CHAPTER 8

TIME-DOMAIN ANALYSIS OF DYNAMIC SYSTEMS

A. Bazoune

8.1 INTRODUCTION

Poles and Zeros of a Transfer Function

Poles: The *poles* of a transfer function are those values of s for which the function is *undefined* (becomes infinite).

Zeros: The *zeros* of a transfer function are those values of s for which the function is *zero*.

Example of the Effect of Pole/Zero Locations

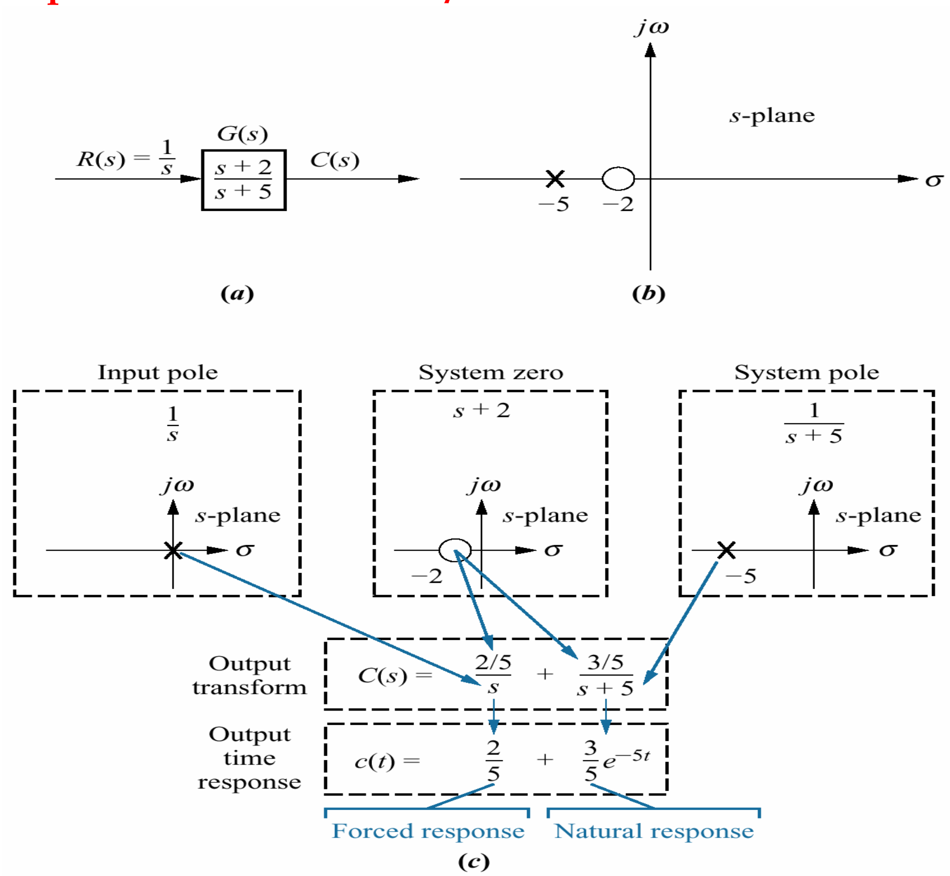
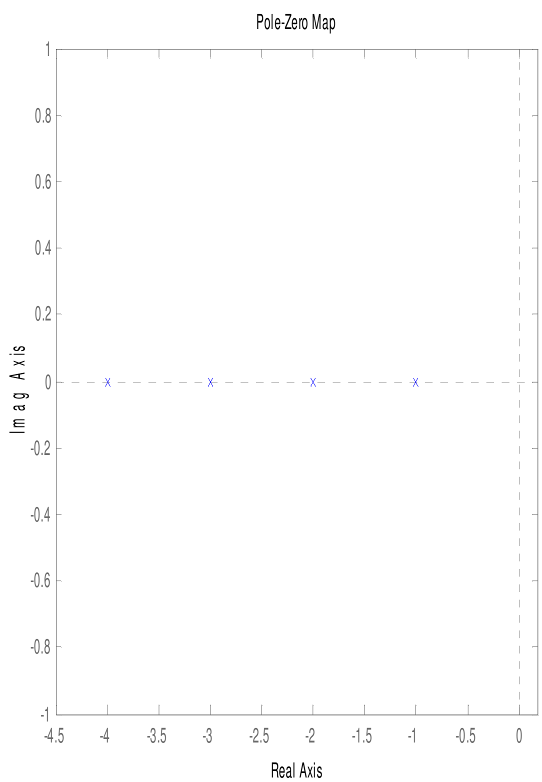


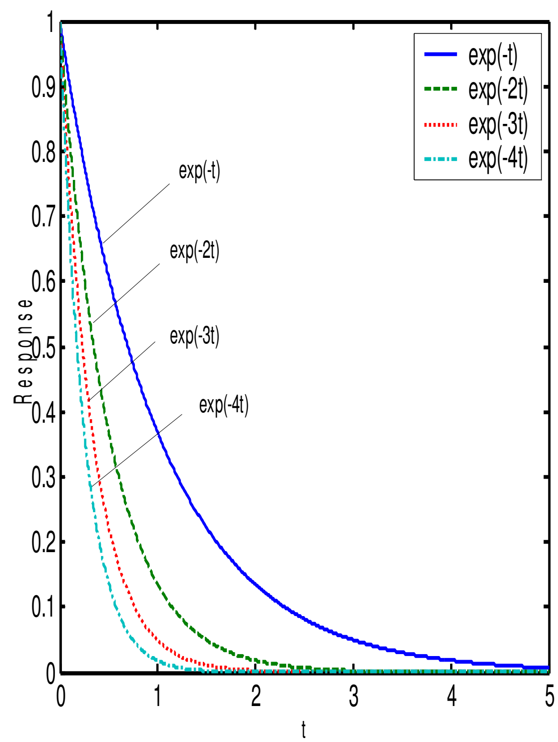
Figure 8-1.

From the development summarized above, the following conclusions can be drawn:

A pole of the input function generates the form of the forced response. (i.e., the pole at the origin generates a step function at the output)

- ► A pole of the transfer function generates the form of the natural response (i.e., the pole at $-\sigma$ generates $e^{-\sigma t}$)
- ► A pole on the real axis generates an exponential response of the form $e^{-\sigma t}$, where $-\sigma$ is the pole location on the real axis. Thus, the pole farther the left a pole is on the negative





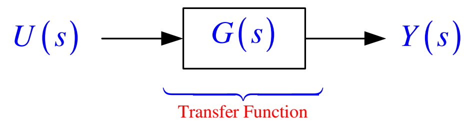
Standard Form of the First Order System Equation

$$\dot{y}+\left(\frac{1}{\tau}\right)y=\frac{B}{\tau}u(t) \qquad (1)$$

The transfer function of the previous system is defined by

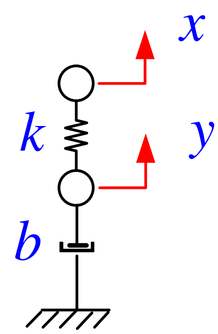
$$G(s)=\frac{Y(s)}{U(s)}=\frac{B/\tau}{s+(1/\tau)}$$

where τ is known as the time constant. It has dimensions of time for all physical systems described by the first order differential equation above. The above equation can be respresented as



Examples

i) Spring-Damper System



Equation of motion

$$b\dot{y} + ky = k\ x$$

or

$$\dot{y} + \frac{1}{\left(\frac{b}{k}\right)} y = \left(\frac{k}{b}\right) x$$

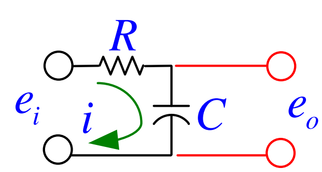
Comparing the above equation with the standard one, so that

$$\tau = b/k = \text{time constant}$$

with the units of

$$[\tau] = [b/k] = \frac{[\text{N.s/m}]}{[\text{N/m}]} = [\text{s}]$$

ii) RC-Circuit



Equation of motion

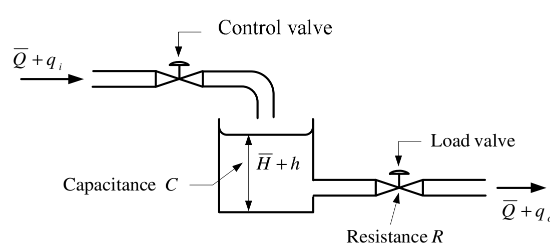
$$RC\dot{e}_o + e_o = e_i$$

$$\tau = RC = \text{time constant}$$

with the units of

$$[\tau] = [RC] = \frac{[\text{V.s}]}{[\text{q}]} \times \frac{[\text{q}]}{[\text{V}]} = [\text{s}]$$

iii) Liquid Level System



Equation of motion

$$\frac{dh}{dt} + \frac{1}{RC} h = \frac{1}{C} q$$

$$\tau = RC = \text{time constant}$$

with the units of

$$[\tau] = [RC] = \frac{[\text{m}]}{\left[\text{m}^3/s\right]} \times \left[\text{m}^2\right] = [\text{s}]$$

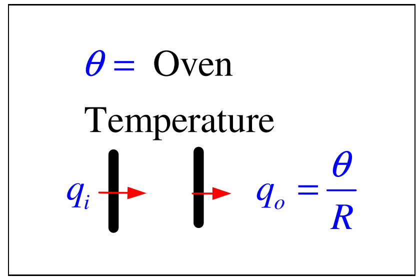
iv) Thermal System

Equation of motion

$$\frac{d\theta}{dt}+\frac{1}{RC}\theta=\frac{1}{C}q_i$$

$$\tau = RC = \text{time constant}$$

with the units of $[\tau]=[RC]=\frac{[{}^{\circ}C]}{[\text{Kcal}/s]}\times\frac{[\text{Kcal}]}{[{}^{\circ}C]}=[\text{s}]$



Thus when a first order system is written in the form of Equation (1) with the coefficient of dy/dt is equal to 1 and the coefficient of the dependant variable is $(1/\tau)$ in which τ represents the time constant of the system and has always the dimension of time regardless of the physical system under consideration.

Interest in Analysis of Dynamic Systems

After obtaining a model of a dynamic system, one then needs to apply test signals to see if the system performs accordingly to certain design specifications put forward by the design of the system. In general, one requires the system to be

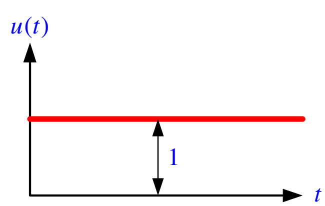
1. stable (system does not grow out unbounded)
2. with a fast response
3. has a small error as possible (steady sate error).

Other performance criteria may also exist but we will be mainly concerned with the above mentioned three. Usually the actual input to the dynamic system is unknown in advance; however, by subjecting the system to standard test signals, we can get an indication of the ability of the system performance under actual operating conditions. For example, the information we gain by analyzing the system stability and its speed of response and steady state error due to various types of standard test signals, will give an indication on the system performance under actual operating condition.

