

CHAPTER 9

FREQUENCY-DOMAIN ANALYSIS OF DYNAMIC SYSTEMS

A. Bazoune

9.1 INTRODUCTION

The term *Frequency Response* refers to the *steady state response* of a system to a *sinusoidal input*.

An input $f(t)$ is periodic with a period τ if $f(t+\tau) = f(t)$ for all values of time t , where τ is a constant called the *period*. Periodic inputs are commonly found in many applications. The most common perhaps is ac voltage, which is sinusoidal. For the common ac frequency of 60 Hz, the period is $\tau = \frac{1}{60}$ s. Rotating unbalanced machinery produces periodic forces on the supporting structures, internal combustion engines produce a periodic torque, and reciprocating pumps produce hydraulic and pneumatic pressures that are periodic.

Frequency response analysis focuses on sinusoidal inputs. A sine function has the form $A \sin \omega t$, where A is the amplitude and ω is its frequency in radians/seconds. Notice that a cosine is simply a sine shifted by 90° or $\pi/2$ rad, as $\cos \omega t = \sin\left(\omega t + \frac{\pi}{2}\right)$

9.2 SINUSOIDAL TRANSFER FUNCTION (STF)

When a sinusoidal input is applied to a LTI system, the system will tend to *vibrate at its own natural frequency, as well as follow the frequency of the input*. In the presence of damping, that portion of motion sustained by the sinusoidal input will gradually die out. As a result, *the response at steady-state is sinusoidal at the same frequency as the input. The steady-state output differs from the input only in the amplitude and the phase angle*. See Figure 9-1 below.

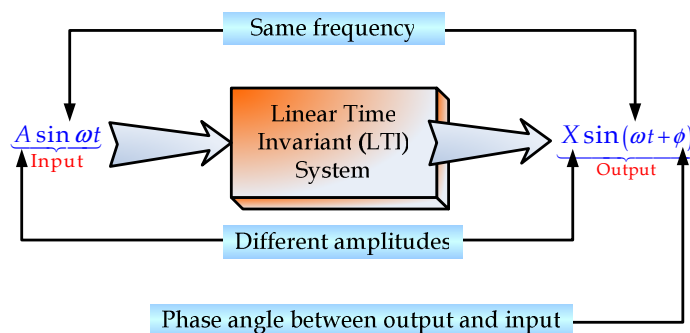


Figure 9-1.

Thus, the output-input amplitude ratio and the phase angle between the output and input sinusoids are the only parameters needed to predict the steady state output of LTI systems when the input is a sinusoid.

$\frac{X}{A}$: output-input amplitude ratio
 ϕ : phase angle between output and input

Forced Vibration without damping

mass system in which the mass is subjected to a sinusoidal input force $p(t) = P_o \sin \omega t$. Let us find the response of the system when it is initially at rest. The equation of motion is

$$m \ddot{x} + k x = P_o \sin \omega t$$

where x is the output, P is the amplitude of the excitation and ω is the forcing (excitation) frequency.

The above equation can be written in the form

$$\ddot{x} + \frac{k}{m} x = \frac{P_o}{m} \sin \omega t \quad (9-1)$$

where $\sqrt{k/m} = \omega_n$ is known as the natural frequency of the system. The solution of Equation (1) consists of the vibration at its natural frequency (**the complementary solution**) and that at the forcing frequency (**the particular solution**) as shown in Figure 9-3. Thus,

$$x(t) = \text{complementary solution} + \text{particular solution}$$

Let us obtain the solution under the condition that the system is at rest. Take LT of both sides of Equation (9-1) for zero initial conditions, i.e., $x(0) = \dot{x}(0) = 0$.

$$\left(s^2 + \frac{k}{m} \right) X(s) = \frac{P_o}{m} \frac{\omega}{s^2 + \omega^2}$$

where $X(s) = \mathcal{L}[x(t)]$. Substituting $k/m = \omega_n^2$ and Solving for $X(s)$ yields

$$X(s) = \frac{P_o}{m} \frac{\omega}{s^2 + \omega^2} \frac{1}{s^2 + \omega_n^2}$$

The above equation can be written in partial fraction as:

$$X(s) = \frac{P_o}{m} \frac{\omega}{s^2 + \omega^2} \frac{1}{s^2 + \omega_n^2} = \frac{A_1 s + B_1}{s^2 + \omega^2} + \frac{A_2 s + B_2}{s^2 + \omega_n^2}$$

Figure 9-2 illustrates a spring-

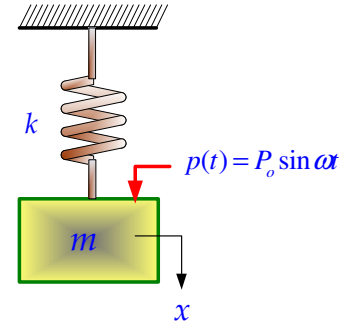


Figure 9-2 Spring-mass system.

where A_1 , A_2 , B_1 and B_2 are left as an exercise for the student.

The expression for $X(s)$ is therefore

$$X(s) = -\frac{P_0 \omega}{k - m\omega^2} \left(\frac{1}{\omega_n} \right) \frac{\omega_n}{s^2 + \omega_n^2} + \frac{P_0}{k - m\omega^2} \frac{\omega}{s^2 + \omega^2}$$

The inverse Laplace Transform of the above equation is given by

$$\begin{aligned} [x(t)] &= \mathcal{L}^{-1}[X(s)] = -\underbrace{\frac{P_0(\omega/\omega_n)}{k - m\omega^2}}_A \sin \omega_n t + \underbrace{\frac{P_0}{k - m\omega^2}}_B \sin \omega t \\ &= \underbrace{A \sin \omega_n t}_{\text{Complementary Solution}} + \underbrace{B \sin \omega t}_{\text{Particular Solution}} \end{aligned} \quad (9-2)$$

where

$$A = -\frac{P_0(\omega/\omega_n)}{k - m\omega^2} = \frac{P_0(\omega/\omega_n)}{\text{Den}} \quad \text{and} \quad B = \frac{P_0}{k - m\omega^2} = \frac{P_0}{\text{Den}}$$

where $\text{Den} = k - m\omega^2$.

- As $\omega \rightarrow 0$, $\lim_{\omega \rightarrow 0} A = 0$ and $\lim_{\omega \rightarrow 0} B = \left(\frac{P_0}{k} \right) \equiv$ static deflection
- As ω increases from zero the denominator $\text{Den} = k - m\omega^2$ becomes small and the amplitude increases, therefore, both A and B increase.
- The expression of the denominator Den can be written as

$$\text{Den} = k - m\omega^2 = m \left[\frac{k}{m} - \omega^2 \right] = m \left[\omega_n^2 - \omega^2 \right] = m \omega_n^2 \left[1 - \frac{\omega^2}{\omega_n^2} \right]$$

- It is clear that when $\omega = \omega_n$ the denominator becomes zero and the amplitude of vibrations increases without bound, therefore **resonance** occurs.

Sinusoidal Transfer Function (STF) *The sinusoidal Transfer Function*

(STF) is defined as the transfer function $G(s)$ in which the variable (s) is replaced by $(j\omega)$.

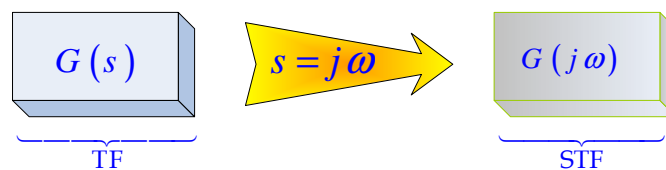


Figure 9-3

Sinusoidal Transfer Function STF

When only the steady-state solution (the particular solution) is wanted, the STF $G(j\omega)$ can simplify the solution. In our discussion, we are concerned with the behavior of stable, LTI system under steady state conditions, i.e., that is after the initial transients died out. We shall see that **sinusoidal inputs produce sinusoidal outputs** in the steady state with the amplitude and phase angle at each frequency ω determined by the magnitude and the angle of $G(j\omega)$, respectively.

Deriving Steady State Output caused by Sinusoidal Input

Figure 9-4 shows an LTI system for which the input $P(s)$ and the output is $X(s)$.

