CHAPTER 4 TRANSFER FUNCTION APPROACH TO MODELING DYNAMIC SYSTEMS

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4.1 INTRODUCTION

Transfer functions (TF)are frequently used to characterize the input-output relationships or systems that can be described by Linear Time-Invariant (LTI) differential equations.

TRANSFER FUNCTION (TF). ***The transfer function (TF)*** of a LTI differential-equation system is defined as the ratio of the Laplace transform (LT) of the output (response function) to the Laplace transform (LT) of the input (driving function) under the assumption that all initial conditions are zero.

Consider the LTI system defined by the differential equation

$$a_0 \overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 \overset{(m)}{x} + b_1 \overset{(m-1)}{x} + \cdots + b_{m-1}\dot{x} + b_m x \quad (n \geq m) \tag{4-1}$$

where y is the output and x is the input. The TF of this system is the ratio of the Laplace-transformed output to the Laplace-transformed input when all initial conditions are zero, or

$$\text{Transfer Function (TF)} = G(s) = \left.\frac{\mathscr{L}[output]}{\mathscr{L}[input]}\right|_{\text{zero initial conditions}} = \frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} \tag{4-2}$$

The above equation can be represented by the following graphical representation:

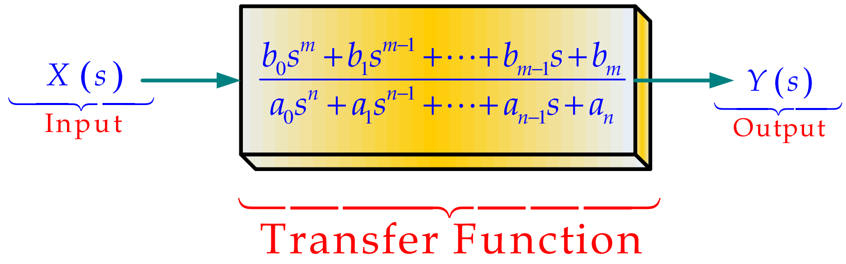


Figure 4-1. Block diagram representation of a transfer function

Comments on the Transfer Function (TF). The applicability of the concept of the Transfer Function (TF) is limited to LTI differential equation systems. The following list gives some important comments concerning the TF of a system described by a LTI differential equation:

1. The TF of a system is a mathematical model of that system, in that it is an operational method of expressing the differential equation that relates the output variable to the input variable.

2. The TF is a property of a system itself, unrelated to the magnitude and nature of the input or driving function.

3. The TF includes the units necessary to relate the input to the output; however it does not provide any information concerning the physical structure of the system. (The TF of many physically different systems can be identified).

4. if the TF of a system is known , the output or response can be studied for various forms of inputs with a view toward understanding the nature of the system.

5. If the TF of a system is unknown, it may be established experimentally by introducing known inputs and studying the output of the system. Once established, a TF gives a full description of the dynamic characteristics of the system, as distinct from its physical description

Example 4-1

Consider the mechanical system shown in Figure 4-2. The displacement x of the mass m is measured from the equilibrium position. In this system, the external force $f(t)$ is input and x is the output.

i) The FBD is shown in the Fig. 4-2.
ii) Apply Newton's second law of motion to a system in translation:

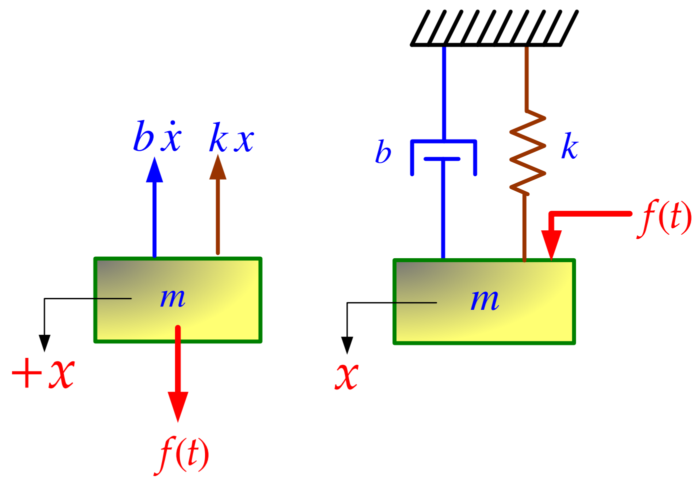


Figure 4-2 Mass -Spring –Damper System and the FBD.

$$\underbrace{\sum F}_{\text{Summation of all forces acting on the system}} = m\ddot{x}$$

$$f(t) - b\dot{x} - kx = m\ddot{x}$$

or

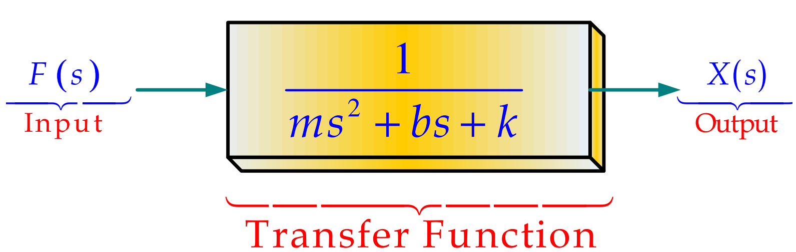
$$m\ddot{x} + b\dot{x} + kx = f(t) \quad \Rightarrow \quad \text{Forced Vibration of a second order system}$$

iii) For zero Initial Conditions (I.C's), taking Laplace Transform (LT) of both sides of the above equation yields

$$\left(m\,s^2 + b\,s + k\right)X(s) = F(s)$$

where $X(s) = \mathscr{L}\left[x(t)\right]$ and $F(s) = \mathscr{L}\left[f(t)\right]$. From Equation (4-2), the TF for the system is

$$\frac{X(s)}{F(s)} = \frac{\text{Output}}{\text{Input}} = \frac{1}{\left(m\,s^2 + b\,s + k\right)}$$



Impulse Response Function.

The TF of a LTI system is

$$\text{Transfer Function (TF)} = G(s) = \frac{Y(s)}{X(s)}$$

where $X(s)$ is the LT of the input $x(t)$ and $Y(s)$ is the LT of the output $y(t)$ and where we assume all I.C's involved are zero. It follows that

$$Y(s) = G(s)X(s) \tag{4-3}$$

Now, consider the ***output (response)*** of the system to a ***unit-impulse*** $\delta(t)$ ***input*** when all the I.C's are zero. Since

$$\mathcal{L}\left[\delta(t)\right] = 1$$

the LT of the output of the system is

$$Y(s) = G(s) \tag{4-4}$$

The inverse LT of the output of the system is given by Equation 4-4 yields the impulse response of the system, i.e;

$$\mathcal{L}^{-1}\left[G(s)\right] = g(t)$$

is called the ***impulse response function*** or the ***weighting function***, of the system. The impulse-response function $g(t)$ is thus the response of a linear system to a unit impulse input when the I.C's are zero. The LT of $g(t)$ gives the TF.