Typical Test Signals

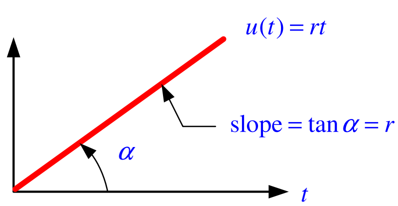
i) Step Input

Very common input to actual dynamic systems. As it represents a sudden change in the value of the reference input



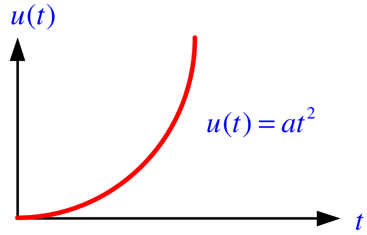
ii) Ramp Input (Constant Velocity)

This represents a situation where the input has a constant rate of increase with time (i.e. constant velocity)



iii) Parabolic Input (Constant Acceleration)

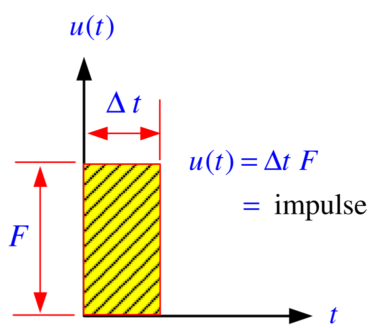
Occurs in situation where the input has a constant acceleration.

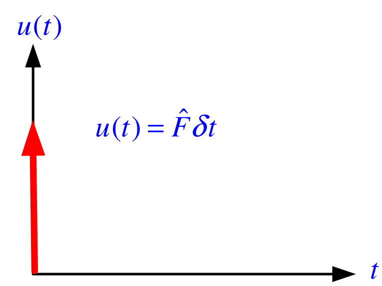


iv) Impulsive Load Input

A large load over a short duration of time $<< 0.1\tau$ = system time constant can be considered as an impulse.

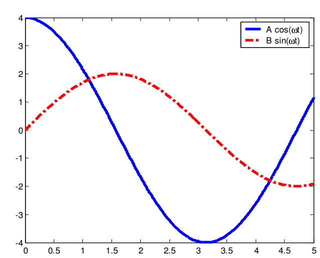
$$\hat{F}\,\delta(t), \quad \hat{F} = F\,\Delta t = \text{ strength of impulse}$$





v) Sinusoidal Load Input

$$u(t) = \begin{cases} A\sin\omega t \\ B\cos\omega t \end{cases}$$



Natural and Forced Response

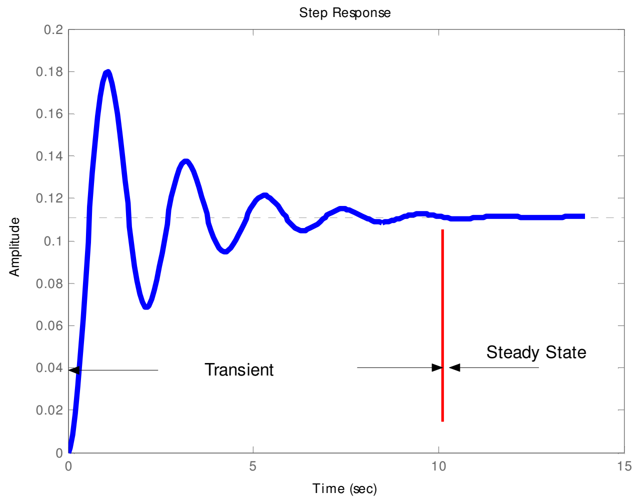
The solution $y(t)$ to the homogeneous differential equation (1) is composed of two parts:

- **Complementary solution**: $y_c(t)$ natural response due to initial conditions.
- **Particular solution** : $y_p(t)$ forced response

such that: $$y(t) = y_p(t) + y_c(t)$$

Transient and Steady State Response

- **Transient response:** response from initial state to final state.
- **Steady sate Response:** response as time approaches ∞.

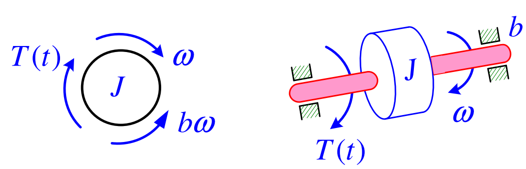


Notice that if $\lim_{t\to\infty} y_c(t) = 0$ and $\lim_{t\to\infty} y_p(t) =$ a bounded function of time then the system is said to be steady state; where the steady state solution is

$$y_{ss} = \lim_{t\to\infty} y_c(t) = \lim_{t\to\infty} y_p(t)$$

8.2 TRANSIENT RESPONSE ANALYSIS OF FIRST-ORDER SYSTEM

Rotor mounted in bearings is shown in the figure below. External torque $T(t)$ is applied to the system.



Apply Newton's second law for a system in rotation

$$\sum M = J\ddot{\theta} = J\dot{\omega}$$

$$J\dot{\omega} + b\omega = T(t) \quad \Rightarrow \quad \dot{\omega} + (b/J)\omega = T(t)/J$$

or

$$\dot{\omega} + \frac{1}{(J/b)}\omega = T^*(t)$$

Define the time constant $\tau = (J/b)$, the previous equation can be written in the form

$$\dot{\omega} + \frac{1}{\tau}\omega = T^*(t), \quad \omega(0) = \omega_\circ$$

which represents the equation of motion as well as the mathematical model of the system shown. It represents a first order system.

Free Response $T(t)=0$

To find the response $\omega(t)$, take LT of both sides of the previous equation.

$$\Big[\underbrace{s\Omega(s)-\omega(0)}_{L[\dot{\omega}]}\Big]+\frac{1}{\tau}\Big[\underbrace{\Omega(s)}_{L[\omega]}\Big]=0$$

$$\left(s+\frac{1}{\tau}\right)\Omega(s)=\omega_\circ \qquad \Rightarrow \qquad \Omega(s)=\omega_\circ \Big/ \left(s+\frac{1}{\tau}\right)$$

Taking inverse LT of the above equation will give the expression of $\omega(t)$

$$\omega(t)=\omega_\circ\, e^{-(b/J)t}=\omega_\circ\, e^{-(t/\tau)}$$

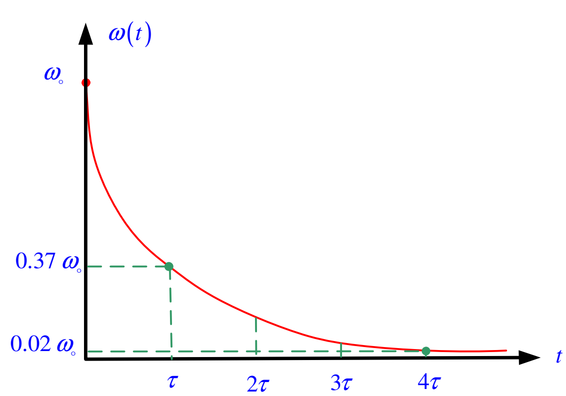
It is clear that the angular velocity decreases exponentially as shown in the figure below. Since $\lim_{t\to\infty} e^{-(t/\tau)}=0$; then for such decaying system, it is convenient to depict the response in terms of a time constant.

A ***time constant*** is that value of time that makes the exponent equal to -1. For this system, time constant $\tau=J/b$. When $t=\tau$, the exponent factor is

$$e^{-(t/\tau)}=e^{-(\tau/\tau)}=e^{-1}=0.368=36.8\ \%$$

This means that when ***time constant*** $=\tau$, the time response is reduced to $36.8\ \%$ of its final value. We also have

$$\tau=J/b=\text{time constant}$$
$$\omega(\tau)=0.37\ \omega_\circ$$
$$\omega(4\tau)=0.02\ \omega_\circ$$



Forced Response $T(t) \neq 0$

Remember

$$\dot{\omega} + \frac{1}{\tau}\omega = T^*(t)$$

To find the response $\omega(t)$, take LT of both sides of the previous equation for zero initial conditions.

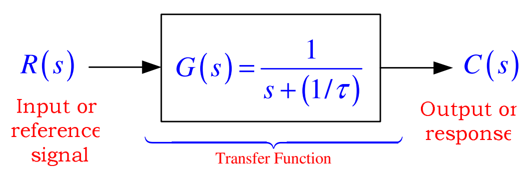
$$\underbrace{s\Omega(s) - \underbrace{\omega(0)}_{=0}}_{L[\dot{\omega}]} + \frac{1}{\tau}\underbrace{\Omega(s)}_{L[\omega]} = T^*(s)$$

$$\left(s + \frac{1}{\tau}\right)\Omega(s) = T^*(s) \qquad \Rightarrow \qquad \frac{\Omega(s)}{T^*(s)} = \frac{1}{\left(s + \frac{1}{\tau}\right)}$$

The above equation can be written in the more general form as

$$\frac{C(s)}{R(s)} = \frac{1}{\left(s + \frac{1}{\tau}\right)}$$

where $C(s)$ is the output or the response and $R(s)$ is the input or reference signal. The above equation can be represented as



i) Impulse Response

In this case, for an unit impulse input of magnitude $r(t) = B\delta(t)$,

$$R(s) = B$$

and the above equation can be written in the form

$$C(s) = \frac{B}{\left(s + \frac{1}{\tau}\right)}$$

from which

$$c(t) = B\, e^{-(1/\tau)t}$$

The figure below shows the response $c(t) = B\, e^{-(1/\tau)t}$. Since we assumed zero I.C's, the output must change instantaneously from 0 at time $t = (0^-)$ to 1 at time $t = (0^+)$.

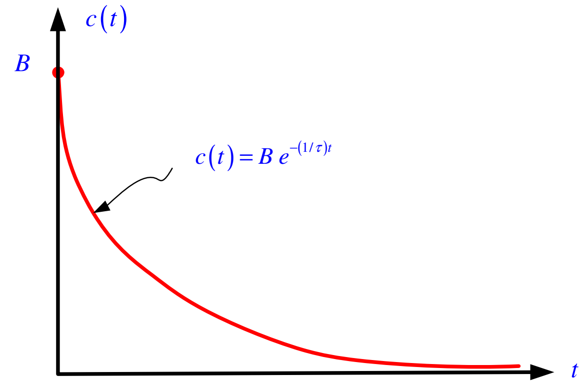


Figure . Impusle response of a first order system

ii) Step Response

In this case, for a unit step input of magnitude B,

$$R(s) = B/s$$

and the above equation can be written in the form

$$C(s) = \frac{B}{s\left(s+\frac{1}{\tau}\right)} = \frac{a_1}{s} + \frac{a_2}{\left(s+\frac{1}{\tau}\right)}$$

where

$$a_1 = \left.\frac{B\not{s}}{\not{s}\left(s+\frac{1}{\tau}\right)}\right|_{s=0} = B\,\tau \quad \text{and} \quad a_2 = \left.\frac{B\left(s+\frac{1}{\tau}\right)}{s\left(s+\frac{1}{\tau}\right)}\right|_{s=-\frac{1}{\tau}} = -B\,\tau$$

Therefore

$$C(s) = \frac{B}{s\left(s+\frac{1}{\tau}\right)} = \frac{B\,\tau}{s} - \frac{B\,\tau}{\left(s+\frac{1}{\tau}\right)}$$

for which

$$c(t) = B\,\tau - B\,\tau e^{-(1/\tau)t} = B\,\tau\left(1-e^{-(1/\tau)t}\right)$$

The response $c(t)$ is represented in the Figure below.