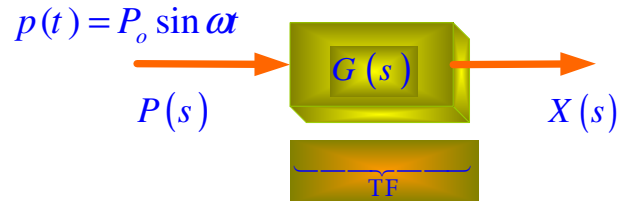


Figure 9-4 Linear Time Invariant (LTI) System

The input $p(t)$ is sinusoidal and is given by

$$p(t) = P_o \sin \omega t$$

We shall show that the output $x(t)$ at steady state is given by

$$x(t) = |G(j\omega)| P_o \sin(\omega t + \phi)$$

where $|G(j\omega)|$ and ϕ are the magnitude and phase angle of $G(j\omega)$, respectively.

Suppose that the transfer function $G(s)$ can be written as a ratio of two polynomials in s ; that is

$$G(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_n)}{(s+s_1)(s+s_2)\cdots(s+s_n)}$$

The Laplace transform $X(s)$ is

$$X(s) = G(s)P(s) \quad (9-3)$$

where $P(s) = \mathcal{L}[p(t)]$.

Let us limit our discussion to stable systems. For such systems, the real parts of the $-s_i$ are negative. The steady state response of a stable linear system to a sinusoidal input does not depend on I. C's, so they can be ignored.

- If $G(s)$ has only distinct poles, then the partial fraction of Equation (9-3) yields

$$\begin{aligned} X(s) &= G(s) \frac{P\omega}{s^2 + \omega^2} \\ &= \frac{a}{s + j\omega} + \frac{\bar{a}}{s - j\omega} + \frac{b_1}{s + s_1} + \frac{b_2}{s + s_2} + \dots + \frac{b_n}{s + s_n} \end{aligned} \quad (9-4)$$

where a and b_i ($i = 1, 2, \dots, n$) are constants and \bar{a} is the complex conjugate of a . The response $x(t)$ can be obtained by taking the inverse Laplace transform of Equation (9-4)

$$[x(t)] = \mathcal{L}^{-1}[X(s)] = ae^{-j\omega t} + \bar{a}e^{j\omega t} + \underbrace{b_1e^{-s_1t} + b_2e^{-s_2t} + \dots + b_n e^{-s_nt}}_{\text{For a stable system these terms } \rightarrow 0 \text{ as } t \rightarrow \infty \text{ since they have negative real part}}$$

- If $G(s)$ involves k multiple poles s_j , then $x(t)$ will involve such terms as $t^h e^{-s_j t}$ (where $h = 0, 1, 2, \dots, k - 1$). Since the real part of the $-s_j$ is negative for a stable system, the terms $t^h e^{-s_j t} \rightarrow 0$ when $t \rightarrow \infty$.

Regardless of whether the system involves multiple poles, the steady state response becomes

$$x(t) = ae^{-j\omega t} + \bar{a}e^{j\omega t} \quad (9-5)$$

where the constants a and \bar{a} can be evaluated from Equation (9-4):

$$\begin{aligned} a &= G(s) \frac{P\omega}{s^2 + \omega^2} (s + j\omega) \Big|_{s=-j\omega} = -\frac{P}{2j} G(-j\omega) \\ \bar{a} &= G(s) \frac{P\omega}{s^2 + \omega^2} (s - j\omega) \Big|_{s=j\omega} = \frac{P}{2j} G(j\omega) \end{aligned}$$

Notice that \bar{a} is the complex conjugate of a . Referring to Figure 9-5, we can write

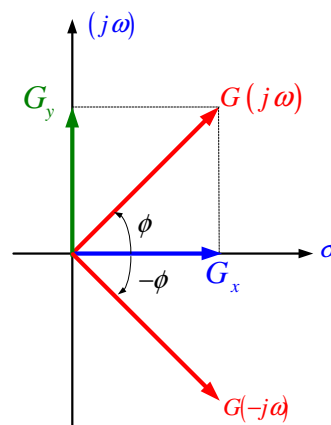


Figure 9-5 Complex function and its complex conjugate.

$$\begin{aligned}
 G(j\omega) &= G_x + G_y \\
 &= |G(j\omega)| \cos \phi + j |G(j\omega)| \sin \phi \\
 &= |G(j\omega)| (\cos \phi + j \sin \phi) \\
 &= |G(j\omega)| e^{j\phi}
 \end{aligned}$$

Notice that $\angle G(j\omega) = \angle e^{j\phi} = \phi$. Similarly,

$$G(-j\omega) = |G(-j\omega)| e^{j\phi} = |G(j\omega)| e^{j\phi}$$

Substitute the expressions of $G(j\omega)$ and $G(-j\omega)$ into the expressions of a and \bar{a} , one can get

$$\begin{aligned}
 a &= -\frac{P}{2j} |G(j\omega)| e^{-j\phi} \\
 \bar{a} &= \frac{P}{2j} |G(-j\omega)| e^{j\phi}
 \end{aligned}$$

Then Equation (9-5) can be written as

$$\begin{aligned}
 x(t) &= |G(j\omega)| P_o \underbrace{\frac{e^{j(\alpha+\phi)} - e^{-j(\alpha+\phi)}}{2j}}_{\sin(\alpha+\phi)} \\
 &= |G(j\omega)| P_o \sin(\alpha + \phi) \\
 &= X \sin(\alpha + \phi)
 \end{aligned} \tag{9-6}$$

where $X = |G(j\omega)| P_o$ and $\phi = \angle G(j\omega)$

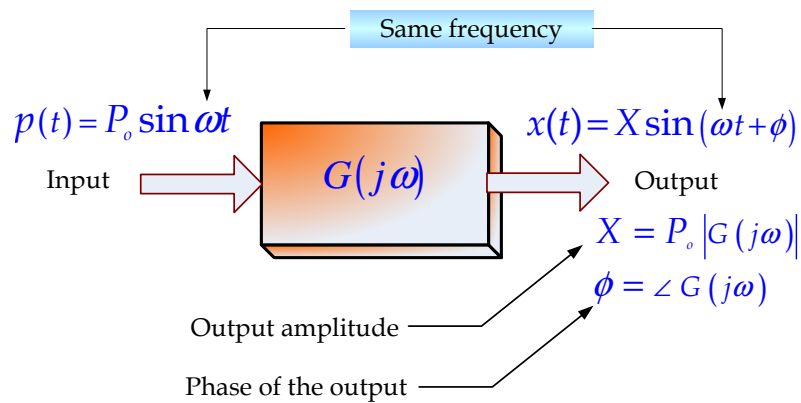


Figure 9-6 Input output relationships for sinusoidal inputs.

Therefore for sinusoidal inputs,

$$|G(j\omega)| = \left| \frac{X(j\omega)}{P(j\omega)} \right| = \frac{\text{amplitude ratio of the output}}{\text{sinusoid to the input sinusoid}} \quad (9-7)$$

$$\begin{aligned} \angle G(j\omega) &= \angle \frac{X(j\omega)}{P(j\omega)} = \tan^{-1} \left[\frac{\text{imaginary part of } G(j\omega)}{\text{real part of } G(j\omega)} \right] \\ &= \frac{\text{phase shift of the output sinusoid}}{\text{with respect to the input sinusoid}} \end{aligned} \quad (9-8)$$

■ Example 9-1 (Textbook Page 437)

Consider the TF

$$\frac{X(s)}{P(s)} = G(s) = \frac{1}{Ts+1}$$

For the sinusoidal input $p(t) = P_o \sin \omega t$, what is the steady-state output $x(t)$.