4.2 Block Diagrams (BD)

Block Diagrams of Dynamic Systems

A Block Diagram (BD) of a dynamic system is a pictorial representation of the functions performed by each component of the system and of the flow signal within the system. Such a diagram depicts the interrelationships that exist among the various components.

- In a BD, all system variables are linked to each other through functional blocks.

- The ***functional block***, or simply ***block***, is a symbol for the mathematical operation on the input signal to the block that produces the output.

- The TF's of the components are usually entered in the corresponding blocks, which are connected by arrows to indicate the direction of the flow signal.

- Notice that a signal can pass only in the direction of the arrows. Thus, a block diagram of a dynamic system explicitly shows a unilateral property.

Figure 4-3 shows an element of a BD . The arrowhead pointing toward the block indicates the input to the block, and the arrowhead leading away from the block represents the output of the block. As mentioned, such arrows represent signals.

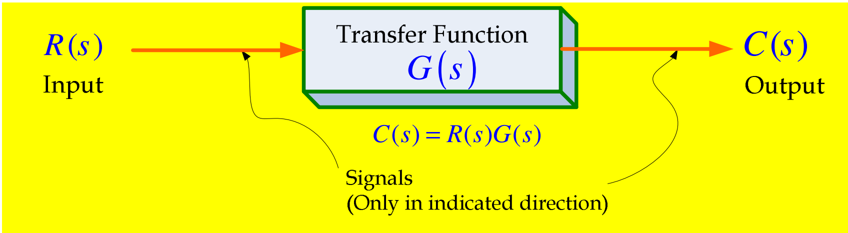


Figure 4-3 Element of a Block Diagram (BD).

Note that

$$[\text{Dimension of the output signal}] = [\text{Dimension of the input signal}] \times [\text{Dimension of the TF}]$$

Notice that in BD the main source of energy is not explicitly shown and that the BD of a given system is not unique. A number of different BD's can be drawn for a system, depending on the point of view of the analysis (See **Example 4-2**).

SUMMING POINT. Figure 4-4 shows a circle with a cross, the symbol that stands for a summing operation. The $(+)$ or $(-)$ sign at each arrowhead indicates whether the associated signal is to be added or subtracted. It is important that the quantities being added or subtracted have the same dimensions and the same units.

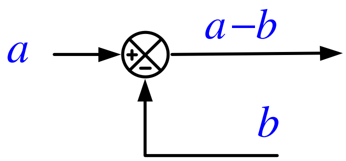


Figure 4-4 Summing point.

BRANCH POINT. A ***branch point*** is a point from which the signal from a block goes concurrently to other blocks or summing points.

BLOCK DIAGRAM OF A CLOSED-LOOP SYSTEM. Figure 4-5 is a BD of closed loop system. The output $C(s)$ is fed back to the summing point, where it is compared to the input $R(s)$. The closed loop nature of the system is indicated clearly by the figure. The output $C(s)$ is obtained by multiplying the TF $G(s)$ by the input of the block, $E(s)$.

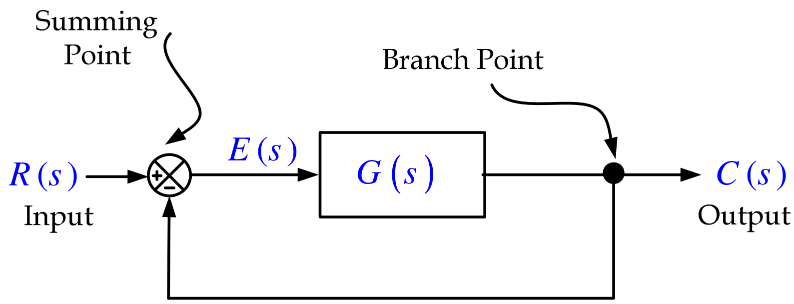


Figure 4-5 Block Diagram of a closed loop system.

Any linear system can be represented by a BD consisting of blocks, summing points, and branch points. When the output is fed back to the summing point for comparison with the input, it is necessary to convert the form of the output signal to that of the input signal. This conversion is accomplished by the feedback element whose transfer function is $H(s)$, as shown in Figure 4-6. Another important role of the feedback element is to modify the output before it is compared with the input. In the figure, the feedback signal that is fed back to the summing point for comparison with the input is $B(s) = H(s)C(s)$.

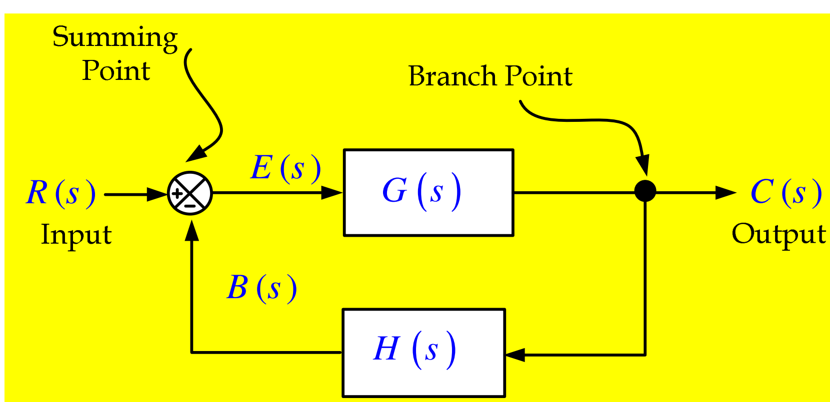
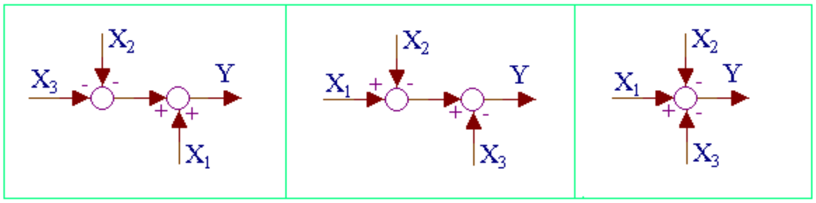


Figure 4-6 Block Diagram of a closed loop-system with feedback element.

