Chapter 7

Fluid Systems and Thermal Systems

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7.1 INTRODUCTION

A fluid system uses one or more fluids to achieve its purpose. Dampers and shock absorbers are examples of fluid systems because they depend on the viscous nature of a fluid to provide damping. In addition to providing damping, other applications of fluid systems include actuators and processes that involve mixing, heating, and cooling of fluids.

Active vehicle suspensions use hydraulic and pneumatic actuators to provide forces to supplement the passive spring and damping elements. Water supply, waste treatment, and other chemical processing applications are examples of a general category of fluid systems called *"liquid-level-systems"*, because they involve regulating the volumes, and therefore the levels of liquids in containers such as tanks.

A fluid might be either a *liquid* or a *gas*. A fluid is said to be *incompressible* if the fluid's *density remains constant* despite changes in the fluid pressure. If the *density changes with pressure*, the fluid is compressible.

7.2 MATHEMATICAL MODELING OF LIQUID LEVEL SYSTEMS

Steady State Flow

Laminar

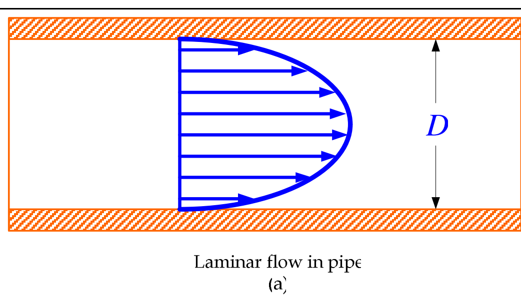


Figure 7.1 (a) Velocity profile for laminar flow

Flow dominated by viscosity forces is called

Turbulent

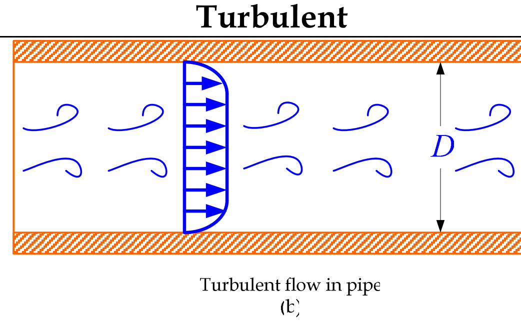


Figure 7.1 (b) Velocity profile for turbulent flow

When inertia forces dominate, the flow is

laminar flow and is characterized by a smooth, parallel line motion of the fluid and low Reynolds number $\mathrm{Re} = \frac{\rho v D}{\mu} < 2000$ where ρ is the mass density of the fluid, μ is the dynamic viscosity of the fluid, v is the average velocity of flow, and D is characteristic length. Friction force is linearly proportional to velocity, $f_f = bv$	called *turbulent flow* and is characterized by an irregular and eddylike motion of the fluid and High Reynolds number $\mathrm{Re} > 4000$. Friction force varies as a power of velocity $f_f = bv^{\alpha}$

Resistance and Capacitance of Liquid-Level Systems.

Consider the flow through a short pipe connecting two tanks as shown in Figure 7-2. The *resistance* for liquid flow in such a pipe or restriction is defined as the change in the level difference (the difference of the liquid levels of the two tanks) necessary to cause a unit change in flow rate; that is,

$$\text{Resistance } R = \frac{\text{Change in level difference}}{\text{Change in flow rate}} \equiv \frac{\text{m}}{\text{m}^3/s}$$

$$R = \frac{\Delta(H_1 - H_2)}{\Delta Q}$$

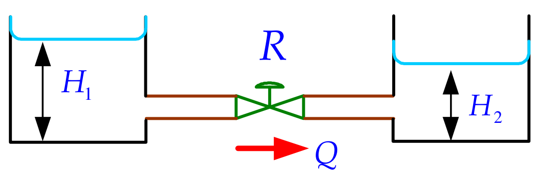


Figure 7-2 Two tanks connected by a short pipe with a valve

Since the relationship between the flow rate and the level difference differs from laminar flow and turbulent flow, we shall consider both cases in what fallows.

Resistance in Laminar Flow.