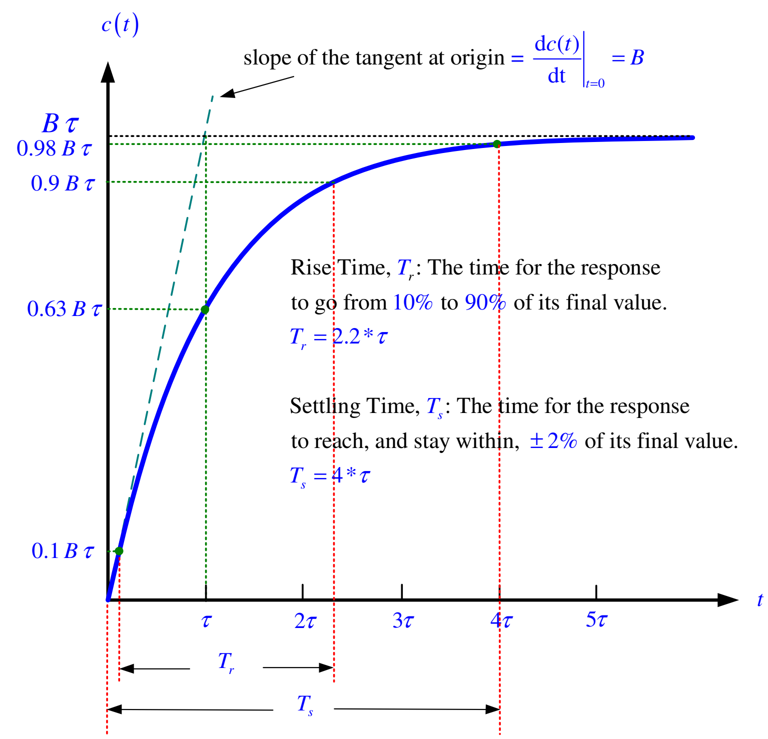


Figure Step response of a first order system

Time constant: It is the time for $e^{-\frac{t}{\tau}}$ to decay to $37\ \%$ of its final value, i.e.,

$$e^{-\frac{t}{\tau}}\bigg|_{t=0} = e^{-1} = 0.37$$

Alternatively, the ***time constant*** is the time it takes for a step response to rise to $63\ \%$ of its final value, i.e.,

$$c(t)\big|_{t=\tau} = B\,\tau\left(1-e^{-(1/\tau)t}\right)\Big|_{t=\tau} = B\,\tau\left(1-e^{-1}\right) = 0.63B\,\tau$$

Rise Time: time for the response to go from $10\ \%$ to $90\ \%$ of its final value. The rise time T_r is found by solving the expression for the step response for the difference in time $c(t)=0.9$ and $c(t)=0.1$, that is

$$B\,\tau\left(1-e^{-(1/\tau)t}\right) = 0.9B\,\tau$$

or

$$\left(1-e^{-(1/\tau)t}\right) = 0.9$$

$$e^{-(1/\tau)t} = 0.1 \quad \Rightarrow \quad -(t/\tau) = \ln(0.1) = -2.302 \quad \Rightarrow \quad t_{0.9} = 2.302\,\tau$$

similarly

$$B\,\tau\left(1-e^{-(1/\tau)t}\right) = 0.1B\,\tau$$

or

$$\left(1-e^{-(1/\tau)t}\right)=0.1$$

$$e^{-(1/\tau)t}=0.9 \quad \Rightarrow \quad -\left(t/\tau\right)=\ln(0.9)=-0.105 \quad \Rightarrow \quad t_{0.1}=0.105\,\tau$$

Hence , the ***rise time*** is

$$T_r=t_{0.9}-t_{0.1}=2.2\,\tau$$

Settling Time: time for the response to reach, and stay within $\pm 2\,\%$ of its final value. The settling time T_s is found by solving the expression $c(t)=0.98$. Thus,

$$c(t)=B\,\tau\left(1-e^{-(1/\tau)t}\right)=0.98B\,\tau$$

or

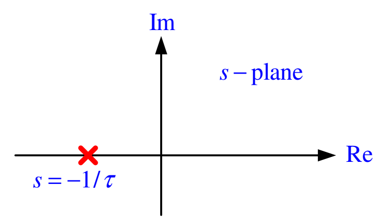
$$\left(1-e^{-(1/\tau)t}\right)=0.98 \quad \Rightarrow \quad e^{-(1/\tau)t}=0.02 \quad \Rightarrow \quad -\left(1/\tau\right)t=\ln(0.02)$$

or

$$t=T_s=\tau\ln(0.02)=4\,\tau$$

Remarks:

1. The smaller the time constant τ, the faster is the response and the furthest is the pole of
$$C(s)=\frac{B}{s\left(s+1/\tau\right)}$$
2. Steady state error e_{ss} due to step input $=0$.
3. The system is stable, i.e., $\lim_{t\to\infty} c(t)=\lim_{t\to\infty} c_p(t)$ provided that the pole $s=-1/\tau$ lies on the left half of the complex plane.



iii) Ramp Response

In this case, for a ramp input of slope B,

$$r(t)=B\,t \quad \Leftrightarrow \quad R(s)=L\left[r(t)\right]=\frac{B}{s^2}$$

and the expression of $C(s)$ above can be written in the form

$$C(s)=\frac{B}{s^2\left(s+\frac{1}{\tau}\right)}=\frac{a_1}{s}+\frac{a_2}{s^2}+\frac{b}{\left(s+\frac{1}{\tau}\right)}$$

where

$$a_1 = \frac{d}{ds}\left[\frac{B\not{s}^2}{\not{s}^2\left(s+\frac{1}{\tau}\right)}\right]_{s=0} = \left[\frac{-B}{\left(s+\frac{1}{\tau}\right)^2}\right]_{s=0} = -B\,\tau^2$$

$$a_2 = \left[\frac{B\not{s}^2}{\not{s}^2\left(s+\frac{1}{\tau}\right)}\right]_{s=0} = \left[\frac{B}{\left(s+\frac{1}{\tau}\right)}\right]_{s=0} = B\,\tau$$

$$b = \left[\frac{B\left(s+\frac{1}{\tau}\right)}{s^2\left(s+\frac{1}{\tau}\right)}\right]_{s=-1/\tau} = \left[\frac{B}{s^2}\right]_{s=-1/\tau} = B\,\tau^2$$

Therefore

$$C(s) = \frac{B}{s^2\left(s+\frac{1}{\tau}\right)} = -\frac{B\,\tau^2}{s} + \frac{B\,\tau}{s^2} + \frac{B\,\tau^2}{\left(s+\frac{1}{\tau}\right)}$$

Hence

$$c(t) = L^{-1}\left[C(s)\right] = -B\,\tau^2 + B\,\tau t + B\,\tau^2 e^{-(1/\tau)t} = B\,\tau\left(t-\tau+\tau e^{-(1/\tau)t}\right)$$

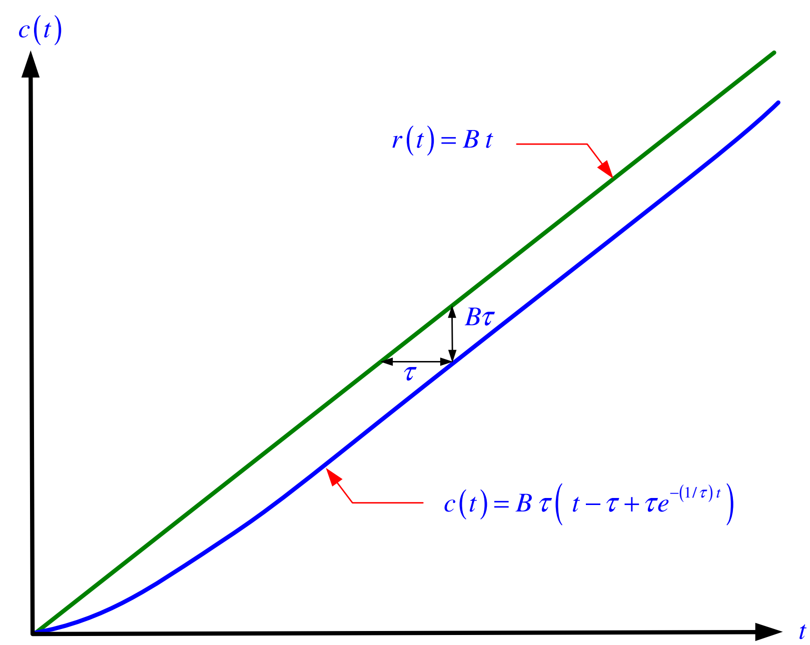
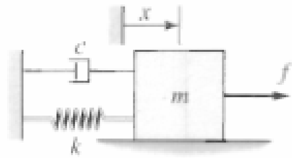


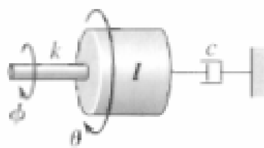
Figure Ramp response of a first order system

8.3 TRANSIENT RESPONSE ANALYSIS OF SECOND-ORDER SYSTEM

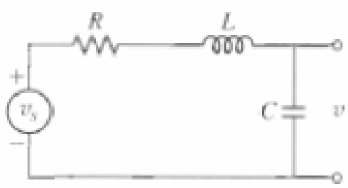
Some Examples of Second Order Systems



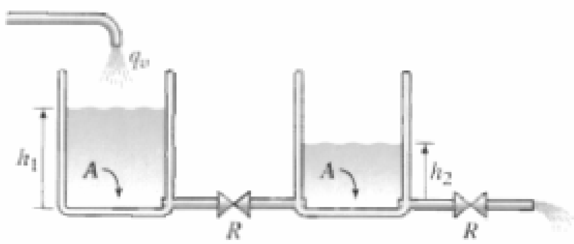
$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f$$



$$I\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt} + k\theta = k\phi$$

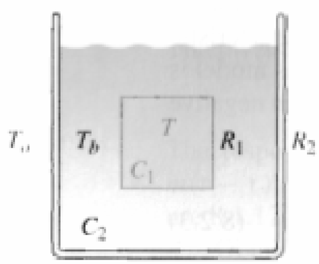


$$LC\frac{d^2v}{dt^2} + RC\frac{dv}{dt} + v = v_s$$



$$RA\frac{dh_1}{dt} + g(h_1 - h_2) = Rq_v$$

$$RA\frac{dh_2}{dt} + g(h_2 - h_1) + gh_2 = 0$$



$$R_1C_1\frac{dT}{dt} + T = T_b$$

$$R_1R_2C_2\frac{dT_b}{dt} + (R_1 + R_2)T_b = R_2T + R_1T_a$$

Free Vibration without damping

Consider the mass spring system shown in Figure 3-11. The equation of motion can be given by

$$m\,\ddot{x} + k\,x = 0$$

or

$$\ddot{x} + \frac{k}{m}x = 0 \quad \Rightarrow \quad \ddot{x} + \omega_n^2\,x = 0$$

where

$$\omega_n = \sqrt{\frac{k}{m}}$$

is the natural frequency of the system and is expressed in rad/s.

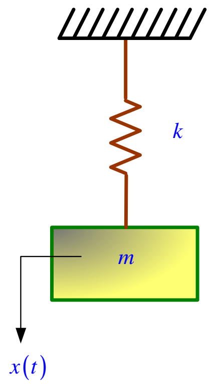


Figure 8-11 Mass Spring System

Taking LT of both sides of the above equation where $x(0) = x_\circ$ and $\dot{x}(0) = \dot{x}_\circ$ gives

$$\underbrace{s^2X(s) - sx(0) - \dot{x}(0)}_{L[\ddot{x}]} + \omega_n^2\,X(s) = 0$$

rearrange to get

$$X(s) = \frac{sx_\circ + \dot{x}_\circ}{s^2 + \omega_n^2}, \quad \Rightarrow \text{Remember poles are} \quad \underbrace{s = \pm j\omega_n}_{\text{complex conjugates}}$$

$$X(s) = \frac{\dot{x}_\circ}{\omega_n}\frac{\omega_n}{s^2 + \omega_n^2} + x_\circ \frac{s}{s^2 + \omega_n^2}$$

and the response $x(t)$ is given by

$$x(t) = \frac{\dot{x}_\circ}{\omega_n}\sin(\omega_n t) + x_\circ \cos(\omega_n t)$$

It is clear that the response $x(t)$ consists of a sine and cosine terms and depends on the values of the initial conditions $x_\circ$ and $\dot{x}_\circ$. Periodic motion such that described by the above equation is called ***simple harmonic motion***.