■ Solution

Substituting $j\omega$ for s in $G(s)$ yields

$$G(j\omega) = \frac{1}{Tj\omega+1}$$

The output-input amplitude ratio is

$$|G(j\omega)| = \frac{1}{\sqrt{T^2\omega^2+1}}$$

and the phase angle ϕ is

$$\phi = \angle G(j\omega) = -\tan^{-1} T\omega$$

So, for the input $p(t) = P_o \sin \omega t$, the steady-state output $x(t)$ can be found as

$$x(t) = \frac{P_o}{\sqrt{T^2\omega^2+1}} \sin(\omega t - \tan^{-1} T\omega) \quad (9-9)$$

■ Example 9-2

Find the steady state response of the following system:

$$\dot{y} + 5y = 4\dot{p} + 12p$$

if the input is $p(t) = 20\sin 4t$

Solution

First obtain the TF

$$G(s) = \frac{Y(s)}{P(s)} = \frac{4s + 12}{s + 5} = 4 \frac{s + 3}{s + 5}$$

From the input $p(t) = 20\sin 4t$, it is clear that $\omega = 4$ rad/s. Therefore, the sinusoidal Transfer function is

$$G(j\omega) = \frac{Y(j\omega)}{G(j\omega)} = 4 \frac{j\omega + 3}{j\omega + 5} = 4 \frac{j4 + 3}{j4 + 5} = 4 \frac{3 + j4}{5 + j4}$$

Then

$$|G(j\omega)| = 4 \frac{|3 + j4|}{|5 + j4|} = 4 \frac{\sqrt{3^2 + (4)^2}}{\sqrt{5^2 + (4)^2}} = 4 \frac{\sqrt{25}}{\sqrt{41}} = 3.123$$

and

$$\begin{aligned} \phi &= \angle G(j\omega) = \angle \left(4 \frac{3 + j\omega}{5 + j\omega} \right) = \angle 4 + \angle(3 + j\omega) - \angle(5 + j\omega) \\ &= 0^\circ + \tan^{-1} \frac{\omega}{3} - \tan^{-1} \frac{\omega}{5} = 0^\circ + \tan^{-1} \frac{4}{3} - \tan^{-1} \frac{4}{5} = 0.253 \text{ rad} \end{aligned}$$

The steady state response is

$$\begin{aligned} y(t) &= |G(j\omega)| P_o \sin(\omega t + \phi) \\ &= 3.123 \times 20 \sin(4t + 0.253) = 62.46 \sin(4t + 0.253) \end{aligned}$$

Example 9-3 (Example 9-2 in the Textbook Page 437-438)

Suppose that a sinusoidal force $p(t) = P_o \sin \omega t$ is applied to the mechanical system shown in Figure 9-7. Assuming that the displacement x is measured from the equilibrium position, find the steady-state output.

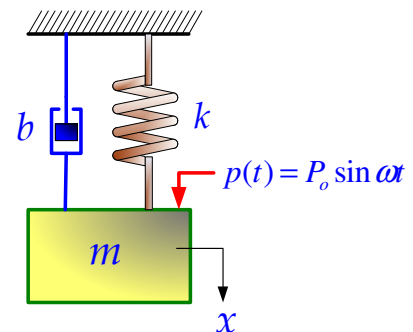


Figure 9-7 Mechanical system

Solution

The equation of motion for the system is

$$m \ddot{x} + b \dot{x} + k x = p(t)$$

The Laplace Transform of this equation, assuming zero I.C's, is

$$(m s^2 + b s + k) X(s) = P(s)$$

where $X(s) = \mathcal{L}[x(t)]$ and $P(s) = \mathcal{L}[p(t)]$. (Notice that the I.C's do not affect the steady state output and so can be taken to be zero). The TF is

$$\frac{X(s)}{P(s)} = G(s) = \frac{1}{(m s^2 + b s + k)}$$

Since the input is a sinusoidal function $p(t) = P_o \sin \omega t$, we can use the STF to obtain the steady-state solution. The STF is

$$\frac{X(j\omega)}{P(j\omega)} = G(j\omega) = \frac{1}{-m \omega^2 + b j \omega + k} = \frac{1}{(k - m \omega^2) + j b \omega}$$

From Equation (9-6), the steady-state output $x(t)$ can be written as

$$x(t) = |G(j\omega)| P_o \sin(\omega t + \phi)$$

where

$$|G(j\omega)| = \frac{1}{\sqrt{(k - m \omega^2)^2 + b^2 \omega^2}}$$

and

$$\phi = \angle G(j\omega) = \angle \frac{1}{(k - m \omega^2) + j b \omega} = -\tan^{-1} \frac{b \omega}{k - m \omega^2}$$

therefore

$$x(t) = \frac{P_o}{\sqrt{(k - m \omega^2)^2 + b^2 \omega^2}} \sin\left(\omega t - \tan^{-1} \frac{b \omega}{k - m \omega^2}\right)$$

Since $\omega_n^2 = k/m$ and $b/k = 2\zeta/\omega_n$, the equation for $x(t)$ can be written as

$$x(t) = \frac{P_o/k}{\sqrt{\left(\frac{k}{k} - \frac{m}{k} \omega^2\right)^2 + \frac{b^2}{k^2} \omega^2}} \sin\left(\omega t - \tan^{-1} \frac{\frac{b}{k} \omega}{\frac{k}{k} - \frac{m}{k} \omega^2}\right)$$

or

$$x(t) = \frac{P_o/k}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + (2\zeta(\omega/\omega_n))^2}} \sin\left(\omega t - \tan^{-1} \frac{2\zeta(\omega/\omega_n)}{1 - (\omega/\omega_n)^2}\right)$$

Let $\beta = \omega/\omega_n \equiv$ frequency ratio, the above equation can be written as

$$x(t) = \frac{x_{st}}{\sqrt{\left[1 - \beta^2\right]^2 + (2\zeta\beta)^2}} \sin\left(\omega t - \tan^{-1} \frac{2\zeta\beta}{1 - \beta^2}\right) \quad (9-10)$$

where $x_{st} = P_o/k$ is the static deflection. Writing the amplitude of $x(t)$ as X , we find that the amplitude ratio X/x_{st} and the phase shift ϕ are

$$\frac{X}{x_{st}} = \frac{1}{\sqrt{\left[1 - \beta^2\right]^2 + (2\zeta\beta)^2}}$$

and

$$\phi = -\tan^{-1} \frac{2\zeta\beta}{1 - \beta^2}$$

The variations of the amplitude ratio X/x_{st} and the phase shift ϕ are shown in figures 9-8 and 9-9 as a function of β for different values of ζ .

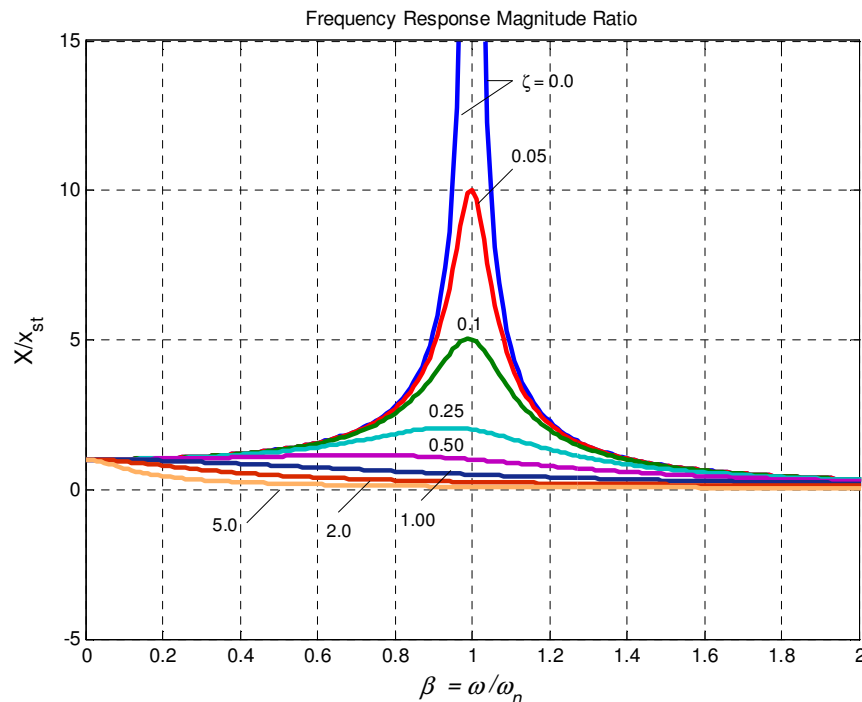


Figure 9-8 Variation of the amplitude ratio X/x_{st} with the frequency ratio β .