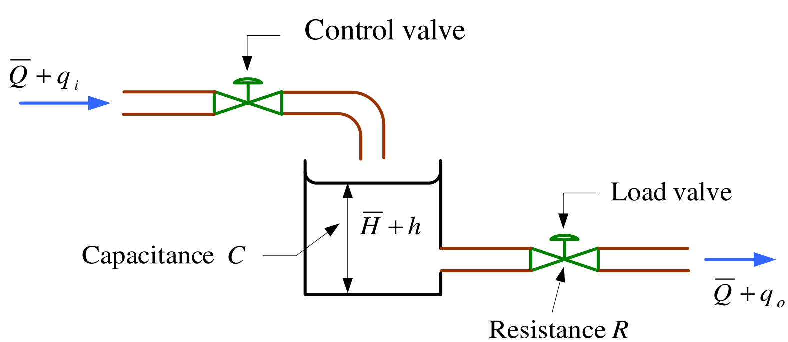


Figure 7-3 (a) Liquid level system;

For *laminar flow*, ($\mathrm{Re} < 2000$), the relationship between the steady-state flow rate and steady-state head at the level of restriction is given by

$$Q = K_l H$$

Where $Q =$ steady-state liquid flow rate, m^3/s, $K_l =$ constant, m^2/s and $H =$ steady-state head, m.

For *laminar flow*, the resistance R_l is

$$R_l = \frac{\mathrm{d}H}{\mathrm{d}Q} = \frac{1}{K_l} = \frac{H}{Q}$$

The laminar-flow resistance is constant and is analogous to the electrical resistance, where. $\text{Height} \quad (H) \rightarrow \text{Voltage} \quad (e)$ and $\text{Steady state flow rate} \quad (Q) \rightarrow \text{Current} \quad (i)$

Resistance in Turbulent Flow.

For *turbulent flow*; ($\mathrm{Re} > 3000$), the steady-state flow rate is given by

$$Q = K_t \sqrt{H} \qquad (7\text{-}1)$$

where $Q =$ steady-state liquid flow rate, m^3/s, $K_t =$ constant, $\mathrm{m}^{2.5}/s$ and $H =$ steady-state head, m.

The resistance R_t for turbulent flow is obtained from

$$R_t = \frac{\mathrm{d}H}{\mathrm{d}Q}$$

Then

$$\mathrm{d}Q = \frac{K_t}{2\sqrt{H}}\mathrm{d}H \Rightarrow \frac{\mathrm{d}Q}{\mathrm{d}H} = \frac{K_t}{2\sqrt{H}} \Rightarrow \frac{\mathrm{d}H}{\mathrm{d}Q} = \frac{2\sqrt{H}}{(Q/\sqrt{H})} = \frac{2H}{Q}$$

Thus

$$R_t = \frac{2H}{Q} \qquad (7\text{-}2)$$

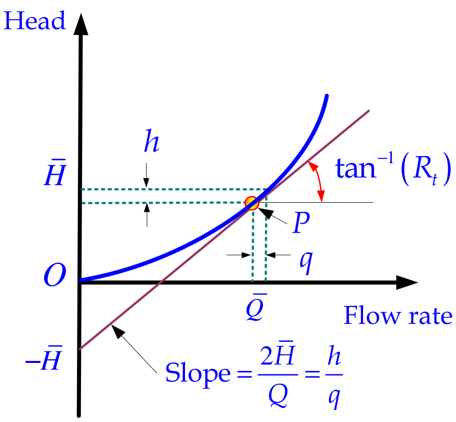


Figure 7-3 (b) curve of head versus flow rate

Capacitance

The *capacitance* of a tank is defined to be the change in quantity of stored liquid necessary to cause a unity change in the potential (head). The potential (head) is the quantity that includes the energy level of the system).

$$\text{Capacitance } C = \frac{\text{Change in liquid stored}}{\text{Change in head}} \equiv \frac{\text{m}^3}{\text{m}} \text{ or } \text{m}^2$$
$$\text{Capacitance } C = \text{Cross-Sectional area (A) of the tank.}$$

or

$$\text{Rate of change of fluid volume in the tank} \equiv \text{flow in} - \text{flow out}$$

$$\frac{dV}{dt} = q_{in} - q_{out} \quad \Rightarrow \quad \frac{d(A \times h)}{dt} = q_{in} - q_{out} \quad \Rightarrow \quad A\frac{dh}{dt} = q_{in} - q_{out}$$
$$C\frac{dh}{dt} = q_{in} - q_{out}$$

$$C\frac{dh}{dt} = q_{in} - q_{out}$$

Mathematical Modeling of Liquid-level Systems.