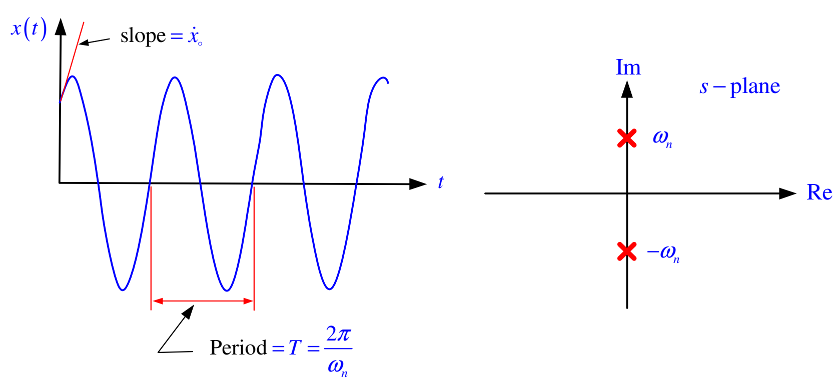


Figure 8-12 Free response of a simple harmonic motion and pole location on the s-plane

if $\dot{x}(0) = \dot{x}_\circ = 0$,

$$x(t) = x_\circ \cos(\omega_n t)$$

Free Vibration with Viscous damping

Damping is always present in actual mechanical systems, although in some cases it may be negligibly small. Consider the mass spring damper system shown in the figure. The equation of motion can be given by

$$m\,\ddot{x} + b\dot{x} + k\,x = 0 \qquad (1)$$

the characteristic equation of the above equation is

$$ms^2 + bs + k = 0 \qquad (2)$$

and the two roots of this equation are

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4\,m\,k}}{2\,m} \qquad (3)$$

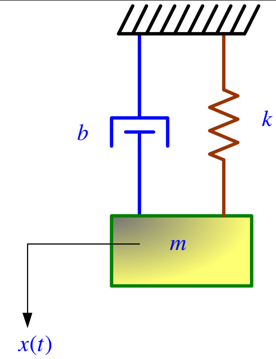


Figure 8-13

We consider three cases:

- $b^2 - 4\,m\,k < 0$ Roots are complex conjugates (***underdamped*** case)
- $b^2 - 4\,m\,k = 0$ Roots are real and repeated $s_1 = s_2$ (***critically damped*** case)
- $b^2 - 4\,m\,k > 0$ Roots are real and distinct (***overdamped*** case)

In solving equation (1) for the response $x(t)$, it is convenient to define

$$\omega_n = \sqrt{\frac{k}{m}} = \text{undamped natural frequency, [rad/s]}$$

and

$$\zeta = \text{dampin gratio} = \frac{\text{actual damping value}}{\text{critical damping value}} = \frac{b}{2\sqrt{km}}$$

and rewrite equation (2) in the form

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0 \qquad (4)$$

which is the standard form equation of a second order system.

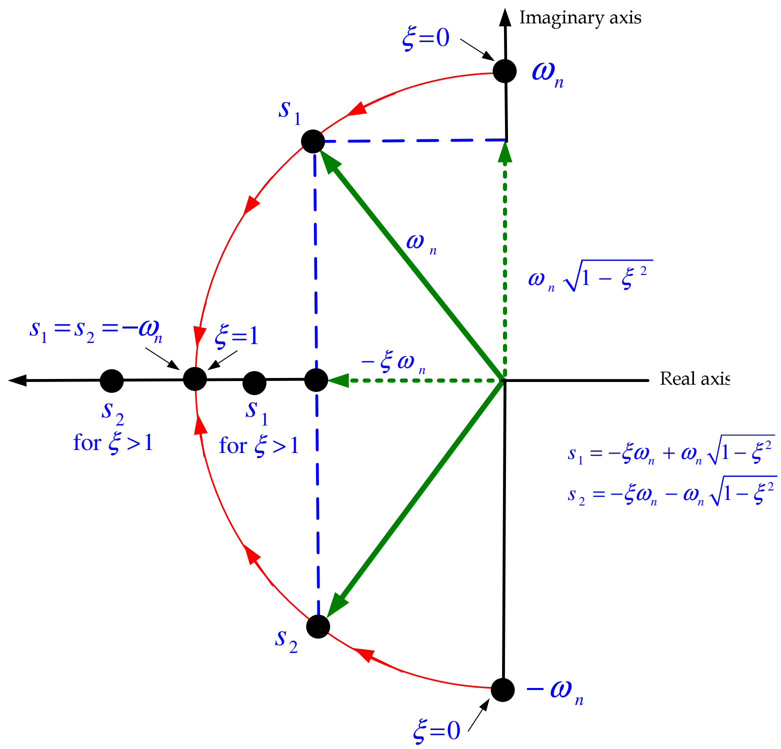
i) Underdamped Case $0 < \xi < 1$ Taking LT of both sides of equation (1) where $x(0) = x_\circ$ and $\dot{x}(0) = \dot{x}_\circ$, and rearrange to get

$$X(s) = \frac{(s + 2\xi\omega_n)x_\circ + \dot{x}_\circ}{s^2 + 2\xi\omega_n s + \omega_n^2} \qquad (5)$$

knowing that equation (4) can be written as

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s + \xi\omega_n)^2 + \left(\omega_n\sqrt{1-\xi^2}\right)^2 = 0$$

wich is a complete square equation. The nature of the roots s_1 and s_2 of equation (4) with varying values of damping ratio ξ can be shown in the complex plane as shown in the figure below. The semicircle represents the locus of the roots s_1 and s_2 for different values of ξ in the range $0 < \xi < 1$



Define

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \text{damped natural frequency (rad/s)}$$

The relationship between ζ and the non-dimensional frequency (ω_d/ω_n) is shown the figure below.

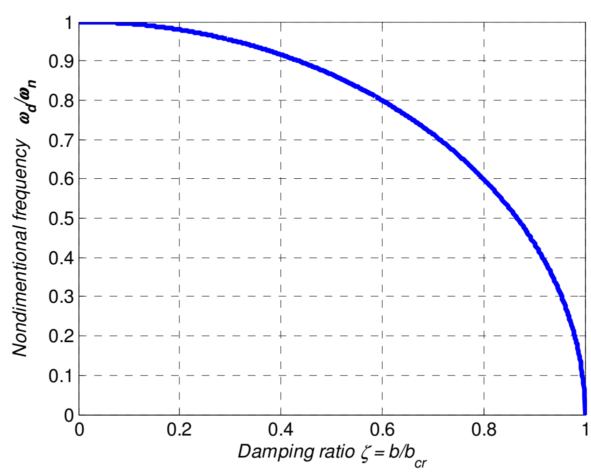


Figure Non-dimensional frequency versus the damping ratio.

Then

$$X(s) = \frac{\xi\omega_n x_\circ + \dot{x}_\circ}{\omega_d} \times \frac{\omega_d}{(s+\xi\omega_n)^2 + (\omega_d)^2} + \frac{(s+\xi\omega_n)x_\circ}{(s+\xi\omega_n)^2 + (\omega_d)^2}$$

from which

$$x(t) = L^{-1}\left[X(s)\right] = \frac{\xi\omega_n x_\circ + \dot{x}_\circ}{\omega_d} e^{-\xi\omega_n t} \sin\omega_d t + x_\circ e^{-\xi\omega_n t} \cos\omega_d t \tag{6}$$

or

$$x(t) = e^{-\xi\omega_n t}\left\{\left(\frac{\xi}{\sqrt{1-\xi^2}} x_\circ + \frac{\dot{x}_\circ}{\omega_d}\right) \sin\omega_d t + x_\circ \cos\omega_d t\right\} \tag{7}$$

If the initial velocity $\dot{x}(0) = 0$, the above equation reduces to

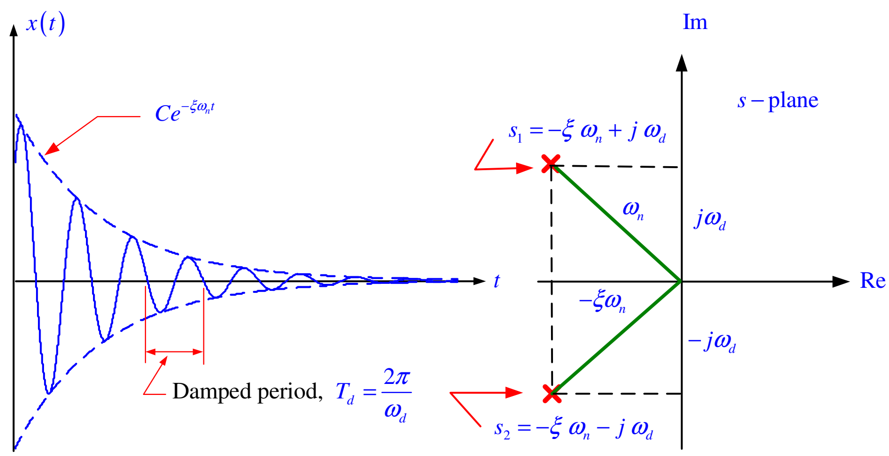
$$x(t) = x_\circ e^{-\xi\omega_n t}\left\{\left(\frac{\xi}{\sqrt{1-\xi^2}}\right) \sin\omega_d t + \cos\omega_d t\right\} \tag{8}$$

or

$$x(t) = C e^{-\xi\omega_n t} \sin(\omega_d t + \phi) \tag{9}$$

where

$$\phi = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi} \qquad \text{and} \qquad C = \frac{x_\circ}{\sqrt{1-\xi^2}} \tag{10}$$



Remarks:

Notice that for this case (undedamped case $0 < \xi < 1$)

1. the response is a decaying sinusoid.
2. the frequency of oscillations is $\omega_d = \left(\omega_n\sqrt{1-\xi^2}\right)$.
3. For positive damping $(\xi > 0)$, the poles s_1 and s_2 have negative real and lie entirely on the let half of the complex plane. As a result the transient response decays with time and the system is said to be ***stable*****.**
4. The rate at which the transient response decays depends on the coefficient $\xi\omega_n$ of t in $e^{-\xi\omega_n t}$. Larger $\xi\omega_n$ (i.e., smaller $1/\xi\omega_n$) leads to faster transient response (i.e., faster decay of $x(t)$). The term $1/\xi\omega_n$ is in this case the time constant of the second

order system. Therefore, the time constant of the second order system can be made smaller (i.e., its speed faster) by moving the real part $-\xi\omega_n$ farther away from the origin of the complex plane.

ii) Critically damped Case $\zeta = 1$ In this case, the poles the poles s_1 and s_2 become

$$s_1 = s_2 = -\xi\omega_n$$

and the response $x(t)$ can be obtained from equation (5). Thus

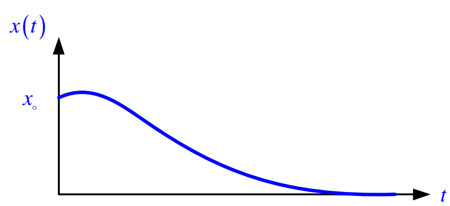
$$X(s) = \frac{(s + 2\omega_n)x_\circ + \dot{x}_\circ}{s^2 + 2\omega_n s + \omega_n^2} = \frac{(s + \omega_n)x_\circ + \omega_n x_\circ + \dot{x}_\circ}{(s + \omega_n)^2}$$

