169	0.07	85.19	0.47	178.84	0.44	2.10
	$2,000	$1,845	0.29	$2,189	0.03	94.97	0.37	199.99	0.34	2.11

Initial Bet	Player's Initial Purse Amount	(i) Player's Cumulative Amount	CoV	(ii) Player's Maximum Amount	CoV	(iii) Number of Wins, i.e., Games	CoV	(v) Number of Spins Played	CoV	(iv) Average Number of Spins/Game
$5	$50	$45	0.76	$80	0.44	7.06	0.98	13.02	0.87	1.85
	$100	$85	0.72	$160	0.39	13.02	0.94	25.74	0.85	1.98
	$200	$161	0.73	$315	0.36	23.98	0.93	49.09	0.88	2.05
	$250	$216	0.57	$371	0.33	25.15	0.96	51.33	0.90	2.04
	$500	$431	0.58	$718	0.26	44.58	0.82	93.05	0.78	2.09
	$1,000	$857	0.54	$1,322	0.16	65.10	0.65	136.73	0.62	2.10
	$2,000	$1,809	0.44	$2,428	0.08	86.11	0.45	180.59	0.43	2.10

Initial Bet	Player's Initial Purse Amount	(i) Player's Cumulative Amount	CoV	(ii) Player's Maximum Amount	CoV	(iii) Number of Wins, i.e., Games	CoV	(v) Number of Spins Played	CoV	(iv) Average Number of Spins/Game
$10	$50	$46	0.66	$73	0.45	3.56	0.85	5.82	0.73	1.63
	$100	$88	0.73	$157	0.42	6.82	0.94	12.65	0.83	1.85
	$200	$169	0.76	$318	0.40	12.84	0.98	25.38	0.90	1.98
	$250	$214	0.55	$363	0.32	12.38	0.94	24.80	0.85	2.00
	$500	$436	0.59	$744	0.33	25.40	0.96	51.83	0.90	2.04
	$1,000	$851	0.57	$1,430	0.25	43.94	0.82	91.49	0.77	2.08
	$2,000	$1,756	0.53	$2,661	0.16	66.83	0.63	139.83	0.60	2.09

Figure 9: Martingale results for 240 spins.