Basic Rules for Reducing Block diagrams

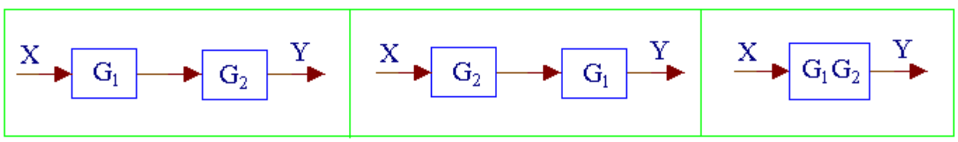
Rule: 1

$Y = X_1 - X_2 - X_3$



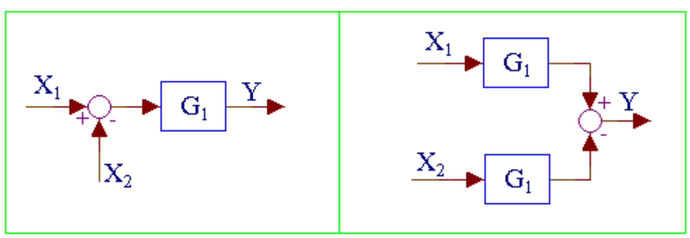
Rule: 2 (Associative and Commutative Properties)

$Y = G_1 G_2 X = G_2 G_1 X$



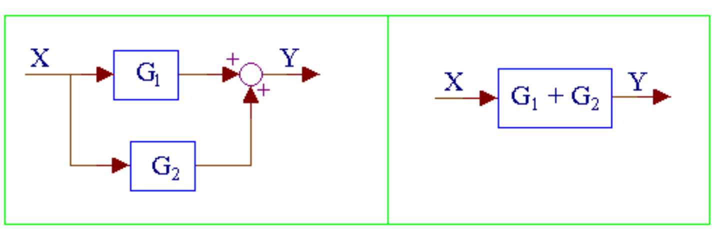
Rule: 3 (Distributive Property)

$Y = G_1(X_1 - X_2) = G_1 X_1 - G_1 X_2$



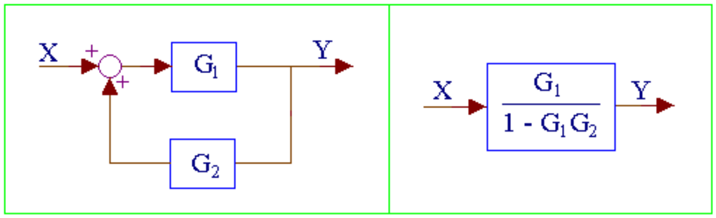
Rule: 4 (Blocks in Parallel)

$Y = X(G_1 + G_2) = G_1 X + G_2 X$



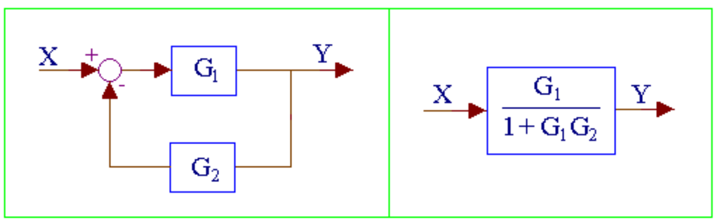
Rule: 5 (Positive Feedback Loop)

$$Y = G_1 X + G_2 G_1 Y = \frac{G_1}{1 - G_1 G_2} X$$



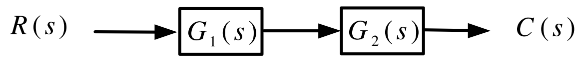
Rule: 6 (Negative Feedback loop)

$$Y = G_1 X - G_2 G_1 Y = \frac{G_1}{1 + G_1 G_2} X$$



MATLAB Implementation

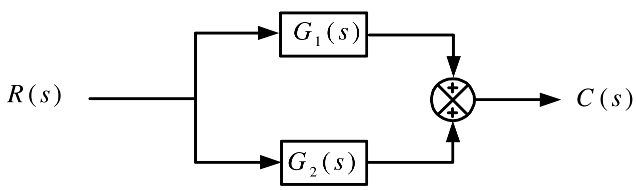
Series Connection



$$T(s)=\frac{C(s)}{R(s)}=\frac{\text{num}}{\text{den}} \qquad G_1(s)=\frac{\text{num1}}{\text{den1}} \qquad G_2(s)=\frac{\text{num2}}{\text{den2}}$$

[num,den]=series(num1,den1,num2,den2)

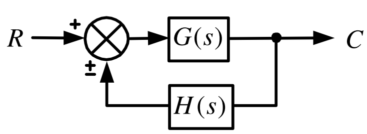
Parallel Connection



$$T(s)=\frac{C(s)}{R(s)}=\frac{\text{num}}{\text{den}} \qquad G_1(s)=\frac{\text{num1}}{\text{den1}} \qquad G_2(s)=\frac{\text{num2}}{\text{den2}}$$

[num,den]=parallel(num1,den1,num2,den2)

Feedback Connection



$$T(s)=\frac{C(s)}{R(s)}=\frac{\text{num}}{\text{den}} \qquad G(s)=\frac{\text{num1}}{\text{den1}} \qquad H(s)=\frac{\text{num2}}{\text{den2}}$$

+1	Positive Feedback
-1	Negative Feedback (default)

[num,den] = feedback(num1,den1,num2,den2,sign)

4.3 Partial-Fraction Expansion with Matlab

MATLAB representation of Transfer Functions (TF). The transfer function of a system is represented by two arrays of numbers. For example, consider a system defined by

$$\frac{Y(s)}{U(s)}=\frac{25}{s^2+4s+25}$$

This system is represented as two arrays, each containing the coefficients of the polynomials in descending powers of s as follows

```
>> num=25;
>> den=[1    4    25];
>> sys=tf(num,den)
```

MATLAB will automatically respond with the display

```
Transfer function:
     25
--------------
s^2 + 4 s + 25
```

Partial-Fraction Expansion with MATLAB. MATLAB allows us to obtain the partial-fraction expansion of the ratio of two polynomials,

$$\frac{B(s)}{A(s)}=\frac{\text{num}}{\text{den}}=\frac{b(1)s^h+b(2)s^{h-1}+\cdots+b(h)}{a(1)s^n+a(2)s^{n-1}+\cdots+a(n)}$$

Where $a(1)\neq 0$, some of $a(i)$ and $b(j)$ may be zero, and num and den are row vectors that specify the numerator and denominator of $B(s)/A(s)$. That is,

```
>> num=[b(1)   b(2)   ….   b(h)];
>> den =[a(1)   a(2)   ….   a(h)];
```

The command

```
>> [r,pk]=residue(num,den);
```

finds the residues, poles and direct terms of a partial fraction expansion of the ratio of the two polynomials $B(s)$ and $A(s)$. The partial fraction expansion of $B(s)/A(s)$ is given by

$$\frac{B(s)}{A(s)} = k(s) + \frac{r(1)}{s-p(1)} + \frac{r(2)}{s-p(2)} + \cdots + \frac{r(n)}{s-p(n)}$$