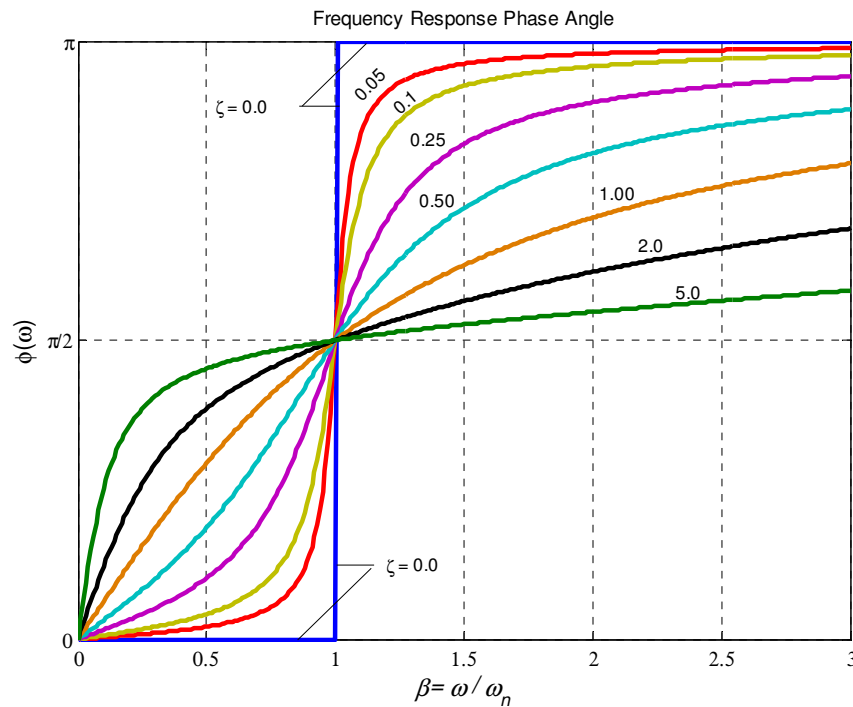


Figure 9-9 Variation of the phase ϕ with the frequency ratio β .

9.3 VIBRATIONS IN ROTATING MECHANICAL SYSTEMS

Vibration due to Rotating Unbalance

Force inputs that excite vibratory motion often arise from rotating unbalance, a condition that arises when the mass center of a rotating rigid body and the center of rotation do not coincide.

Figure 9-10 shows an unbalanced machine resting on shock mounts.

Assume that the rotor is rotating at a constant speed ω rad/s and that the unbalanced mass m is located at a distance r from the center of rotation. Then the unbalanced mass will produce a centrifugal force $m\omega^2 r$.

The equation of motion for the system is

$$M \ddot{x} + b\dot{x} + kx = p(t) \tag{9-11}$$

where

$$p(t) = m\omega^2 r \sin \omega t$$

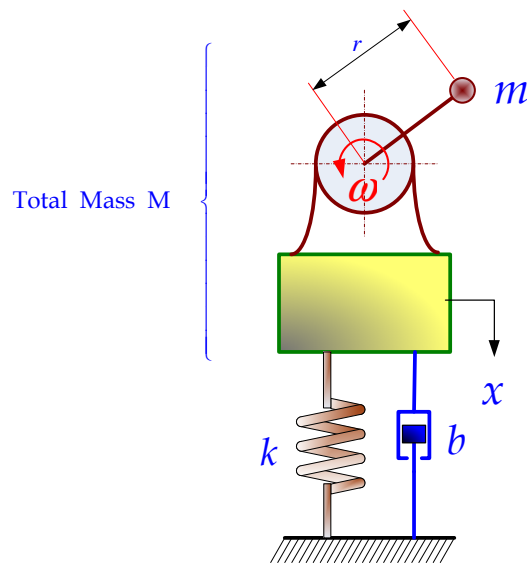


Figure 9-10 Unbalance machine resting on shock mounts

Is the force applied to the system. Take LT of both sides of Equation. (9-11), assuming zero I.C's, we have

$$(M s^2 + bs + k) X(s) = P(s)$$

or

$$\frac{X(s)}{P(s)} = G(s) = \frac{1}{(M s^2 + bs + k)}$$

The STF is

$$\frac{X(j\omega)}{P(j\omega)} = G(j\omega) = \frac{1}{(k - M \omega^2) + jb\omega}$$

For the sinusoidal forcing function $p(t)$, the steady-state output is obtained from Equation. (9-6) as

$$\begin{aligned} x(t) &= X \sin(\omega t + \phi) \\ &= |G(j\omega)| m \omega^2 r \sin\left(\omega t - \tan^{-1} \frac{b\omega}{k - M \omega^2}\right) \\ &= \frac{m \omega^2 r}{\sqrt{(k - M \omega^2)^2 + b^2 \omega^2}} \sin\left(\omega t - \tan^{-1} \frac{b\omega}{k - M \omega^2}\right) \end{aligned}$$

Divide the numerator and denominator of the amplitude and those of the phase angle by k and substitute $\omega_n^2 = k / M$ and $b / M = 2\zeta\omega_n$ into the result, the steady-state output becomes

$$x(t) = \frac{m \omega^2 r / k}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + (2\zeta(\omega/\omega_n))^2}} \sin\left(\omega t - \tan^{-1} \frac{2\zeta(\omega/\omega_n)}{1 - (\omega/\omega_n)^2}\right)$$

or

$$x(t) = \frac{m \omega^2 r / k}{\sqrt{[1 - \beta^2]^2 + (2\zeta\beta)^2}} \sin\left(\omega t - \tan^{-1} \frac{2\zeta\beta}{1 - \beta^2}\right)$$

where $\beta = \omega/\omega_n$.

9.4 VIBRATION ISOLATION

Vibration isolation is a process by which vibratory effects are minimized or eliminated. The function of a vibration isolator is:

- to reduce the magnitude of force transmitted from a machine to its foundation.
- or to reduce the magnitude of motion transmitted from a vibratory foundation to a machine.

Figure 9-11(a) illustrates the case in which the source of vibration is a vibrating force originating within the machine (force excitation). The isolator reduces the force transmitted to the foundation. In Figure 9-11(b) the source of vibration is a vibrating motion of the foundation (motion excitation). The isolator reduces the vibration amplitude of the machine.

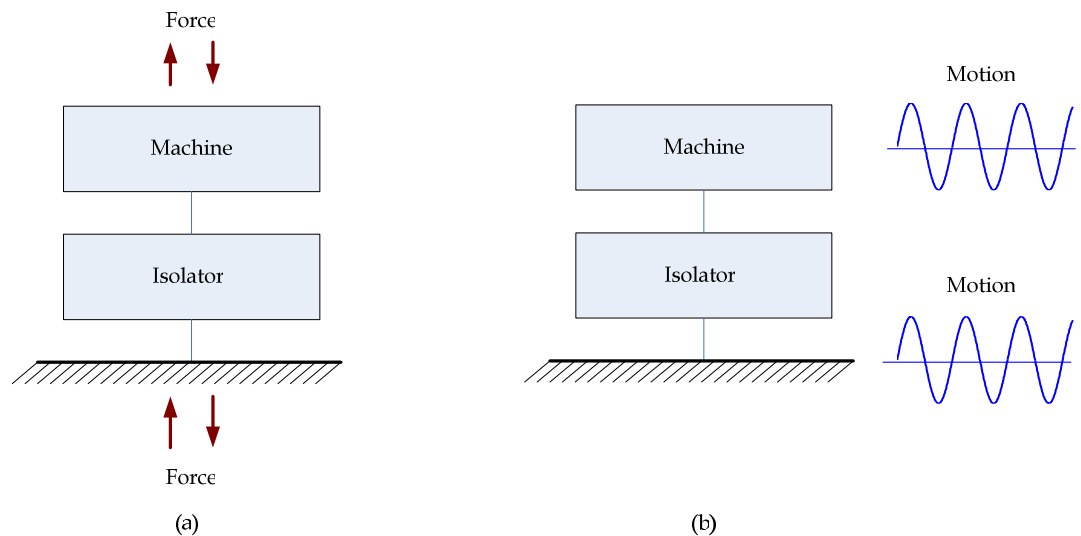


Figure 9-11 Vibration isolation. (a) Force excitation; (b) Motion excitation.