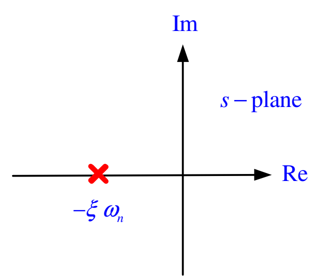
$$= \frac{x_\circ}{(s + \omega_n)} + \frac{\omega_n x_\circ + \dot{x}_\circ}{(s + \omega_n)^2}$$

from which

$$x(t) = x_\circ e^{-\xi\omega_n t} + (\omega_n x_\circ + \dot{x}_\circ)te^{-\xi\omega_n t}$$

which is decaying exponentially as shown in the figure below





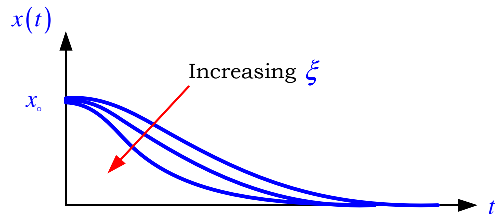
iii) Overdamped Case $\xi > 1$ In this case, the poles the poles s_1 and s_2 are both real

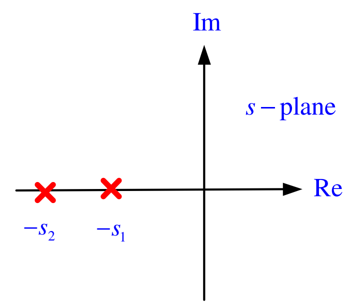
$$s_1 = -\xi\omega_n + \omega_n\sqrt{\xi^2 - 1}$$
$$s_2 = -\xi\omega_n - \omega_n\sqrt{\xi^2 - 1}$$

and the response $x(t)$ becomes

$$x(t) = \left\{\frac{-\xi + \sqrt{\xi^2 - 1}}{2\sqrt{\xi^2 - 1}}x_\circ - \frac{\dot{x}_\circ}{2\omega_n\sqrt{\xi^2 - 1}}\right\}e^{-\left(\xi\omega_n + \omega_n\sqrt{\xi^2 - 1}\right)t}$$
$$+ \left\{\frac{\xi + \sqrt{\xi^2 - 1}}{2\sqrt{\xi^2 - 1}}x_\circ + \frac{\dot{x}_\circ}{2\omega_n\sqrt{\xi^2 - 1}}\right\}e^{-\left(\xi\omega_n - \omega_n\sqrt{\xi^2 - 1}\right)t}$$

where the response is shown in the figure below





Remarks:

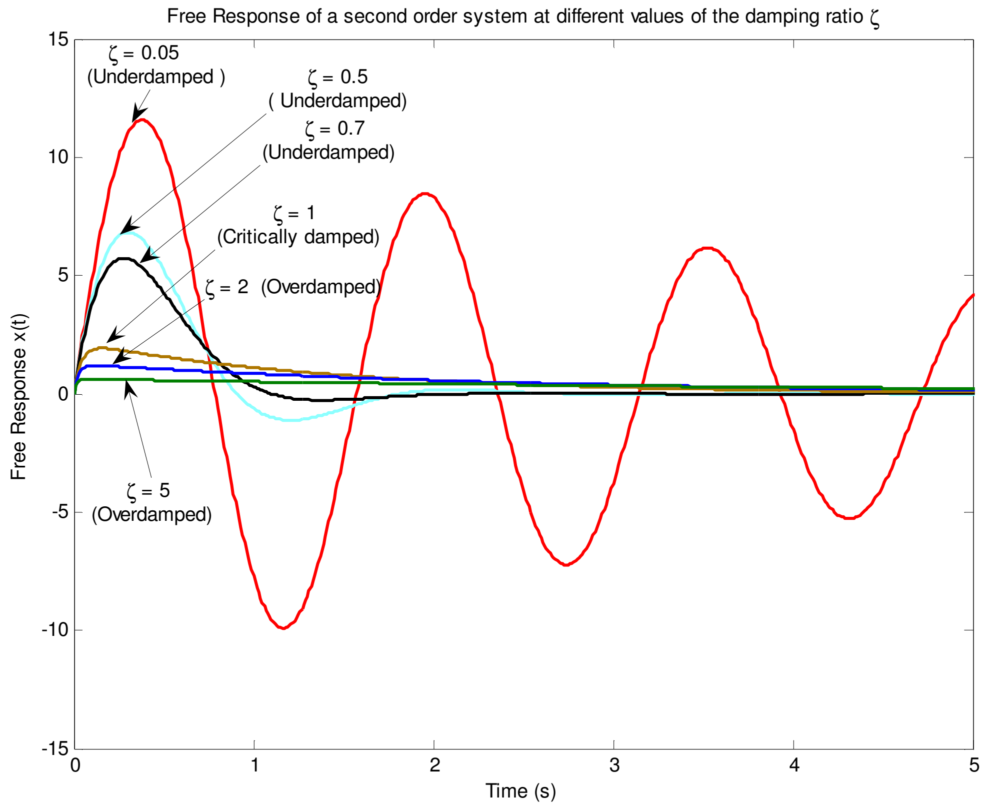
The response in this case *(overdamped case* $\xi > 1$*)* is similar to that of the first order system and is the sum of two exponentials. The first has a time constant $1/s_1$ and the other $1/s_2$. The difference between these two time constants increases as the ξ increases so that the exponential term corresponding to the smaller one (i.e., $1/s_2$) decays much faster than that corresponding to $1/s_1$. Under such case the second order system may be approximated by a first order one with time constant equals to $1/s_1$. From study of the first order system we found that the response remains within $2\ \%$ of its final value in $t > 4$ time constants $= 4\tau$ ($\tau =$ time constant). For a second order underdamped system $\tau = 1/\zeta\omega_n$ and the time required for the solution to remain within $2\ \%$ of its final value is called the settling time T_s which from above is given by

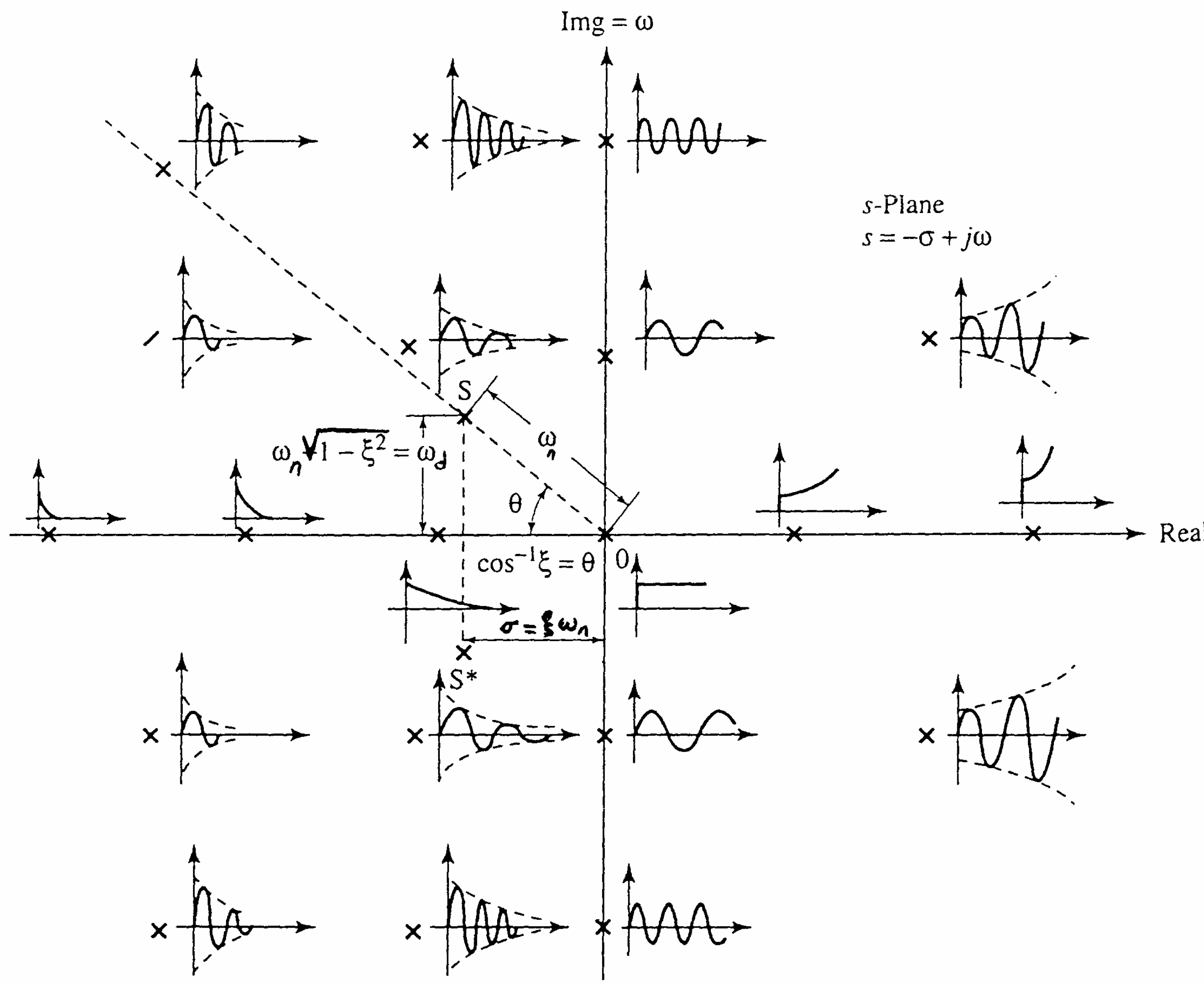
$$T_s = 4\tau = \left(4/\zeta\omega_n\right)$$

Free Response of a Second Order System by MATLAB

MATLAB PROGRAM:

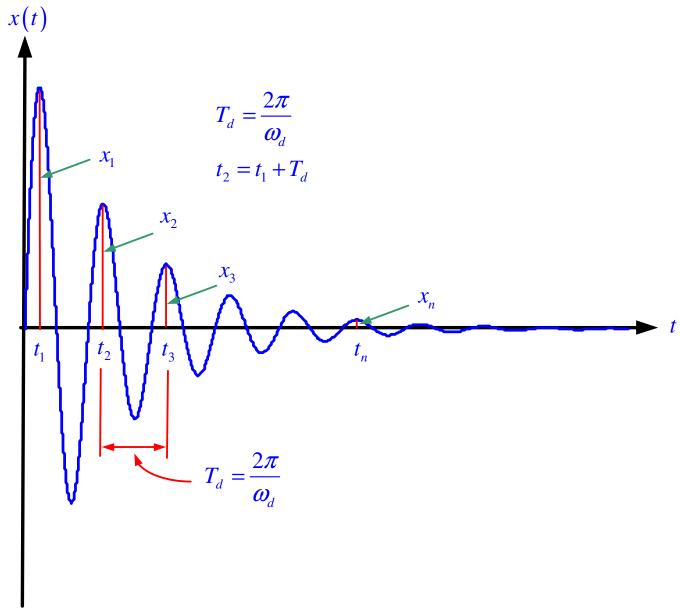
```
>> wn=1;
>> zeta=[0.2   0.5   0.7   1   2   5];
>> for k=1:6
num=[0   0   wn^2];
den=[1   2*zeta(k)   wn^2];
sys=tf(num,den)
impulse(sys);
hold on
>> end
```