As an example, consider the function

$$\frac{B(s)}{A(s)} = \frac{\text{num}}{\text{den}} = \frac{s^4 + 8s^3 + 16s^2 + 9s + 6}{s^3 + 6s^2 + 11s + 6}$$

```
>> num=[1    8    16    9    6];
>> den=[1    6    11    6];
>> [r,p,k]=residue(num,den)
```

gives the residues r, poles p and direct terms k as follows

```
r=-6.0000
  -4.0000
   3.0000

p=-3.0000
   -2.0000
   -1.0000

k= 1   2
```

Therefore , the partial-fraction expansion of $B(s)/A(s)$ is:

$$\frac{B(s)}{A(s)} = \frac{\text{num}}{\text{den}} = \frac{s^4 + 8s^3 + 16s^2 + 9s + 6}{s^3 + 6s^2 + 11s + 6} = s + 2 - \frac{6}{s+3} - \frac{4}{s+2} + \frac{3}{s+1}$$

The command

```
[num,den]=residue(r,p,k)
```

where r,p and k are outputs , converts the partial-fraction expansion back to the polynomial ratio $B(s)/A(s)$ as shown below

```
>> r =[-6    -4    3];
>> p =[-3    -2    -1];
>> k=[1    2];
>> [num,den]=residue(r,p,k)

num =

    1     8     16     9     6
```

den =

    1   6   11   6

Example 4-3 (Textbook Page 114-115)

Consider the spring-mass-dashpot system mounted on a massless cart as shown in Figure 4-7.

1. Obtain the mathematical model of the system.
2. If $m = 10\,\text{kg}$, $b = 20\,\text{N-s/m}$ and $k = 100\,\text{N/m}$. Find the response $y(t)$ for a unit step input.

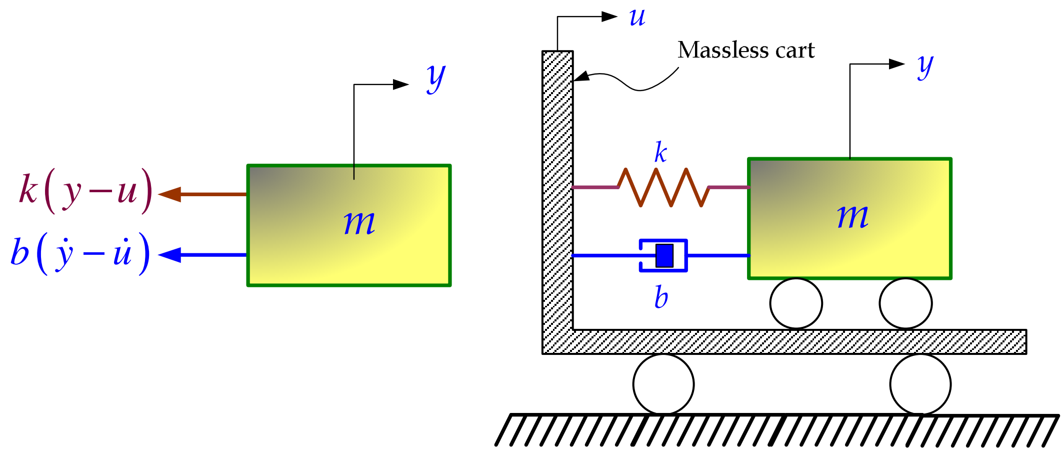


Figure 4-7 Spring-mass-dashpot system mounted on a cart and its FBD.

1. Apply Newton's second law for a system in translation

$$\underbrace{\sum F}_{\text{Summation of all forces acting on the system}} = m\ddot{y} \quad \Rightarrow \quad -b(\dot{y}-\dot{u}) - k(y-u) = m\ddot{y}$$

or

$$m\ddot{y} + b\dot{y} + ky = b\dot{u} + ku$$

The latter equation represents the mathematical model of the system under consideration.

2. For zero I. C's, taking LT of both sides of the above equation gives

$$\left(ms^2 + bs + k\right)Y(s) = (bs + k)U(s)$$

Taking the ratio of $Y(s)$ to $U(s)$, we find the TF of the system

$$\text{Transfer Function (TF)} = \frac{Y(s)}{U(s)} = \frac{(bs+k)}{(ms^2+bs+k)}$$

3. Next, we shall obtain analytical solution of the response to the unit-step input. Substituting the given numerical values for the mass, spring and dashpot elements gives

$$\frac{Y(s)}{U(s)} = \frac{20s+100}{10s^2+20s+100} = \frac{2s+10}{s^2+2s+10}$$

Since the input u is a unit step function,

$$U(s) = \frac{1}{s}$$

The output $Y(s)$ becomes

$$Y(s) = \frac{1}{s}\frac{2s+10}{s^2+2s+10} = \frac{2s+10}{s^3+2s^2+10s}$$

4. To obtain the inverse LT of $Y(s)$, we need to express $Y(s)$ into partial fractions. Use MATLAB for that

```
>> num=[2    10];
>> den=[1    2    10    0];
>> [r,p,k]=residue(num,den)

r =
  -0.5000 - 0.1667i
  -0.5000 + 0.1667i
   1.0000

p =
  -1.0000 + 3.0000i
  -1.0000 - 3.0000i
        0

k =

     []
```

Therefore, $Y(s)$ becomes

$$Y(s) = \frac{-0.5 - j0.1667}{s+1-j3} + \frac{-0.5 + j0.1667}{s+1+j3} + \frac{1}{s}$$

Since $Y(s)$ involves complex-conjugate poles, it is convenient to combine two complex conjugate terms into one as follows

$$\frac{-0.5 - j0.1667}{s+1-j3} + \frac{-0.5 + j0.1667}{s+1+j3} = \frac{-s}{(s+1)^2 + 3^2}$$

Then $Y(s)$ can be expanded as

$$Y(s) = \frac{1}{s} - \frac{s}{(s+1)^2 + 3^2} = \frac{1}{s} - \frac{s+1-1}{(s+1)^2 + 3^2}$$
$$= \frac{1}{s} - \frac{s+1}{(s+1)^2 + 3^2} + \frac{1}{3}\frac{3}{(s+1)^2 + 3^2}$$

5. The inverse LT of $Y(s)$ is obtained as

$$y(t) = \mathcal{L}^{-1}\left[Y(s)\right] = 1 - e^{-t}\cos 3t + \frac{1}{3}e^{-t}\sin 3t.$$

Example 4-4 (Textbook Page 117-119)

Consider the mechanical system shown in Figure 4-8. The system is initially at rest. The displacements x and y are measured from their respective equilibrium positions. Assuming that $p(t)$ is a step input and the displacement $x(t)$ is the output.