Isolation Systems

- **Passive:** It consists of a resilient member (stiffness) and energy dissipater (damping) that have constant properties. Examples are metal springs, cork, felt, pneumatic springs, elastomer (rubber) springs.
- **Active:** External power is required for the isolator to perform its function. An active isolator is comprised of a servomechanism with a sensor, signal processor and an actuator.

A typical vibration isolator is shown in Figure 9-12. (In a simple vibration isolator, a single element like synthetic rubber can perform the functions of both the load-supporting means and the energy-dissipating means).

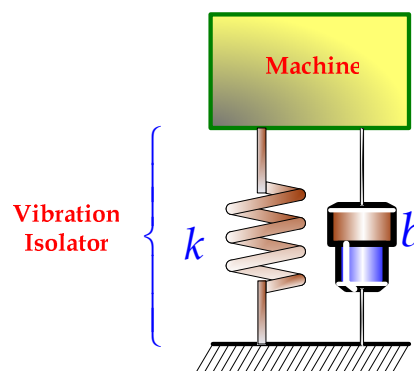


Figure 9-12 Vibration isolator.

Practical Examples.

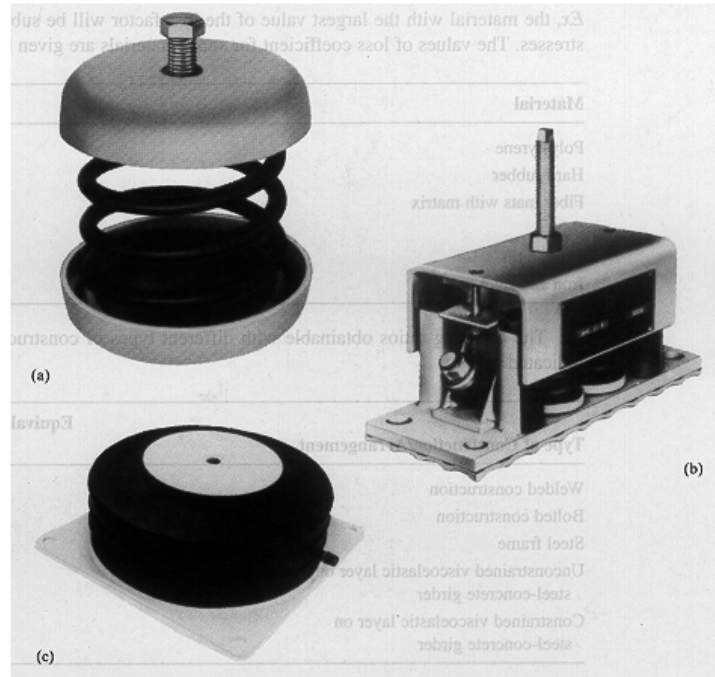


Figure 9-13 (a)- Undamped spring mount;(b)- Damped spring mount;(c)- Pneumatic rubber Mount.

Transmissibility. Transmissibility is a measure of the reduction of a transmitted force or motion afforded by an isolator

Transmissibility for Force excitation. For the system shown in Figure 9-8, the source of vibration is a vibrating force resulting from the unbalance of the machine. The transmissibility in this case is the force amplitude ratio and is given by

$$\text{Transmissibility} = TR = \frac{F_t}{F_0} = \frac{\text{Amplitude of the transmitted force}}{\text{Amplitude of the excitatory force}}$$

Let us find the transmissibility of this system in terms of the damping ratio ζ and the frequency ratio $\beta = \omega/\omega_n$.

The excitation force (in the vertical direction) is caused by the unbalanced mass of the machine and is

$$p(t) = m \omega^2 r \sin \omega t = F_0 \sin \omega t$$

The equation of motion for the system is equation (9-11), rewritten here for convenience:

$$M \ddot{x} + b\dot{x} + k x = p(t) \quad (9-12)$$

where M is the total mass of the machine including the unbalance mass m . The force $f(t)$ transmitted to the foundation is the sum of the damper and spring forces, or

$$f(t) = b\dot{x} + kx = F_i \sin(\omega t + \phi) \quad (9-13)$$

Taking the LT of Equations. (9-12) and (9-13), assuming zero I. C's, gives

$$(M s^2 + bs + k) X(s) = P(s)$$

$$(bs + k) X(s) = F(s)$$

where $X(s) = \mathcal{L}[x(t)]$, $P(s) = \mathcal{L}[p(t)]$ and $F(s) = \mathcal{L}[f(t)]$. Hence,

$$\frac{X(s)}{P(s)} = \frac{1}{M s^2 + bs + k}$$

$$\frac{F(s)}{X(s)} = \frac{1}{bs + k}$$

Eliminating $X(s)$ from the above two equations yields

$$\frac{F(s)}{P(s)} = \frac{F(s)}{X(s)} \frac{X(s)}{P(s)} = \frac{bs + k}{M s^2 + bs + k}$$

The STF is thus

$$\frac{F(j\omega)}{P(j\omega)} = \frac{bj\omega + k}{-M\omega^2 + bj\omega + k} = \frac{(b/M)j\omega + (k/M)}{-\omega^2 + (b/M)j\omega + (k/M)}$$

Substituting $k/M = \omega_n^2$ and $b/M = 2\zeta\omega_n$ into the last equation, and simplifying, we obtain

$$\frac{F(j\omega)}{P(j\omega)} = \frac{1 + j(2\zeta\omega/\omega_n)}{1 - (\omega/\omega_n)^2 + j(2\zeta\omega/\omega_n)}$$

From which it follows that

$$\left| \frac{F(j\omega)}{P(j\omega)} \right| = \frac{\sqrt{1 + (2\zeta\omega/\omega_n)^2}}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + (2\zeta\omega/\omega_n)^2}} = \frac{\sqrt{1 + (2\zeta\beta)^2}}{\sqrt{(1 - \beta^2)^2 + (2\zeta\beta)^2}}$$

where $\beta = \omega/\omega_n$. Noting that the amplitude of the excitatory force is $F_0 = |P(j\omega)|$ and that the amplitude of the transmitted force is $F_t = |F(j\omega)|$, we obtain the transmissibility:

$$\text{TR} = \frac{F_t}{F_0} = \left| \frac{F(j\omega)}{P(j\omega)} \right| = \frac{\sqrt{1 + (2\zeta\beta)^2}}{\sqrt{(1 - \beta^2)^2 + (2\zeta\beta)^2}} \quad (9-14)$$

which depends on β and ζ only. Figure 9-14 shows plots of TR for different values of ζ as a function of β . It immediately follows from Figure 9-14 that the condition $\beta > \sqrt{2}$ must be met in order that $TR < 1$, which means the transmitted force amplitude is less than the excitation force amplitude.