Consider the system shown in figure 7-3(a). If the operating condition as to the head and flow rate varies little for the period considered, a mathematical model can easily be found in terms of resistance and capacitance. Assume turbulent flow, and define

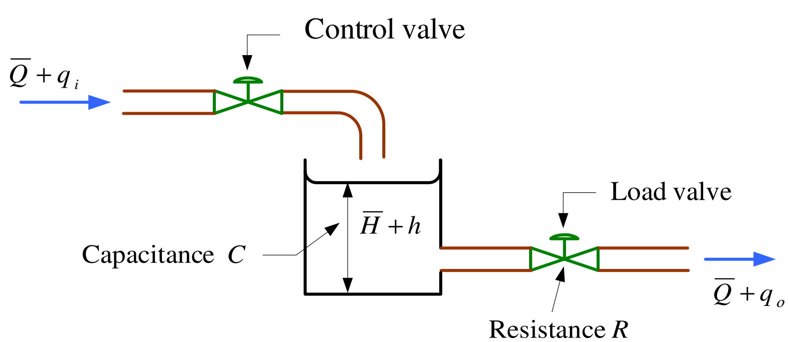


Figure 7-3 (a) Liquid level system;

$\bar{H} =$ steady-state head (before any change has occurred), m.
$h =$ small deviation of head from its steady-state value, m.
$\bar{Q} =$ steady-state flow rate (before any change has occurred), m³/s.
$q_i =$ small deviation of inflow rate from its steady-state value, m³/s.
$q_o =$ small deviation of outflow rate from its steady-state value, m³/s.

Tank: The rate of change in liquid stored in the tank is equal to the flow in minus flow out

$$C\frac{dh}{dt} = q_i - q_o$$

or

$$C\,dh = (q_i - q_o)\,dt \tag{7-3}$$

where C is the capacitance of the tank. In the present system, we define h and q_o as small deviations from steady state head and steady state outflow rate, respectively. Thus,
$dH = h, \qquad dQ = q_o$

Resistance R: The resistance R may be written as

$$R = \frac{dH}{dQ} = \frac{h}{q_o} \quad \Rightarrow \quad q_o = \frac{h}{R}$$

Substitute $q_o = \dfrac{h}{R}$ into Equation (7-3), we obtain

$$C\frac{dh}{dt} = q_i - \frac{h}{R}$$

or

$$RC\frac{dh}{dt} + h = Rq_i \tag{7-4}$$

Notice that RC is the time constant of the system. Equation (7-4) is a linearized mathematical model for the system when h is considered the system output. If q_o rather than h, is considered the system output, then substituting $h = Rq_o$ in the above equation gives

$$RC\frac{dq_o}{dt} + q_o = q_i \tag{7-5}$$

Analogous Systems.

The liquid level system considered here is analogous to the electrical system shown in Figure 7-4(a). It is also analogous to the mechanical system shown in Figure 7-4(b). For the electrical system, a mathematical model is

$$RC\frac{de_o}{dt} + e_o = e_i \tag{7-6}$$

For the mechanical system, a mathematical model is

$$\frac{b}{k}\frac{dx_o}{dt} + x_o = x_i \tag{7-7}$$

Equations (7-5), (7-6) and (7-7) are of the same form; thus they are analogous.

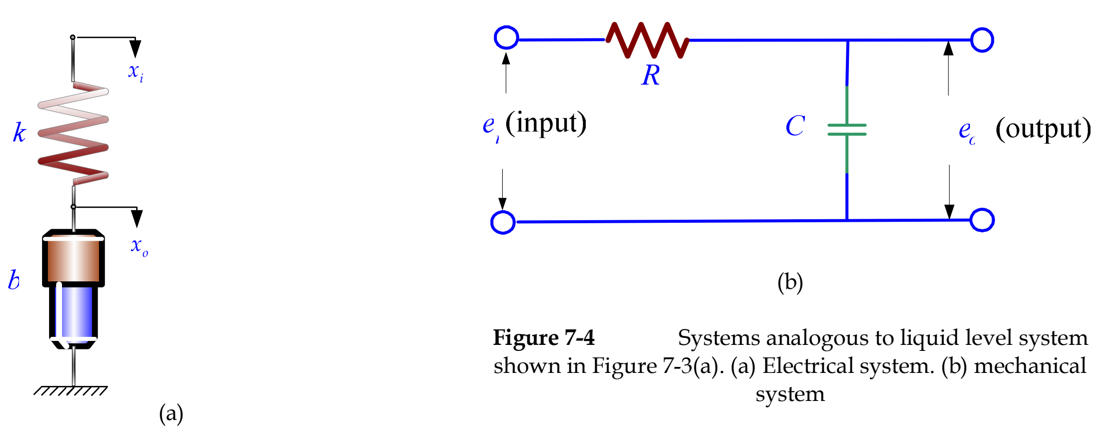
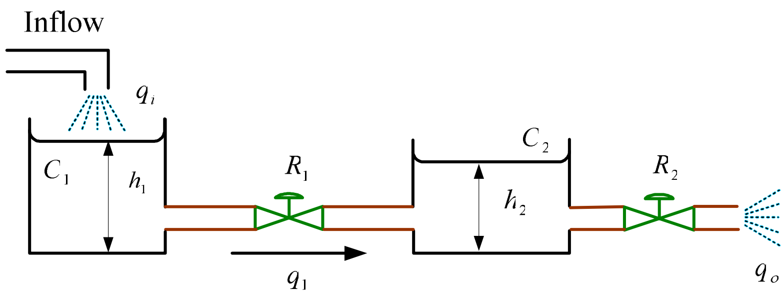