1. Obtain the transfer function of the system.
2. If $m = 0.1\,\text{kg}$, $b_2 = 0.4\,\text{N-s/m}$ and $k_1 = 6\,\text{N/m}$, $k_2 = 4\,\text{N/m}$, and $p(t)$ is a step force of magnitude $10\,\text{N}$, obtain an analytical solution of $x(t)$.

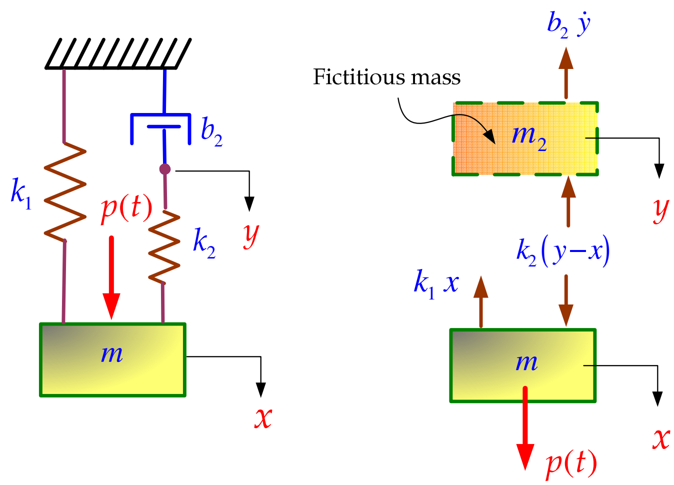


Figure 4-8 Mechanical system and its FBD.

1. Put a fictitious mass m_2.
2. Draw the FBD as shown.
3. Apply Newton second for translational motion for mass m

$$\underbrace{\sum F}_{\text{Summation of all forces acting on the mass } m} = m\ddot{x} \quad \Rightarrow \quad p(t) - k_1 x + k_2(y - x) = m\ddot{x}$$

or

$$m\ddot{x} + (k_1 + k_2)x - k_2 y = p(t) \qquad (1)$$

4. Apply Newton second for translational motion for mass m_2

$$\underbrace{\sum F}_{\text{Summation of all forces acting on the mass } m_2} = m_2\ddot{y} \quad \Rightarrow \quad -b_2\dot{y} - k_2(y - x) = 0$$

or

$$-k_2 x + (b_2\dot{y} + k_2 y) = 0 \qquad (2)$$

5. For zero I. C's taking LT of both sides of Eqs. (1) and (2), gives

$$\left[ms^2 + (k_1 + k_2)\right]X(s) - k_2 Y(s) = P(s) \qquad (3)$$

$$-k_2 X(s) + \left[b_2 s + k_2\right]Y(s) = 0 \qquad (4)$$

Equations (3) and (4) constitute a system of 2 equations with 2 unknowns $X(s)$ and $Y(s)$.[1]

6. Solving Eq. (3) for $Y(s)$ gives

$$Y(s) = \frac{k_2}{[b_2 s + k_2]} X(s) \tag{5}$$

7. Substitute Eq. (5) into Eq. (3) we get

$$\left[ms^2 + (k_1 + k_2)\right] X(s) - \frac{k_2^2}{[b_2 s + k_2]} X(s) = P(s)$$

or

$$\text{Transfer Function (TF)} = \frac{X(s)}{P(s)} = \frac{b_2 s + k_2}{mb_2 s^3 + mk_2 s^2 + (k_1 + k_2) b_2 s + k_1 k_2}$$

which represents a third order system.

8. Next, we shall obtain analytical solution of the response to a step input of magnitude $10\,\mathrm{N}$. Substituting the given numerical values for the mass, springs and dashpot elements gives

$$\frac{X(s)}{P(s)} = \frac{0.4s + 4}{0.04s^3 + 0.4s^2 + 4s + 24}$$
$$= \frac{10s + 100}{s^3 + 10s^2 + 100s + 600}$$

Since the input p is a step function of magnitude $10\,\mathrm{N}$, then

$$U(s) = \frac{10}{s}$$

The output $X(s)$ becomes

$$\frac{X(s)}{P(s)} = \frac{10s + 100}{s^3 + 10s^2 + 100s + 600} \times \frac{10}{s}$$

[1] See Appendix at the end of this chapter

9. To obtain the inverse LT of $X(s)$, we need to express $X(s)$ into partial fractions. Use MATLAB for that

```
>> num=[100    1000];
>> den=[1    10    100    600    0];
>> [r,p,k]=residue(num,den)

r =

  -0.6845 + 0.2233i
  -0.6845 - 0.2233i
  -0.2977
   1.6667

p =

  -1.2898 + 8.8991i
  -1.2898 - 8.8991i
  -7.4204
        0

k =

     []
```

Therefore, $X(s)$ becomes

$$X(s)=\frac{-0.6845+j0.2233}{s+1.2898-j8.8991}+\frac{-0.6845-j0.2233}{s+1.2898+j8.8991}-\frac{0.2977}{s+7.4204}+\frac{1.6667}{s}$$

$$=\left[\frac{-1.3690(s+1.2898)-3.9743}{(s+1.2898)^2+8.8991^2}\right]-\frac{0.2977}{s+7.4204}+\frac{1.6667}{s}$$

10. The inverse LT of $X(s)$ is obtained as

$$x(t)=\mathscr{L}^{-1}\left[X(s)\right]=-1.3690e^{-1.2898t}\cos(8.8991t)$$
$$-\underbrace{\frac{3.9743}{8.8991}}_{=0.4466}e^{-1.2898t}\sin(8.8991t)-0.2977e^{-7.4204t}+1.6667.$$

From the preceding examples, we have seen that once the TF $X(s)/U(s) = G(s)$ of a system is obtained, the response of the system to any input can be determined by taking the inverse LT of $X(s)$, or

$$x(t) = \mathscr{L}^{-1}\left[X(s)\right] = \mathscr{L}^{-1}\left[G(s)U(s)\right]$$

Finding the inverse LT of $G(s)$ may be time consuming if the TF $G(s)$ of the system is complicated, eventhough the input $U(s)$ may be a simple function of time. Unless, for some reason, the analytical solution is needed, we should use a computer to get a numerical solution.

4.4 Transient Response Analysis with Matlab

MATLAB Representation of Transfer-Functions (TF) Systems.

Figure 4-1 shows a block with a TF. Such a block represents a system or an element of a system. To simplify our presentation, we shall call the block with a TF a system. MATLAB uses sys to represent such a system. The statement

```
>> sys=tf(num,den)
```

represents the system. For example, consider the system

$$\frac{Y(s)}{X(s)} = \frac{2s+25}{s^2+4s+25}$$

This system is represented as two arrays, each containing the coefficients of the polynomials in descending powers of s as follows

```
>> num=[2    25];
>> den=[1    4    25];
>> sys=tf(num,den)
```

MATLAB will automatically respond with the display

```
Transfer function:
   2 s + 25
--------------
s^2 + 4 s + 25
```

Step Response. The step function plots the unit step response, assuming the I.C's are zero. The basic syntax is step(sys), where sys is the LTI object defined previously.