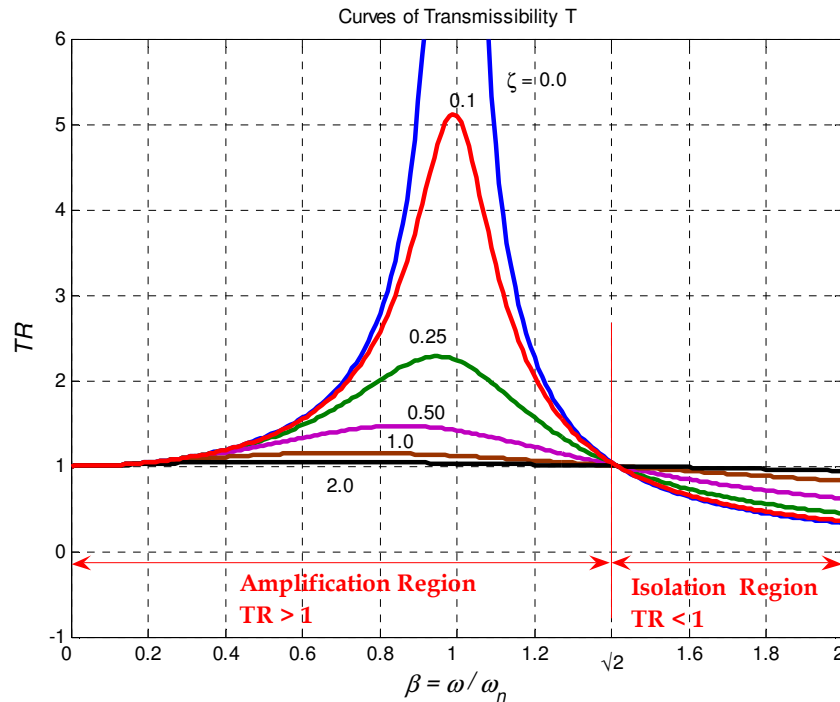


Figure 9-14 Curves of transmissibility TR versus $\beta = \omega/\omega_n$

Therefore, in order to achieve transmitted force reduction, often called *suppression*, it is important to design the spring constant k such that ω_n satisfies the condition that

$$\beta = \omega/\omega_n > \sqrt{2} \quad \text{or} \quad \omega_n = \sqrt{\frac{k}{m}} < \frac{\omega}{\sqrt{2}}$$

for a given mass M and a specified forcing frequency ω . When $\beta = \sqrt{2}$, however the Transmissibility is equal to **unity** regardless of the value of ζ .

Figure 9-14 shows some curves of the transmissibility versus $\beta = \omega/\omega_n$. It is clear that:

- all the curves pass through a critical point where $TR = 1$ and $\beta = \sqrt{2}$.
- For $\beta < \sqrt{2}$, as the damping ratio ζ increases, the transmissibility at resonance decreases.
- For $\beta > \sqrt{2}$, as ζ increases, the transmissibility increases.
- For $\beta < \sqrt{2}$, or $\omega < \sqrt{2}\omega_n$, increasing damping improves the vibration isolation.
- For $\beta > \sqrt{2}$, or $\omega > \sqrt{2}\omega_n$, increasing damping adversely affects the vibration isolation

Notice that since $|P(j\omega)| = F_0 = m\omega^2 r$, the amplitude of the force transmitted to the foundation is

$$F_t = |F(j\omega)| = \frac{m\omega^2 r \sqrt{1 + (2\zeta\beta)^2}}{\sqrt{(1 - \beta^2)^2 + (2\zeta\beta)^2}} \quad (9-15)$$

Example 9-4

Suppose a machine is mounted on an elastic bearing, which in turns sits on a rigid foundation. The bearing damping is negligible. In operation the machine generates a harmonic force having a frequency of 1000 rpm. If the mass of the machine is 50 kg, find the condition on the equivalent spring constant k of the elastic bearing for suppression of the transmitted force. Also find the percentage of the dynamic force, generated by the machine, that is transmitted into the foundation if the stiffness of the bearing is $k = 200 \text{ kN/m}$.

Solution

The condition for suppression of the transmitted force is given by $\beta = \omega/\omega_n > \sqrt{2}$ or

$$\omega_n = \sqrt{\frac{k}{m}} < \frac{\omega}{\sqrt{2}}$$

The forcing frequency is $\omega = 1000 \text{ rpm} = (1000)(2\pi)/60 = 104.7 \text{ rad/s}$. Substituting the values of ω and m into the above equation gives

$$\omega_n = \sqrt{\frac{k}{m}} < \frac{104.7}{\sqrt{2}} = 74.05 \text{ rad/s}$$

or

$$k < m (74.05)^2 = (50)(74.05)^2 = 274 \text{ kN/m}$$

which is the desired condition on the stiffness of the bearing.

When the stiffness of the foundation is $k = 200 \text{ kN/m}$, the natural frequency of the machine-bearing system is given by

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{200000}{50}} = 63.2 \text{ rad/s}$$

Therefore, the frequency ratio is $\beta = \omega/\omega_n = 104.7/63.2 = 1.657$. Substitute this value of β and $\zeta = 0$ into Equation (9-14) yields

$$\text{TR} = \frac{\sqrt{1 + (2\zeta\beta)^2}}{\sqrt{(1 - \beta^2)^2 + (2\zeta\beta)^2}} = \frac{1}{\sqrt{(1 - 1.657^2)^2}} = 0.573$$

Therefore 57.3% of the machine-generated dynamic force is transmitted into the foundation. An assessment of whether this is an adequate reduction must be based upon the information not provided in the problem statement.

Example 9-5 (Textbook Page 445)

In the system shown in Figure 9-8 and shown below for convenience, the mass, $M = 15 \text{ kg}$, $b = 450 \text{ N-s/m}$, $k = 6000 \text{ N/m}$, $m = 0.005 \text{ kg}$, $r = 0.2 \text{ m}$ and $\omega = 16 \text{ rad/s}$, what is the force transmitted to the foundation?

Solution

The equation of motion for the system is

$$15 \ddot{x} + 450 \dot{x} + 6000 x = (0.005)(16)^2 (0.2) \sin 16t$$

Consequently,

$$\omega_n = \sqrt{\frac{6000}{15}} = 20 \text{ rad/s}$$

$$2\zeta\omega_n = \frac{450}{15} \Rightarrow \zeta = \frac{450}{15 \times 2 \times 20} = 0.75$$

We can find that $\beta = \omega/\omega_n = 16/20 = 0.8$. From Equation. (9-15), we have

$$F_t = \frac{m\omega^2 r \sqrt{1+(2\zeta\beta)^2}}{\sqrt{(1-\beta^2)^2 + (2\zeta\beta)^2}} = \frac{(0.005)(16)^2 (0.2) \sqrt{1+(2 \times 0.75 \times 0.8)^2}}{\sqrt{(1-0.8^2)^2 + (2 \times 0.75 \times 0.8)^2}} = 0.319 \text{ N}$$

Automobile Suspension System.

Figure 9-15(a) shows an automobile system. Figure 9-15(b) is a schematic diagram of an automobile suspension system. As the car moves along the road, the vertical displacements at the tire act as motion excitation to the automobile suspension system. The motion of this system consists of a translational motion of the center of mass and a rotational motion about the center of mass. A complete analysis of the suspension system would be very involved.

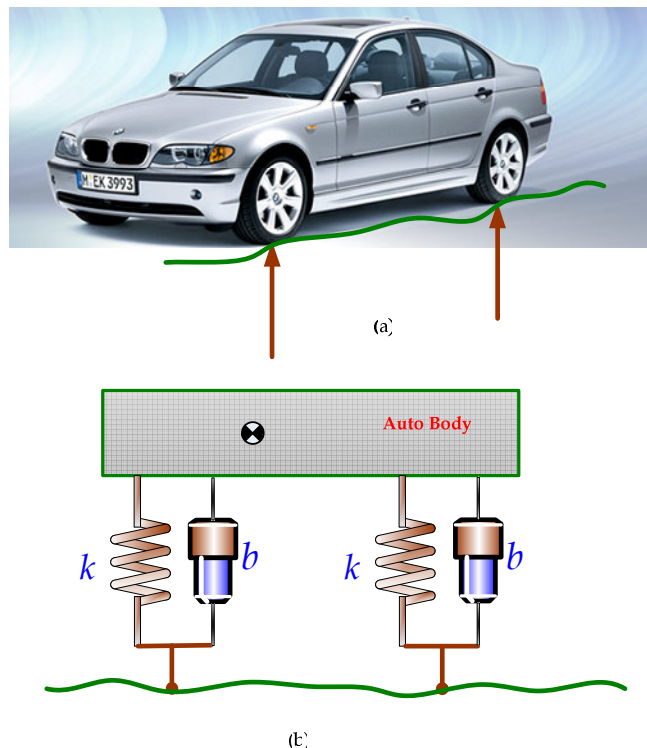


Figure 9-15 (a) Automobile system; (b) Schematic diagram of an automobile suspension system.

A highly simplified version appears in Figure 9-16. Let us analyze this simple model when the motion input is sinusoidal. We shall derive the transmissibility motion excitation system.