Figure 7-4 Systems analogous to liquid level system shown in Figure 7-3(a). (a) Electrical system. (b) mechanical system

Liquid-Level System with Interaction.

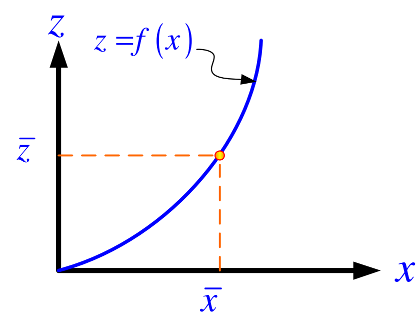


7.3 LINEARIZATION OF NONLINEAR SYSTEMS

Linearization of $z = f(x)$ about a point $(\bar{x}, \bar{z})$.

Consider a nonlinear system whose input is x and output is z, the relationship between z and x may be written as

$$z = f(x) \tag{7-21}$$



If the normal operating condition corresponds to a point $(\bar{x}, \bar{z})$, then Equation (7-21) can be expanded into a Taylor series about this point as follows:

$$z = f(x) = f(\bar{x}) + \left.\frac{df}{dx}\right|_{x=\bar{x}} (x-\bar{x}) + \frac{1}{2!}\left.\frac{d^2 f}{dx^2}\right|_{x=\bar{x}} (x-\bar{x})^2 + \cdots \tag{7-22}$$

where the derivatives df/dx, $d^2 f/dx^2$ are evaluated at the operating point, $x = \bar{x}$, $z = \bar{z}$. If the variation $(x-\bar{x})$ is small, we can neglect the higher-order terms in $(x-\bar{x})$. Noting that $\bar{z} = f(\bar{x})$, Equation (7-22) can be written

$$z = f(\bar{x}) + \left.\frac{df}{dx}\right|_{x=\bar{x}} (x-\bar{x})$$

Noting that $\bar{z} = f(\bar{x})$, we can write Equation (7-22) as

$$z - \bar{z} = m(x-\bar{x}) \tag{7-23}$$

where

$$m = \left.\frac{df}{dx}\right|_{x=\bar{x}}$$

Equation (7-23) indicates that $z-\bar{z}$ is proportional to $x-\bar{x}$. The equation is a linear mathematical model for the nonlinear system given by Equation (7-21) near the operating point $x-\bar{x}$, $z-\bar{z}$. Equation (7-23) represents an equation of the tangent line to the curve $z=f(x)$ at the operating point $(\bar{x},\bar{z})$ with a slope m

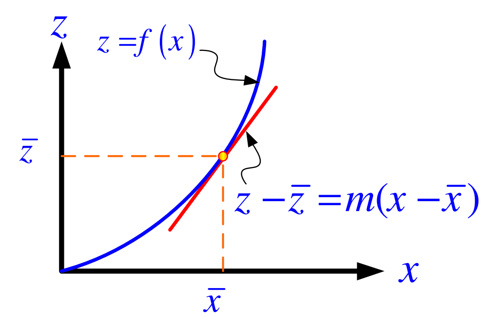


Figure 1. Linearization of the function $y=f(x)$ about the point $(\bar{x},\bar{z})$

Linearization of $z=f(x,y)$ about a point $(\bar{x},\bar{y},\bar{z})$.