The basic syntax commands are summarized below

Command (Basic Syntax)	Use
>> step(sys)	generates a plot of a unit step response and displays a response curve on the screen. The computation time interval Δt and the time span of the response tf are determined automatically by MATLAB.
>> step(sys,tf)	generates a plot of a unit step response and displays a response curve on the screen for the specified final time tf. The computation time interval Δt is determined automatically by MATLAB.
>> step(sys,t)	generates a plot of a unit step response and displays a response curve on the screen for the user specified time t where $t = 0 : \Delta t : tf$.
>> [y,t]=step(sys,…)	Returns the output y, and the time array t used for the simulation. No plot is drawn. The array y is $p \times q \times m$ where p is length(t), q is the number of outputs, and m is the number of inputs.
>> step(sys1, sys2,…,t)	Plots the step response of multiple LTI systems on a single plot. The time vector t is optional. You can specify line color, line style and marker for each system.

The steady state response and the time to reach that steady state are automatically determined. The steady state response is indicated by horizontal dotted line.

For more details in this topic: type doc step or help step at MATLAB prompt >>

Example 4-5 (Textbook Page 121-122)

Consider again the spring-mass-dashpot system mounted on a cart as shown in Figure 4-7. (See Example 4-3). The transfer function of the system is

$$\text{Transfer Function (TF)} = \frac{Y(s)}{U(s)} = \frac{(bs+k)}{(ms^2+bs+k)}$$

For $m = 10\ \text{kg}$, $b = 20\ \text{N-s/m}$ and $k = 100\ \text{N/m}$. Find the response $y(t)$ for a unit step input $u(t) = 1(t)$.

MATLAB PROGRAM:

```
>> m=10; b=20; k=100;
>> num=[b    k];
>> den=[m    b    k];
>> sys=tf(num,den);
>> step(sys)
>> grid
```

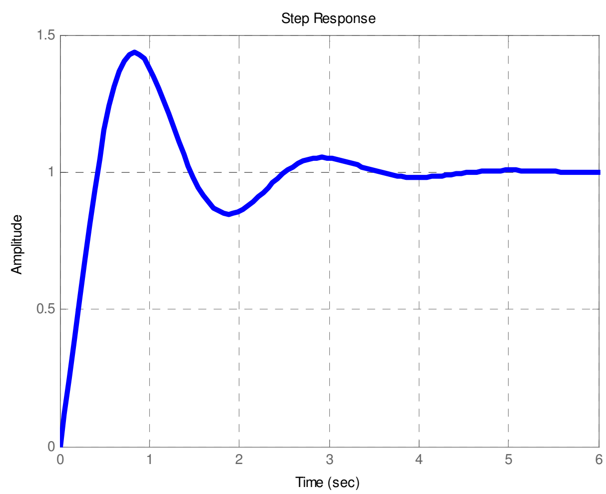


Figure 4-10 Unit step response curve

Example 4-6 (Textbook Page 123-124)

Consider again the mechanical system shown in Figure 4-8. (See Example 4-4). The transfer functions of the system are (See Appendix)

$$\frac{X(s)}{P(s)} = \frac{b_2 s + k_2}{mb_2 s^3 + mk_2 s^2 + (k_1 + k_2) b_2 s + k_1 k_2}$$

and

$$\frac{Y(s)}{P(s)} = \frac{k_2}{mb_2 s^3 + mk_2 s^2 + (k_1 + k_2) b_2 s + k_1 k_2}$$

For $m = 0.10\,\text{kg}$, $b_2 = 0.4\,\text{N-s/m}$ and $k_1 = 6\,\text{N/m}$, $k_2 = 4\,\text{N/m}$ and $p(t)$ is a step input of magnitude $10\,\text{N}$. Obtain the responses $x(t)$ and $y(t)$.

MATLAB PROGRAM:

```
>> m=0.1; b2=0.4; k1=6;k2=4;
>> num1=[b2   k2]
>> num2=[k2]
>> den=[m*b2   m*k2   k1*b2+k2*b2   k1*k2]
>> sys1=tf(num1,den)
>> sys2=tf(num2,den)
>> step(10*sys1,'r:',10*sys2,'b')
>> grid
>> gtext('x(t)');gtext('y(t)')
```

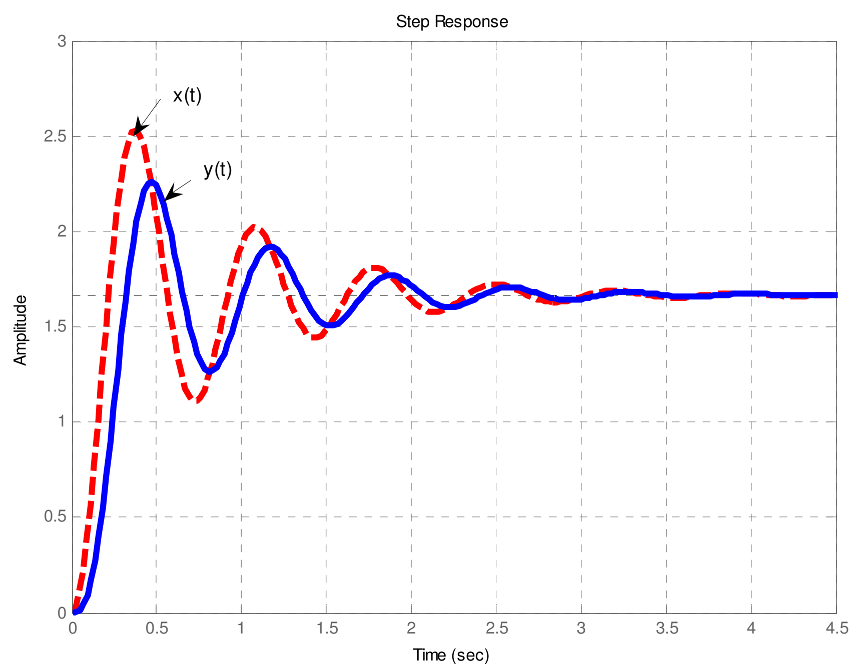


Figure 4-11 Step response curves $x(t)$ and $y(t)$

Impulse Response. The impluse function plots the unit-impulse response, assuming the I.C's are zero. The basic syntax is impulse(sys), where sys is the LTI object.