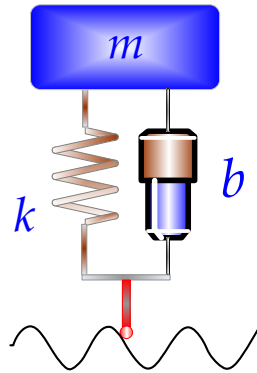


Figure 9-16 Simplified version of the automobile suspension system of Figure 9-13.

Transmissibility for Motion Excitation. In the mechanical system shown in Figure 9-17, the motion of the body is in the vertical direction only. The motion $p(t)$ of point A is the input to the system; the vertical motion $x(t)$ of the body is the output. The displacement $x(t)$ is measured from the equilibrium position in the absence of the input $p(t)$. We assume that $p(t)$ is sinusoidal, or

$$p(t) = P_o \sin \omega t$$

The equation of motion for the system is

$$m\ddot{x} + b(\dot{x} - \dot{p}) + k(x - p) = 0$$

or

$$m\ddot{x} + b\dot{x} + kx = b\dot{p} + kp$$

Take LT of both sides of the above equation, assuming zero I.C's

$$(ms^2 + bs + k)X(s) = (bs + k)P(s)$$

Hence,

$$\frac{X(s)}{P(s)} = \frac{(bs + k)}{(ms^2 + bs + k)}$$

The STF is then

$$\frac{X(j\omega)}{P(j\omega)} = \frac{(bj\omega + k)}{(k - m\omega^2 + jb\omega)}$$

The steady-state output $x(t)$ has the amplitude $|X(j\omega)|$. The input amplitude is $|P(j\omega)|$. The transmissibility TR in this case is the displacement amplitude ratio and is given by

$$\text{TR} = \frac{\text{Amplitude of the output displacement}}{\text{Amplitude of the input displacement}}$$

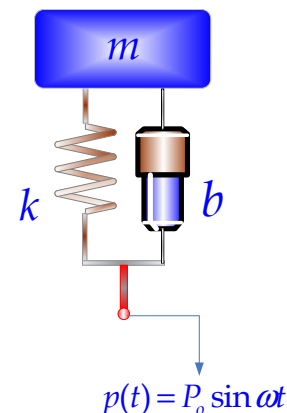


Figure 9-17 Mechanical system

Thus

$$\frac{|X(j\omega)|}{|P(j\omega)|} = \frac{\sqrt{b^2\omega^2 + k^2}}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}$$

Substituting $k/m = \omega_n^2$ and $b/m = 2\zeta\omega_n$ into the last equation, and simplifying, we obtain

$$\text{TR} = \frac{\sqrt{1 + (2\zeta\beta)^2}}{\sqrt{(1 - \beta^2)^2 + (2\zeta\beta)^2}} \quad (9-16)$$

where $\beta = \omega/\omega_n$. This equation is identical to Equation 9-14.

Example 9-6 (Example 9-4 in the Textbook Page 447)

A rigid body is mounted on an isolator to reduce vibratory effects. Assume that the mass of the rigid body is 500 kg, the damping coefficient of the isolator is very small ($\zeta = 0.01$), and the spring constant of the isolator is 12,500 N/m. Find the percentage of motion transmitted to the body if the frequency of the motion excitation of the base of the isolator is 20 rad/s.

Solution

The undamped natural frequency ω_n of the system is .

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{12,500}{500}} = 5 \text{ rad/s}$$

so

$$\beta = \omega/\omega_n = 20/5 = 4$$

Substituting $\zeta = 0.01$ and $\beta = 4$ into Equation 9-16, we get

$$\text{TR} = \frac{\sqrt{1 + (2\zeta\beta)^2}}{\sqrt{(1 - \beta^2)^2 + (2\zeta\beta)^2}} = \frac{\sqrt{1 + (2 \times 0.01 \times 4)^2}}{\sqrt{(1 - 4^2)^2 + (2 \times 0.01 \times 4)^2}} = 0.0669$$

The isolator thus reduces the vibratory motion of the rigid body to 6.69% of the vibratory motion of the base of the isolator.

9.5 DYNAMIC VIBRATION ABSORBERS

If a mechanical system operates near a critical frequency, the amplitude of vibration increases to a degree that cannot be tolerated, because the machine might break down or might transmit too much vibration to the surrounding machines. This section discusses a way to reduce vibrations near a specified operating frequency that is close to the natural frequency (i.e., the critical frequency) of the system by the use of a dynamic vibration absorber.

This section is covered with more details in the Lab. Refer to Laboratory notes.

<http://www.wcsscience.com/tacoma/bridge.html>

9.6 FREE VIBRATION IN MULTI-DEGREES-OF-FREEDOM SYSTEMS

Two-degrees-of-freedom system. A *two-degrees-of-freedom system* requires two independent coordinates to specify the system's configuration. Consider the mechanical system shown in Figure 9-18, which illustrates the two-degrees-of-freedom case.

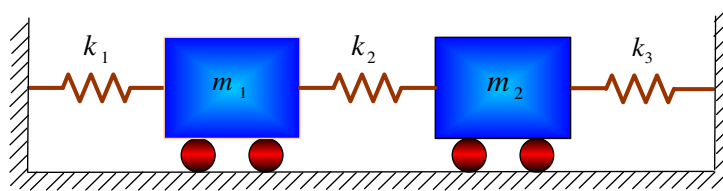


Figure 9-18 Mechanical system with two-degrees-of-freedom.

Let us derive the mathematical model of this system. Apply Newton's second law to mass m_1 and mass m_2 , we have

$$\begin{aligned} \text{Mass } m_1 &\Rightarrow \sum F = m_1 \ddot{x}_1 \Rightarrow -k_1 x_1 - k_2 (x_1 - x_2) = m_1 \ddot{x}_1 \\ \text{Mass } m_2 &\Rightarrow \sum F = m_2 \ddot{x}_2 \Rightarrow -k_3 x_2 - k_2 (x_2 - x_1) = m_2 \ddot{x}_2 \end{aligned}$$

Rearranging terms yields

$$m_1 \ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = 0 \quad (9-19)$$

$$m_2 \ddot{x}_2 + (k_2 + k_3)x_2 - k_2 x_1 = 0 \quad (9-20)$$

The above equations represent a mathematical model of the system.

Free Vibrations in two-degrees-of-freedom system. Consider the mechanical system shown in Figure 9-19, which is a special case of the system shown in Figure 9-18.

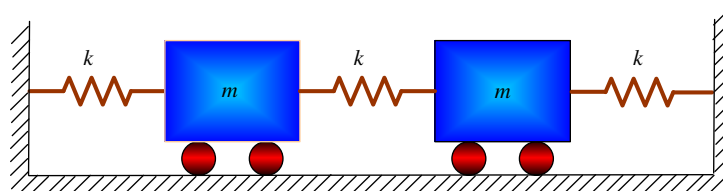


Figure 9-19 Mechanical system with two-degrees-of-freedom.