Next, consider a nonlinear system whose output z is function of two inputs x and y such that

$$z=f(x,y) \tag{7-24}$$

To obtain a linear mathematical model for this nonlinear system about an operating point $(\bar{x},\bar{y},\bar{z})$, we expand Equation (7-24) into a Taylor series about this point as follows:

$$z=f(\bar{x},\bar{y})+\left[\frac{\partial f}{\partial x}(x-\bar{x})+\frac{\partial f}{\partial y}(y-\bar{y})\right]$$
$$+\frac{1}{2!}\left[\frac{\partial^2 f}{\partial x^2}(x-\bar{x})^2+2\frac{\partial^2 f}{\partial x\ \partial y}(x-\bar{x})(y-\bar{y})+\frac{\partial^2 f}{\partial y^2}(y-\bar{y})^2\right]+\ldots$$

where the partial derivatives are evaluated at the operating point, $x=\bar{x}$, $y=\bar{y}$, and $z=\bar{z}$. Near this point, the higher-order terms may be neglected. Noting that $\bar{z}=f(\bar{x},\bar{y})$, a linear mathematical model of this nonlinear system near the operating point $x=\bar{x}$, $y=\bar{y}$, and $z=\bar{z}$ is

$$z-\bar{z}=m(x-\bar{x})+n(y-\bar{y})$$

where

$$m=\frac{\partial f}{\partial x}\Big|_{x=\bar{x},\ y=\bar{y}}$$
$$n=\frac{\partial f}{\partial y}\Big|_{x=\bar{x},\ y=\bar{y}}$$

Example 7-3 (Textbook Page 336)

Linearize the nonlinear equation $z=xy$ in the region $5\le x\le 7,\ 10\le y\le 12$. Find the error if the linearized equation is used to calculate the value of z when $x=5$ and $y=10$.

Solution

Since the region considered is given by $5 \le x \le 7,\ 10 \le y \le 12$, choose $\bar{x} = 6,\ \bar{y} = 11$. Then $\bar{z} = \bar{x}\,\bar{y} = 66$. Let us obtain a linearized equation for the nonlinear equation near a point $\bar{x} = 6,\ \bar{y} = 11$, and $\bar{z} = 66$.

Expanding the nonlinear equation into a Taylor's series about the point $x = \bar{x}$, $y = \bar{y}$ and $z = \bar{z}$ and neglecting the higher order terms, we have

$$z - \bar{z} = m(x - \bar{x}) + n(y - \bar{y})$$

where

$$m = \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x},\ y=\bar{y}} = \frac{\partial}{\partial x}[xy]_{x=\bar{x},\ y=\bar{y}} = y|_{x=\bar{x},\ y=\bar{y}} = \bar{y} = 11$$

$$m = \left.\frac{\partial f}{\partial y}\right|_{x=\bar{x},\ y=\bar{y}} = \frac{\partial}{\partial y}[xy]_{x=\bar{x},\ y=\bar{y}} = x|_{x=\bar{x},\ y=\bar{y}} = \bar{x} = 6$$

Hence the linearized equation is

$$z - 66 = 11(x-6) + 6(y-11)$$

or

$$z = 11x + 6y - 66$$

When $x = 5$ and $y = 10$, the value of z given by the linearized equation is

$$z = 11x + 6y - 66 = 11\times(5) + 6(10) - 66 = 55 + 60 - 66 = 49$$

The exact value is $z = xy = (5)\times(10) = 55$. The error is thus $50 - 49 = 1$. In terms of percentage, the error is $z = \dfrac{50-49}{49}\times 100 = 2\%$

Example 7-4 (Textbook Page 336)

Consider the liquid level-system shown in Figure 7-8. At steady state, the inflow rate is $Q_i = \bar{Q}$, the outflow rate is $Q_o = \bar{Q}$, and the head is $H = \bar{H}$. Assume that the flow is turbulent. Then

$$Q_o = K\sqrt{H}$$

For this system, we have

$$C\frac{dH}{dt} = Q_i - Q_o = Q_i - K\sqrt{H}$$

Where C is the capacitance of the tank. Let us define

$$\frac{dH}{dt} = \frac{1}{C}Q_i - \frac{K\sqrt{H}}{C} = f(H, Q_i) \qquad (7\text{-}25)$$

Assume that the system operates near the steady-state condition $\left(\bar{H},\bar{Q}\right)$. That is , $H=\bar{H}+h$ and $Q=\bar{Q}+q_i$, where h and q_i are small quantities (either positive or negative). At steady-state operation, $dH/dt=0$. Hence, $f\left(\bar{H},\bar{Q}\right)=0$.

Solution

Since the region considered is given by