The basic syntax commands are summarized below

Command (Basic Syntax)	Use
>> impulse(sys)	generates a plot of a unit impulse response

	and displays a response curve on the screen. The computation time interval Δt and the time span of the response tf are determined automatically by MATLAB.
>> impulse(sys,tf)	generates a plot of a unit impulse response and displays a response curve on the screen for the specified final time tf. The computation time interval Δt is determined automatically by MATLAB.
>> impulse(sys,t)	generates a plot of a unit impulse response and displays a response curve on the screen for the user specified time t where $t = 0 : \Delta t : tf$.
>> [y,t]=impulse(sys,…)	Returns the output y, and the time array t used for the simulation. No plot is drawn. The array y is $p \times q \times m$ where p is length(t), q is the number of outputs, and m is the number of inputs.
>>impulse(sys1, sys2,…,t)	Plots the impulse response of multiple LTI systems on a single plot. The time vector t is optional. You can specify line color, line style and marker for each system.

The steady state response and the time to reach that steady state are automatically determined. The steady state response is indicated by horizontal dotted line.

For more details in this topic: type doc impulse or help impulse at MATLAB prompt >>

Impulse Input. The impulse response of a mechanical system can be observed when the system is subjected to a very large force for a very short time, for instance, when the mass of a spring-mass-dashpot system is hit by a hammer or a bullet. Mathematically, such an impulse input can be expressed by an impulse function.

The impulse function is a mathematical function without any actual physical counterpart. However, as shown in Figure 4-12 (a), if the actual input lasts for a short time Δt but has a magnitude h, so that the area $h\,\Delta t$ in a time plot is not negligible, it can be approximated by an impulse function. The impulse input is usually denoted by a vertical arrow, as shown in Figure 4-12 (b), to indicate that it has a very short duration and a very large height.

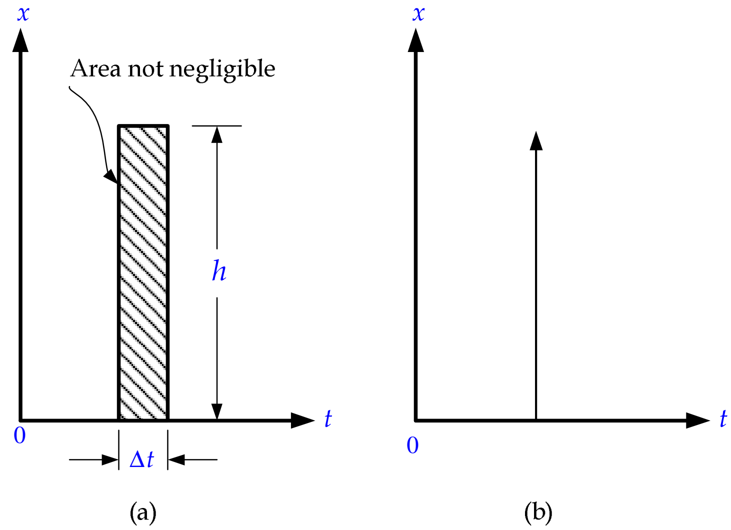


Figure 4-12 Impulse inputs

Example 4-7

Consider the previous Example 4-6 but with an impulse input of magnitude 10 N.

MATLAB PROGRAM:

```
>> m=0.1; b2=0.4; k1=6;k2=4;
>> num1=[b2    k2]
>> num2=[k2]
>> den=[m*b2    m*k2    k1*b2+k2*b2    k1*k2]
>> sys1=tf(num1,den)
>> sys2=tf(num2,den)
>> impulse(10*sys1,'r:',10*sys2,'b')
>> grid
>> gtext('x(t)');gtext('y(t)')
```

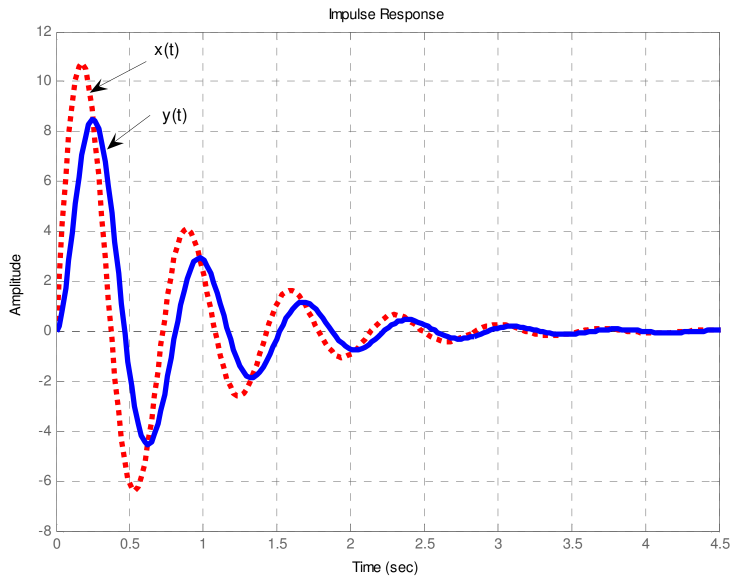


Figure 4-13 Impulse response curves $x(t)$ and $y(t)$

Obtaining response to arbitrary input. The lsim function plots the response of the system to an arbitrary input. The basic syntax commands is summarized below

Command (Basic Syntax)	Use
>> lsim(sys,u,t)	produces a plot of the time response of the LTI model sys to the input time history t,u. The vector t specifies the time samples for the simulation and consists of regularly spaced time samples. $t = 0 : \Delta t : tf$ The matrix u must have as many rows as time samples (length(t)) and as many columns as system inputs. Each row u(i,:) specifies the input value(s) at the time sample t(i).

For more details in this topic: type doc lsim or help lsim at MATLAB prompt >>

Example 4-8

Consider the mass-spring-dashpot system mounted on a cart of Example 4-3 The TF of the system is

$$\frac{Y(s)}{U(s)} = \frac{(bs+k)}{(ms^2+bs+k)}$$

where $Y(s)$ is the output $U(s)$ is the input . Assume that $m = 10\,\text{kg}$, $b = 20\,\text{N-s/m}$ and $k = 100\,\text{N/m}$. Find the response $y(t)$ for a ramp input with a slope of 2, ($r(t) = 2t$).

MATLAB PROGRAM:

```
>> m=10; b=20; k=100;
>> num=[b    k];den=[m    b    k];
>> sys=tf(num,den);
>> t=[0:0.001:3];
>> u=2*t;
>> lsim(sys,u,t);grid;gtext('x(t)');gtext('y(t)')
```

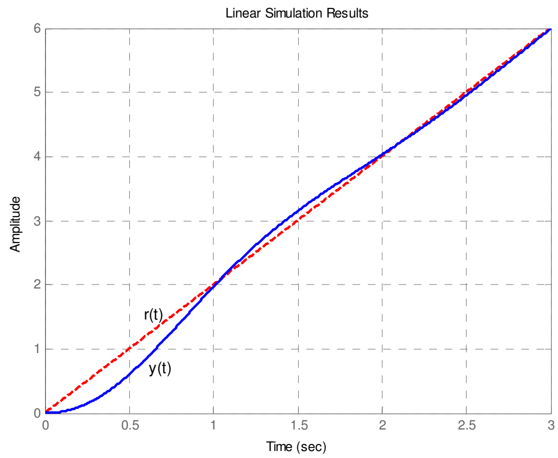


Figure 4-14 Response for a ramp input $r(t) = 2t$

Example 4-9

Find the response $y(t)$ of the previous Example 4-8 if the input is shown by the Figure below.