The equations of motion for the system of Figure 9-17 can be obtained by substituting $m_1 = m_2 = m$ and $k_1 = k_2 = k_3 = k$ into Equations (9-19) and (9-20), yielding

$$m \ddot{x}_1 + 2kx_1 - kx_2 = 0 \quad (9-21)$$

$$m \ddot{x}_2 + 2kx_2 - kx_1 = 0 \quad (9-22)$$

To find the natural frequencies of the vibration, we assume that the motion is harmonic. That is, we assume that

$$x_1 = A \sin \omega t, \quad x_2 = B \sin \omega t$$

Then

$$\ddot{x}_1 = -A \omega^2 \sin \omega t, \quad \ddot{x}_2 = -B \omega^2 \sin \omega t$$

If the preceding expressions are substituted into Equations (9-21) and (9-22), the resulting equations are

$$(-mA \omega^2 + 2kA - kB) \sin \omega t = 0$$

$$(-mB \omega^2 + 2kB - kA) \sin \omega t = 0$$

Since these equations must be satisfied at all times and since $\sin \omega t$ cannot be zero at all times, the quantities in parentheses must be equal to zero. Thus,

$$-mA \omega^2 + 2kA - kB = 0$$

$$-mB \omega^2 + 2kB - kA = 0$$

Rearranging terms we have

$$(2k - m \omega^2)A - kB = 0 \quad (9-23)$$

$$-kA + (2k - m \omega^2)B = 0 \quad (9-24)$$

For constants A and B to be nonzero, the determinant of the coefficients of Equations (9-23) and (9-24) must vanish, or

$$\begin{vmatrix} (2k - m \omega^2) & -k \\ -k & (2k - m \omega^2) \end{vmatrix} = 0$$

or

$$(2k - m \omega^2)^2 - k^2 = 0$$

or

$$\omega^4 - 4 \frac{k}{m} \omega^2 + 3 \frac{k^2}{m^2} = 0 \quad (9-25)$$

Equation (9-25) can be factored as

$$\left(\omega^2 - \frac{k}{m} \right) \left(\omega^2 - 3 \frac{k}{m} \right) = 0$$

from which

$$\omega^2 = \frac{k}{m}, \quad \omega^2 = 3\frac{k}{m}$$

Consequently, ω^2 has two values, the first representing the first natural frequency ω_1 (first mode) and the second representing the second natural frequency ω_2 (second mode):

$$\omega_1 = \sqrt{\frac{k}{m}}, \quad \omega_2 = \sqrt{\frac{3k}{m}}$$

Notice that in the one-degree-of-freedom system only one natural frequency exists, whereas the two degrees-of-freedom system has two natural frequencies.

Notice that, from Equation (9-23), we have

$$\frac{B}{A} = \frac{k}{2k - m\omega^2} \quad (9-26)$$

also, from Equation (9-24), we obtain

$$\frac{A}{B} = \frac{(2k - m\omega^2)}{k} \quad (9-27)$$

If we substitute $\omega^2 = k/m$ (first mode) into either Equation (9-26) or (9-27), we obtain, in both cases,

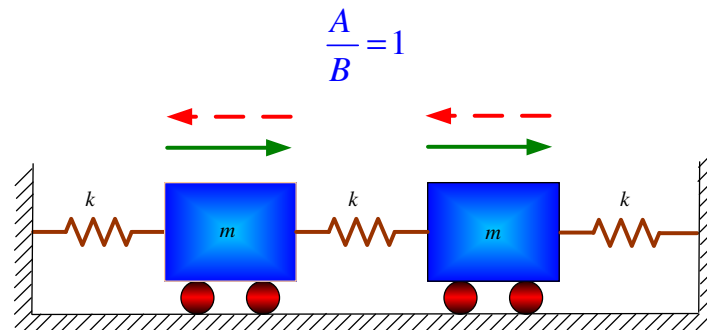


Figure 9-20 Mode shape corresponding the first mode

If we substitute $\omega^2 = 3k/m$ (second mode) into either Equation (9-26) or (9-27), we have

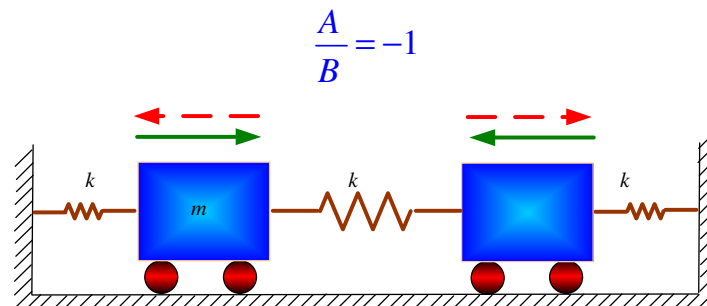


Figure 9-21 Mode shape corresponding the second mode.

If the system vibrates at either of its two natural frequencies, the two masses must vibrate at the same frequency.

Example 9-7

In the system shown in Figure 22, the displacements x_1 , x_2 and x_3 are measured from the static-equilibrium position of the system.

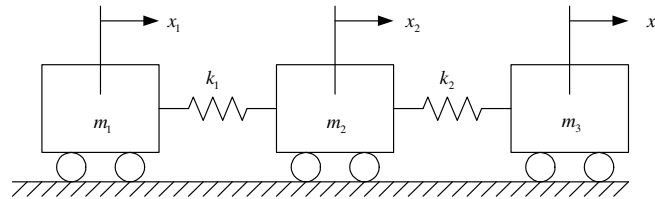


Figure 9-22 System with three degrees of freedom.

Solution

The Free Body Diagram (FBD) of the above system is shown in Figure 2.

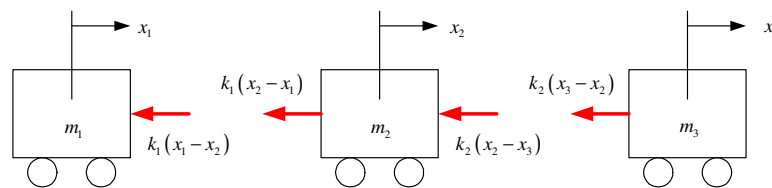


Figure 9-23 FBD of the system above.

Neglect the gravitational force on the three masses and apply Newton's second law of motion for a system in translation, one can get

$$\sum F = m\ddot{x} \quad (1)$$

$$\text{Mass } m_1, \quad m_1\ddot{x}_1 + k_1(x_1 - x_2) = 0 \quad (2)$$

$$\text{Mass } m_2, \quad m_2\ddot{x}_2 + k_1(x_2 - x_1) + k_2(x_2 - x_3) = 0 \quad (3)$$

$$\text{Mass } m_3, \quad m_3\ddot{x}_3 + k_2(x_3 - x_2) = 0 \quad (4)$$

In matrix form, the previous system can be written as

$$\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{pmatrix} k_1 & -k_1 & 0 \\ -k_1 & (k_1 + k_2) & -k_2 \\ 0 & -k_2 & k_2 \end{pmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (5)$$

or in more compact form

$$[M]\{\ddot{x}\} + [K]\{x\} = \{0\} \quad (6)$$

where

$$[M] = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \equiv \text{mass matrix of the system}$$

$$[K] = \begin{pmatrix} k_1 & -k_1 & 0 \\ -k_1 & (k_1 + k_2) & -k_2 \\ 0 & -k_2 & k_2 \end{pmatrix} \equiv \text{stiffness matrix of the system} \quad (7)$$

$$\{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \equiv \text{vector of displacement coordinates}$$

Assume a solution of the form

$$\{x\} = \{v\} e^{i\omega t} \quad (8)$$

Substitution of eq. (8) into (6) yields the following associated eigenvalue problem

$$\begin{pmatrix} k_1 & -k_1 & 0 \\ -k_1 & (k_1 + k_2) & -k_2 \\ 0 & -k_2 & k_2 \end{pmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \lambda \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} \quad (9)$$

where $\lambda = \omega^2$. Assume that $k_1 = k_2 = k = 1 \text{ N/m}$, and $m_1 = m_2 = m_3 = m = 1 \text{ kg}$. The script that determines the eigenvalues and associated eigenvectors is

MATLAB PROGRAM:

```
>> k=1; k1=k;k2=k;
>> m=1; m1=m;m2=m;
>> K=[k -k 0;-k 2*k -k;0 -k k];
>> M=[m 0 0;0 m 0;0 0 m];
>> [Modes, Eigenvalues]=eig(K,M)
```

Execution of the script gives

Modes =

```
-0.5774 -0.7071 0.4082
-0.5774 0.0000 -0.8165
-0.5774 0.7071 0.4082
```

Eigenvalues =

0.0000	1.0000	0.0000
0.0000	1.0000	0.0000
0.0000	0.0000	3.0000

since $\lambda = \omega^2$, then the frequencies are $\omega_1 = \sqrt{3} = 1.732 \text{ rad/s}$, $\omega_2 = \sqrt{1} = 1 \text{ rad/s}$ and $\omega_3 = \sqrt{0} = 0 \text{ rad/s}$.

When the system above is examined, it is found that, since the masses at each end are not restrained, a **rigid-body mode** in which all masses move in the same direction by the same amount is possible. This is reflected in the corresponding vibration mode, which is depicted in the third column of the matrix of vibration modes. The springs are neither stretched nor compressed in this case.