Free Response of a second order system at different values of the damping ratio ζ
ζ = 0.05
(Underdamped)
ζ = 0.5
(Underdamped)
ζ = 0.7
(Underdamped)
ζ = 1
(Critically damped)
ζ = 2 (Overdamped)
ζ = 5
(Overdamped)
Free Response x(t)
Time (s)
15
10
5
0
-5
-10
-15
0
1
2
3
4
5



Experimental Determination of damping ratio (Logarithmic Decrement)

It is sometimes necessary to determine the damping ratios and damped natural frequencies of recorders and other instruments. To determine the damping ratio and damped natural frequency of a system experimentally, a record of decaying or damped oscillations, such as that shown in the Figure below is needed.



$$x_1 = x_1(t) = C\, e^{-\xi\omega_n t_1} \cos(\omega_d\, t_1 - \phi)$$
$$x_2 = x_2(t) = C\, e^{-\xi\omega_n t_2} \cos(\omega_d\, t_2 - \phi)$$

The ratio $\frac{x_2}{x_1}$ is equal to

$$\frac{x_2}{x_1} = e^{\xi\omega_n(t_2 - t_1)} \times \frac{\cos(\omega_d\, t_2 - \phi)}{\cos(\omega_d\, t_1 - \phi)}$$

since t_1 and t_2 are selected T_d seconds apart, one can write

$$\begin{aligned}\cos(\omega_d\, t_2 - \phi) &= \cos(\omega_d\,(t_1 + T_d) - \phi)\\ &= \cos(\omega_d\, t_1 + \omega_d T_d - \phi)\\ &= \cos(\omega_d\, t_1 + 2\pi - \phi)\\ &= \cos(\omega_d\, t_1 - \phi)\end{aligned}$$

Hence

$$\frac{x_1}{x_2} = e^{\xi\omega_n T_d}$$

The Logarithmic Decrement δ is defined as the natural logarithm of the ratio of any ***two successive displacement amplitudes*** , so that by taking the natural logarithm of both sides of the above equation

$$\delta = \ln\left(\frac{x_j}{x_{j+1}}\right) = \zeta\, \omega_n\, T_d = \zeta\, \omega_n \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \qquad (*)$$

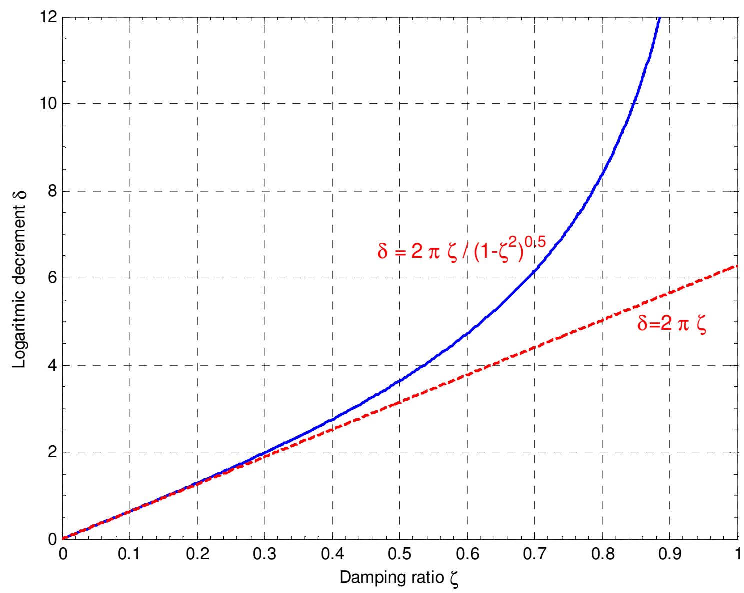
solving the above equation for ζ ,

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}$$

Notice from Eq. (*) that if $\zeta << 1$ (that is, very low damping, which quantitatively means $b << b_{cr}$), $\sqrt{1-\zeta^2} \approx 1$ and thus

$$\delta = 2\pi\zeta \qquad (**)$$

The figure below shows a comparison between Eqs. (*) and (**) versus the damping ratio ζ.



For non-successive amplitudes, say for amplitudes x_1 and x_{n+1}, where n is an integer, we observe that

$$\frac{x_1}{x_{n+1}} = \frac{x_1}{x_2}\frac{x_2}{x_3}\frac{x_3}{x_4}\frac{x_4}{x_5}\cdots\frac{x_n}{x_{n+1}}$$

Taking the natural logarithm of both sides of the above equation gives

$$\ln\left(\frac{x_1}{x_{n+1}}\right) = \ln\left(\frac{x_1}{x_2}\right) + \ln\left(\frac{x_2}{x_3}\right) + \cdots + \ln\left(\frac{x_n}{x_{n+1}}\right)$$
$$= \delta + \delta + \cdots + \delta = n\delta$$

So

$$\delta = \frac{1}{n}\ln\left(\frac{x_1}{x_{n+1}}\right) \qquad \text{or} \qquad \delta = \frac{1}{n-1}\ln\left(\frac{x_1}{x_n}\right)$$

Table-1 Logarithmic decrement for Various Types of Structures

Types of Structures	Approximate Range of

	Logarithmic Decrement, δ
Multistory Steel Buildings	0.02 → 0.10
Steel Bridges	0.05 → 0.15
Multistory Concrete Buildings	0.10 → 0.20
Concrete Bridges	0.1 0→ 0.30
Machinery Foundations	0.4 0→ 0.60

Example

: Logarithmic Decrement and Damping Coefficient

The following data are given for the mass spring dashpot model sketched in Figure 8-9: mass = 5 kg, spring constant = 2 kN/m. It is found experimentally that the motion of the system is periodically oscillatory and the amplitude of the oscillation on the fifth cycle is one-third of its initial value. Find the dashpot constant c of the system.

Because the amplitude x_5 is equal to $\frac{x_1}{3}$, substituting $\frac{x_1}{x_5} = 3$ and $n = 5$ into Eq. (8-142) gives

$$\delta = \frac{1}{4}\ln 3 = 0.275 \qquad \text{(a)}$$

The damping factor ζ can be found by substituting the result in Eq. (a) into Eq. (8-136) as

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} = \frac{0.275}{\sqrt{4\pi^2 + 0.275^2}} = 0.0437 \qquad \text{(b)}$$

The dashpot constant c can be found from Eqs. (8-83) and (8-84) as

$$c = c_c\zeta = 2m\omega_n\zeta = 2m\sqrt{\frac{k}{m}}\zeta = 2\zeta\sqrt{\frac{km^2}{m}} = 2\zeta\sqrt{km} \qquad \text{(c)}$$

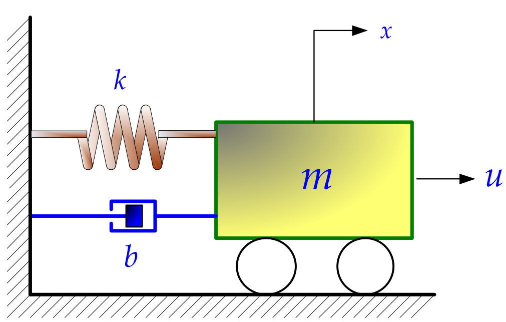
Substituting the given values for m and k and the computed value for ζ into Eq. (c) gives

$$c = 2(0.0437)\sqrt{(2000)(5)} = 8.74 \ \ (\text{N}\cdot\text{s/m}) \qquad \text{(d)}$$

which is the sought-after dashpot constant.

Step Response of a Second Order system:

Consider the mechanical system shown in the Figure below. Assume that the system is at rest for $t < 0$. At $t = 0$, the force $u = a \bullet 1(t)$ [where a is a constant and $1(t)$ is a step force of magnitude 1 N] is applied to the mass m. The displacement is measured from the equilibrium position before the input force u is applied. Assume that the system is underdamped $(\zeta < 1)$



The equation of motion for the system is

$$m\,\ddot{x} + b\dot{x} + k\,x = a \bullet 1(t)$$

The TF for the system is

$$\frac{X(s)}{U(s)} = \frac{1}{m\,s^2 + bs + k}$$

Hence

$$\frac{X(s)}{\mathscr{L}\left[1(t)\right]} = \frac{a}{m\,s^2 + bs + k} = \frac{\frac{a}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}}$$

Define

$$\omega_n = \sqrt{\frac{k}{m}} = \text{undamped natural frequency, } \left[\text{rad/s}\right]$$

and

$$\zeta = \text{dampin gratio} = \frac{\text{actual damping value}}{\text{critical damping value}} = \frac{b}{2\sqrt{km}}$$

Then

$$\frac{X(s)}{\mathscr{L}\left[1(t)\right]} = \frac{a}{m\omega_n^2}\left(\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}\right) \tag{8-16}$$

Hence,

$$X(s) = \frac{a}{m\omega_n^2}\left(\frac{1}{s}\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right)$$

$$= \frac{a}{m\omega_n^2}\left(\frac{1}{s} - \frac{s + 2\zeta\omega_n}{\underbrace{s^2 + 2\zeta\omega_n s + \omega_n^2}_{\Downarrow}}\right)$$

$$= \left(s + \zeta\omega_n\right)^2 + \left(\omega_n\sqrt{1-\zeta^2}\right)^2 = \left(s + \zeta\omega_n\right)^2 + \omega_d^2$$

$$= \frac{a}{m\omega_n^2}\left(\frac{1}{s} - \frac{\zeta\omega_n}{\left(s + \zeta\omega_n\right)^2 + \omega_d^2} - \frac{s + \zeta\omega_n}{\left(s + \zeta\omega_n\right)^2 + \omega_d^2}\right)$$

$$= \frac{a}{m\omega_n^2}\left(\frac{1}{s} - \frac{1}{\omega_d}\frac{\zeta\omega_n * \omega_d}{\left(s + \zeta\omega_n\right)^2 + \omega_d^2} - \frac{s + \zeta\omega_n}{\left(s + \zeta\omega_n\right)^2 + \omega_d^2}\right)$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$. The inverse Laplace transform of the last equation gives

$$x(t) = \frac{a}{m\omega_n^2}\left(1 - \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin\omega_d t - e^{-\zeta\omega_n t}\cos\omega_d t\right)$$

$$= \frac{a}{m\omega_n^2}\left[1 - e^{-\zeta\omega_n t}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t + \cos\omega_d t\right)\right]$$

$$= \frac{a}{m\omega_n^2}\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right]$$

The response starts from $x(0)=0$ and reaches $x(\infty) = a/m\omega_n^2$. The general shape of the response curve is shown in the figure below.