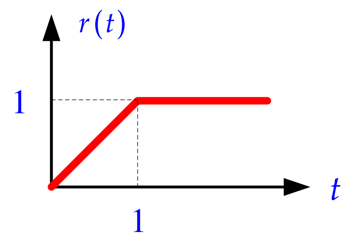


Figure 4-15 Arbitrary input

MATLAB PROGRAM:

```
>> m=10; b=20; k=100;
>> num=[b    k]
>> den=[m    b    k]
```

```
>> sys=tf(num,den)
>> t=[0:0.001:5];
>> for k=1:length(t)
>> if t(k) <= 1
        r(k) =t(k);
        else
        r(k)=1;
        end
        end
>> y=lsim(sys,r,t);
>> plot(t,y,t,r,'r:');grid;
>> xlabel('Time (sec)');
>> gtext('r(t)');gtext('y(t)')
```

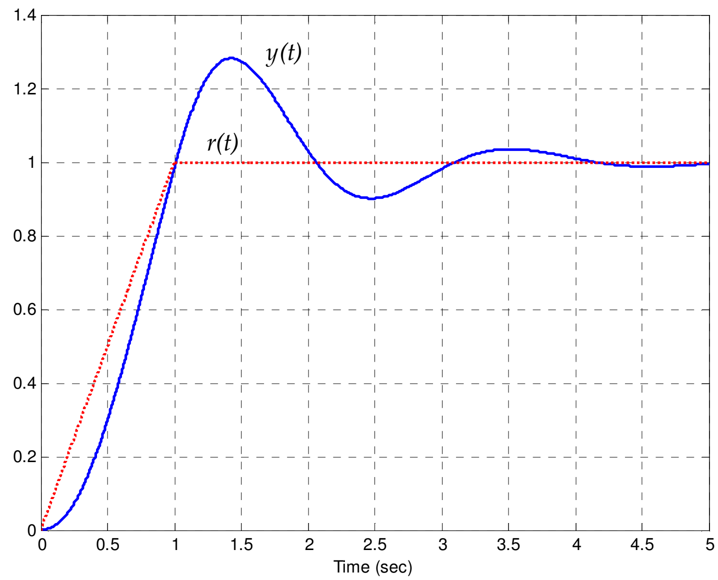


Figure 4-16 Response for an arbitrary input

APPENDIX

The transfer functions $G_1(s) = X(s)/U(s)$ and $G_2(s) = Y(s)/U(s)$ can be found directly by solving the system of Equations (3) and (4) with the unknowns $X(s)$ and $Y(s)$. Rewrite Eqs (3) and (4) as

$$\left[ms^2 + (k_1 + k_2)\right]X(s) - k_2 Y(s) = P(s) \quad (3)$$

$$-k_2 X(s) + [b_2 s + k_2]Y(s) = 0 \quad (4)$$

Method of Substitution:

The above system can be solved by substituting of one of the unknowns from one equation into the other. For instance, from Eq. (4)

$$Y(s) = \frac{k_2}{[b_2 s + k_2]}X(s) \quad (5)$$

Substitute Eq. (5) into Eq. (3) we get

$$\left[ms^2 + (k_1 + k_2)\right]X(s) - \frac{k_2^2}{[b_2 s + k_2]}X(s) = P(s)$$

or

$$\frac{X(s)}{P(s)} = \frac{b_2 s + k_2}{mb_2 s^3 + mk_2 s^2 + (k_1 + k_2)b_2 s + k_1 k_2}$$

or

$$X(s) = \frac{P(s)(b_2 s + k_2)}{mb_2 s^3 + mk_2 s^2 + (k_1 + k_2)b_2 s + k_1 k_2} \quad (6)$$

Substitute $X(s)$ from (6) into (5), we get

$$Y(s) = \frac{k_2}{[b_2 s + k_2]}X(s) = \frac{k_2}{[b_2 s + k_2]}\left\{\underbrace{\frac{P(s)(b_2 s + k_2)}{mb_2 s^3 + mk_2 s^2 + (k_1 + k_2)b_2 s + k_1 k_2}}_{X(s)}\right\}$$

Finally

$$\frac{Y(s)}{P(s)}=\frac{k_2}{mb_2s^3+mk_2s^2+(k_1+k_2)b_2\,s+k_1k_2}$$

Cramer's Rule:

Eqs. (3) and (4) represent a system of two equations with two unknowns $X(s)$ and $Y(s)$. The above system can be written in the form

$$a_{11}X(s)+a_{12}Y(s)=P(s)$$
$$a_{21}X(s)+a_{22}Y(s)=0$$
(7)

Where it is clear that

$$a_{11}=\left[ms^2+(k_1+k_2)\right]$$
$$a_{12}=a_{21}=-k_2$$
$$a_{22}=(b_2s+k_2)$$

The solution to system (7) is

$$X(s)=\frac{\begin{vmatrix}P(s) & a_{12}\\ 0 & a_{22}\end{vmatrix}}{\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}}=\frac{P(s)a_{22}}{a_{11}a_{22}-a_{12}a_{12}} \quad\Rightarrow\quad \frac{X(s)}{P(s)}=\frac{a_{22}}{a_{11}a_{22}-a_{12}a_{12}}$$

$$Y(s)=\frac{\begin{vmatrix}a_{11} & P(s)\\ a_{21} & 0\end{vmatrix}}{\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}}=\frac{-P(s)a_{21}}{a_{11}a_{22}-a_{12}a_{12}} \quad\Rightarrow\quad \frac{X(s)}{P(s)}=\frac{-a_{21}}{a_{11}a_{22}-a_{12}a_{12}}$$

or

$$\frac{X(s)}{P(s)}=\frac{a_{22}}{a_{11}a_{22}-a_{12}a_{12}}=\frac{(b_2s+k_2)}{\left[ms^2+(k_1+k_2)\right](b_2s+k_2)-k^2}$$

$$\frac{Y(s)}{P(s)}=\frac{-a_{21}}{a_{11}a_{22}-a_{12}a_{12}}=\frac{k_2}{\left[ms^2+(k_1+k_2)\right](b_2s+k_2)-k^2}$$

Therefore it does not appear in the equation of motion.