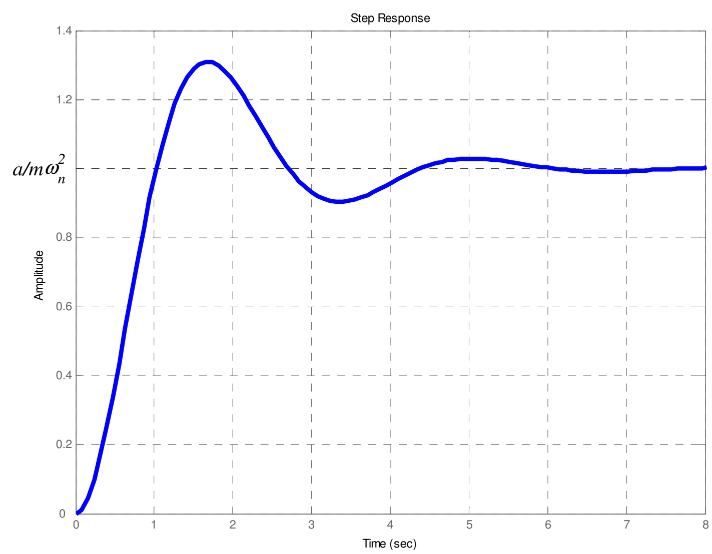
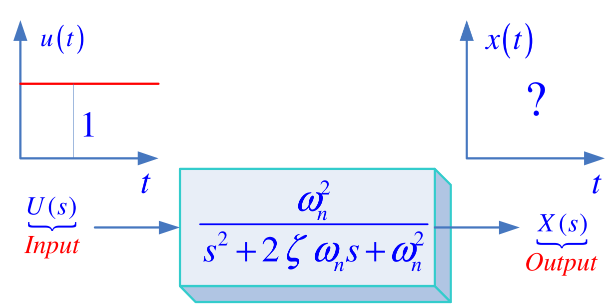


Figure 8-11 Step response of a second order system. The response curve shown corresponds to the case where $\zeta = 0.7$ and $\omega_n = 2$ rad/s.

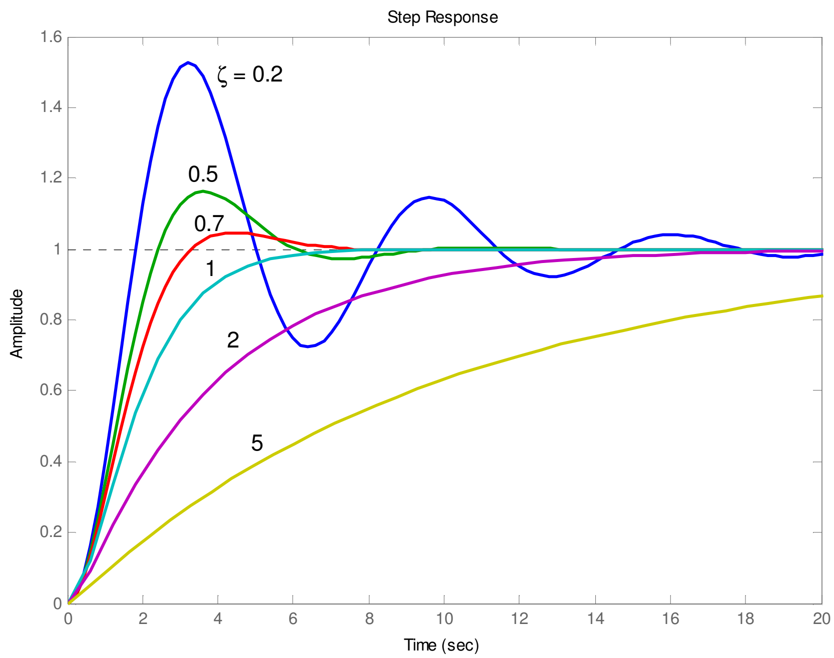
Assume that the system is underdamped $(\zeta < 1)$

MATLAB PROGRAM:	

`>> wn=1;`
`>> zeta=[0.2 0.5 0.7 1 2 5];`
`>> for k=1:6`
`num=[0 0 wn^2];`
`den=[1 2*zeta(k) wn^2];`
`sys=tf(num,den)`
`step(sys);`
`hold on`
	

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

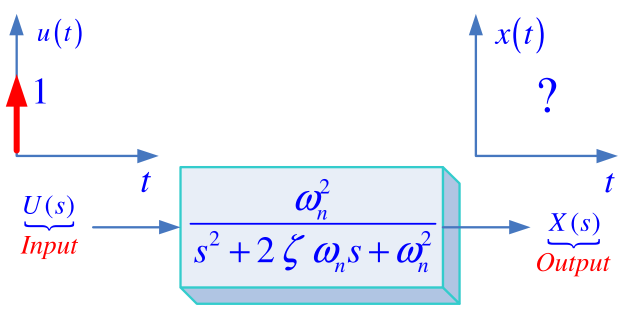
>> end	



Impulse Response of a Second Order system:

MATLAB PROGRAM:

```
>> wn=1;
>> zeta=[0.2  0.5  0.7  1  2  5];
>> for k=1:6
       num=[0  0  wn^2];
       den=[1  2*zeta(k)  wn^2];
       sys=tf(num,den)
       impulse(sys);
       hold on
>> end
```



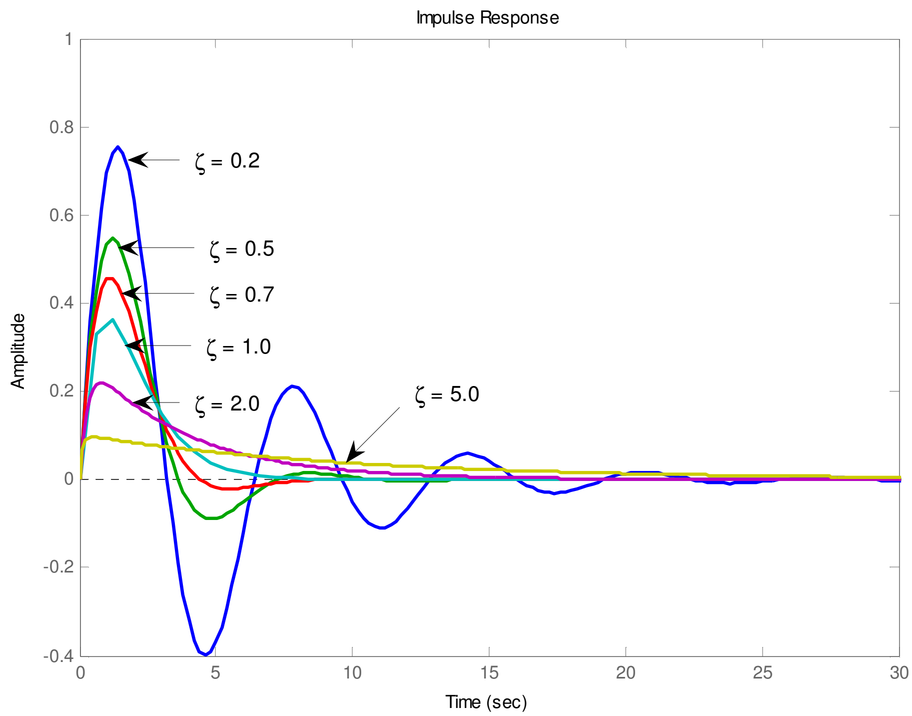

Impulse Response
Amplitude
Time (sec)
ζ = 0.2
ζ = 0.5
ζ = 0.7
ζ = 1.0
ζ = 2.0
ζ = 5.0
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
0
5
10
15
20
25
30

Table 8.1 The Response of the First Order Linear System

$\dot{y} + \frac{1}{\tau} y = u_\circ(t), \quad y(0) = y_\circ$ where $y(t)$ = output and $u_\circ(t)$ = input

Input	Response $y(t)$ if $y(0) = y_\circ$	Response $y(t)$ if $y(0) = 0$
$u_\circ(t) = 0$	$y(t) = y_\circ\, e^{-(t/\tau)}$	-
$u_\circ(t) = B\,t$ (Ramp of Slope B)	$y(t) = B\,\tau\left(t - \tau + \tau e^{-(1/\tau)t}\right) + y_\circ\, e^{-(t/\tau)}$	$y(t) = B\,\tau\left(t - \tau + \tau e^{-(1/\tau)t}\right)$
$u_\circ(t) = B$, (Step of magnitude B)	$y(t) = y_\circ\, e^{-(t/\tau)} + B\,\tau\left(1 - e^{-(1/\tau)t}\right)$	$y(t) = B\,\tau\left(1 - e^{-(1/\tau)t}\right)$
$u_\circ(t) = B * \delta(t)$ (Impulse of magnitude B)	$y(t) = (B + y_\circ)\, e^{-(1/\tau)t}$	$y(t) = B\, e^{-(1/\tau)t}$

Table 8.2 The Free Response of the Second Order Linear System

$$m\ddot{x}+b\dot{x}+kx=0,\quad x(0)=x_\circ \text{ and } \dot{x}(0)=\dot{x}_\circ$$

The natural frequency $\omega_n=\sqrt{\dfrac{k}{m}}$ rad/s

The damping ratio $\zeta=\dfrac{b}{b_{cr}}=\dfrac{b}{2\sqrt{mk}}$

The damped frequency $\omega_d=\omega_n\sqrt{1-\zeta^2}$ for $0\le\zeta<1$

Damping ratio	**Response** $y(t)$
$0\le\zeta<1$	$x(t)=e^{-\zeta\omega_n t}\left\{\left[\dfrac{\zeta}{\sqrt{1-\zeta^2}}x_\circ+\dfrac{\dot{x}_\circ}{\omega_d}\right]\sin\omega_d t+x_\circ\cos\omega_d t\right\}$ If $\dot{x}(0)=\dot{x}_\circ=0$, this simplifies to $x(t)=x_\circ e^{-\zeta\omega_n t}\left\{\dfrac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t+\cos\omega_d t\right\}$ $=\dfrac{x_\circ}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left\{\omega_d t+\tan^{-1}\dfrac{\sqrt{1-\zeta^2}}{\zeta}\right\}$ $=\dfrac{x_\circ}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos\left\{\omega_d t-\tan^{-1}\dfrac{\zeta}{\sqrt{1-\zeta^2}}\right\}$
$\zeta=1$	$x(t)=x_\circ e^{-\omega_n t}+(\omega_n x_\circ+\dot{x}_\circ)te^{-\omega_n t}$ If $\dot{x}(0)=\dot{x}_\circ=0$, this simplifies to $x(t)=x_\circ(1+\omega_n t)e^{-\omega_n t}$
$\zeta>1$	$x(t)=\left\{\dfrac{-\zeta+\sqrt{\zeta^2-1}}{2\sqrt{\zeta^2-1}}x_\circ-\dfrac{\dot{x}_\circ}{2\omega_n\sqrt{\zeta^2-1}}\right\}e^{-\left(\zeta\omega_n+\omega_n\sqrt{\zeta^2-1}\right)t}$ $+\left\{\dfrac{\xi+\sqrt{\zeta^2-1}}{2\sqrt{\zeta^2-1}}x_\circ+\dfrac{\dot{x}_\circ}{2\omega_n\sqrt{\zeta^2-1}}\right\}e^{-\left(\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}\right)t}$ If $\dot{x}(0)=\dot{x}_\circ=0$, this simplifies to $x(t)=\dfrac{x_\circ}{2\sqrt{\zeta^2-1}}\left\{\left(-\zeta+\sqrt{\zeta^2-1}\right)e^{-\left(\zeta\omega_n+\omega_n\sqrt{\zeta^2-1}\right)t}+\left(\xi+\sqrt{\zeta^2-1}\right)e^{-\left(\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}\right)t}\right\}$

Table 8.3 The Forced Response of the Second Order Linear System

$$m\ddot{x}+b\dot{x}+kx=f(t),\quad x(0)=0 \text{ and } \dot{x}(0)=0$$

The natural frequency $\omega_n=\sqrt{\frac{k}{m}}$ rad/s

The damping ratio $\zeta=\frac{b}{b_{cr}}=\frac{b}{2\sqrt{mk}}$

The damped frequency $\omega_d=\omega_n\sqrt{1-\zeta^2}$ for $0\le\zeta<1$

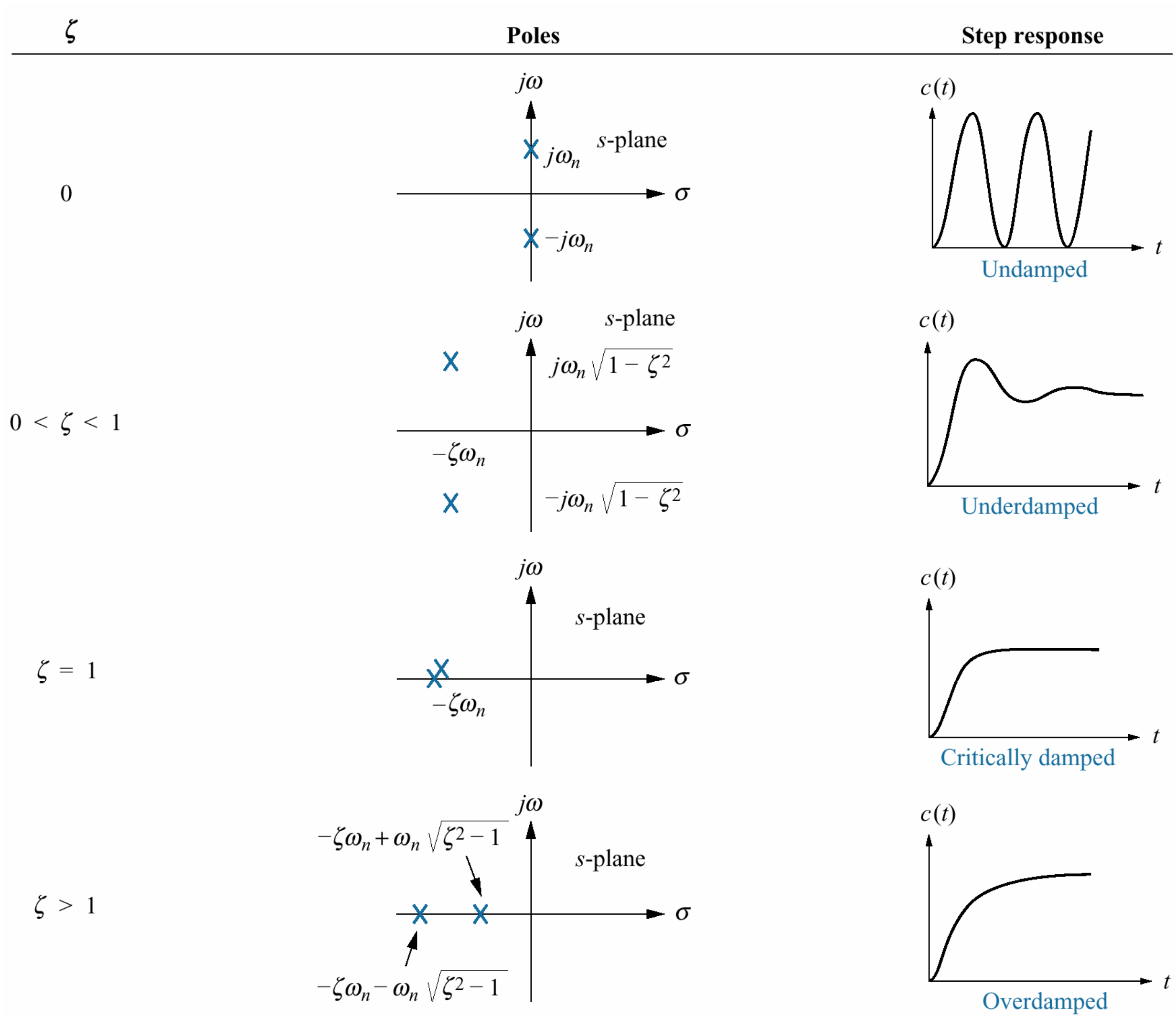
The phase angle $\psi=\tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$ for $0\le\zeta<1$

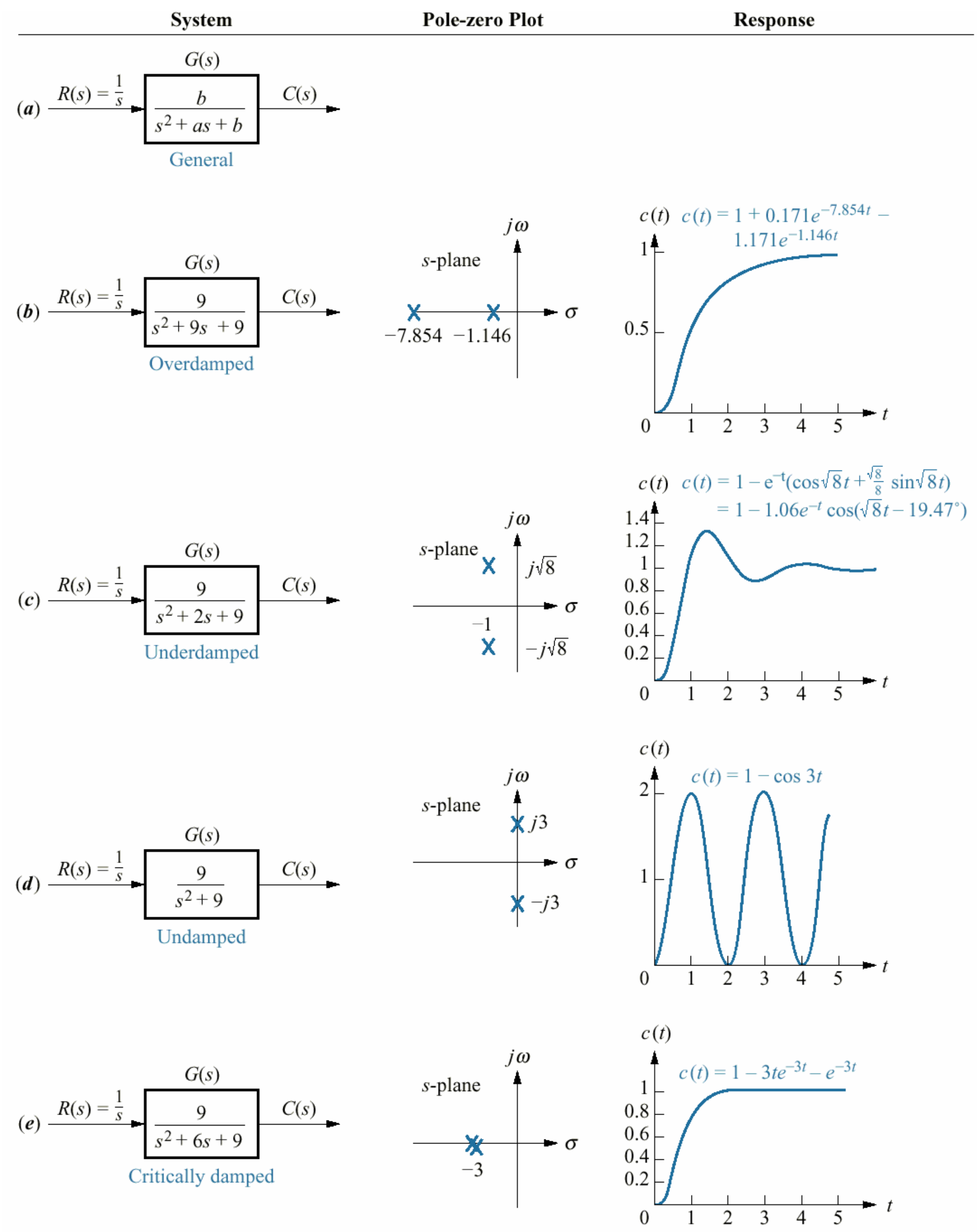
For overdamped systems $\zeta>1$, the time constants are $\tau_1=1/\left(\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}\right)$ and $\tau_2=1/\left(\zeta\omega_n+\omega_n\sqrt{\zeta^2-1}\right)$,

Inputs $\begin{cases} u_r(t)\equiv\text{ramp with slope 1}\\ u_s(t)\equiv\text{unit step}\\ \delta(t)\equiv\text{unit impulse}\end{cases}$

Damping ratio	Input	Response $y(t)$
$0\le\zeta<1$	$f(t)=u_r(t)$	$y_r(t)=\frac{1}{\omega_n^2}\left[t+\frac{e^{-\zeta\omega_n t}}{\omega_n}\left(2\zeta\cos\omega_d t+\frac{2\zeta^2-1}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)-\frac{2\zeta}{\omega_n}\right]$
	$f(t)=u_s(t)$	$y_s(t)=\frac{1}{\omega_n^2}\left[1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos(\omega_d t-\psi)\right]$
	$f(t)=\delta(t)$	$y_\delta(t)=\frac{e^{-\zeta\omega_n t}}{\omega_n\sqrt{1-\zeta^2}}\sin\omega_d t$
$\zeta=1$	$f(t)=u_r(t)$	$y_r(t)=\frac{1}{\omega_n^2}\left[t+\frac{2}{\omega_n}e^{-\omega_n t}+te^{-\omega_n t}-\frac{2}{\omega_n}\right]$
	$f(t)=u_s(t)$	$y_s(t)=\frac{1}{\omega_n^2}\left[1-e^{-\omega_n t}-\omega_n te^{-\omega_n t}\right]$
	$f(t)=\delta(t)$	$y_\delta(t)=te^{-\omega_n t}$
$\zeta>1$	$f(t)=u_r(t)$	$y_r(t)=\frac{1}{\omega_n^2}\left[t+\frac{\omega_n}{2\sqrt{1-\zeta^2}}\left(\tau_1^2e^{-t/\tau_1}-\tau_2^2e^{-t/\tau_2}\right)-\frac{2\zeta}{\omega_n}\right]$

	$f(t) = u_s(t)$	$y_s(t) = \frac{1}{\omega_n^2}\left[1 - \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}\left(\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2}\right)\right]$
	$f(t) = \delta(t)$	$y_\delta(t) = \frac{1}{2\omega_n\sqrt{\zeta^2 - 1}}\left(e^{-t/\tau_1} - e^{-t/\tau_2}\right)$




System
Pole-zero Plot
Response
(a)
$R(s) = \frac{1}{s}$
G(s)
$\frac{b}{s^2 + as + b}$
C(s)
General
(b)
$R(s) = \frac{1}{s}$
G(s)
$\frac{9}{s^2 + 9s + 9}$
C(s)
Overdamped
s-plane
jω
σ
−7.854 −1.146
c(t)
$c(t) = 1 + 0.171e^{-7.854t} - 1.171e^{-1.146t}$
(c)
$R(s) = \frac{1}{s}$
G(s)
$\frac{9}{s^2 + 2s + 9}$
C(s)
Underdamped
s-plane
jω
$j\sqrt{8}$
−1
$-j\sqrt{8}$
σ
c(t)
$c(t) = 1 - e^{-t}(\cos\sqrt{8}t + \frac{\sqrt{8}}{8}\sin\sqrt{8}t)$
$= 1 - 1.06e^{-t}\cos(\sqrt{8}t - 19.47^\circ)$
(d)
$R(s) = \frac{1}{s}$
G(s)
$\frac{9}{s^2 + 9}$
C(s)
Undamped
s-plane
jω
j3
−j3
σ
c(t)
$c(t) = 1 - \cos 3t$
(e)
$R(s) = \frac{1}{s}$
G(s)
$\frac{9}{s^2 + 6s + 9}$
C(s)
Critically damped
s-plane
jω
−3
σ
c(t)
$c(t) = 1 - 3te^{-3t} - e^{-3t}$
t