CHAPTER 3
MECHANICAL SYSTEMS

A. BAZOUNE

3.1 INTRODUCTION

Mathematical Modeling and response analysis of mechanical systems are the subjects of this chapter.

3.2 MECHANICAL ELEMENTS

Any mechanical system consists of mechanical elements. There are three types of basic elements in mechanical systems:

- **Inertia elements**
- **Spring elements**
- **Dampers elements**

INERTIA ELEMENTS. ***Mass*** and ***moment of inertia***. Inertia may be defined as the change in force (torque) required to make a unit change in acceleration (angular acceleration). That is,

$$\text{inertia (mass)} = \frac{\text{change in force}}{\text{change in acceleration}} \quad \frac{\text{N}}{\text{m/s}^2} \quad \text{or} \quad \text{kg}$$

$$\text{inertia (moment of inertia)} = \frac{\text{change in torque}}{\text{change in ang. accel.}} \quad \frac{\text{N-m}}{\text{rad/s}^2} \quad \text{or} \quad \text{kg}$$

SPRING ELEMENTS. A linear ***spring*** is a mechanical element that can be deformed by external force or torque such that the deformation is directly proportional to the force or torque applied to the element.

TRANSLATIONAL SPRINGS

For translational motion, (Fig 3-1(a)), the force that arises in the spring is proportional to x and is given by

$$F = k\,x \tag{3-1}$$

where x is the elongation of the spring and k is a proportionality constant called the ***spring constant*** and has units of [force/displacement]=[N/m] in SI units.

At point P, the spring force F acts opposite to the direction of the force F applied at point P.

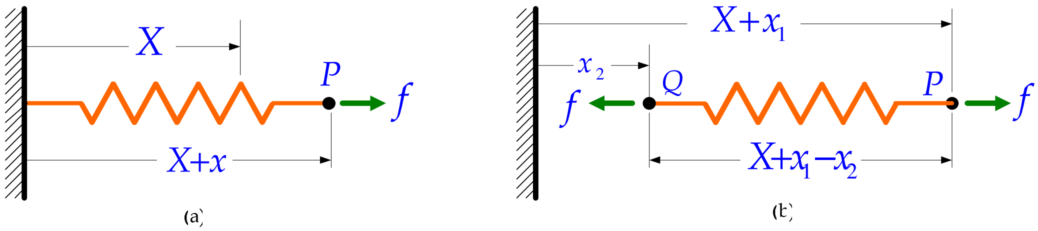


Figure 3-1 (a) One end of the spring is deflected; (b) both ends of the spring are deflected. (X is the natural length of the spring)

Figure 3-1(b) shows the case where both ends P and Q of the spring are deflected due to the forces f applied at each end. The net elongation of the spring is $x_1 - x_2$. The force acting in the spring is then

$$F = k\left(x_1 - x_2\right) \tag{3-2}$$

Notice that the displacement $X + x_1$ and x_2 of the ends of the spring are measured relative to the same reference frame.

PRACTICAL EXAMPLES.

Pictures of various types of real-world springs are found below.

Torsional Springs

Consider the torsional spring shown in Figure 3-2 (a), where one end is fixed and a torque τ is applied to the other end. The ***angular displacement*** of the free end is θ. The torque T in the torsional spring is

$$T = k_T \, \theta \tag{3-3}$$

where θ is ***the angular displacement*** and k_T is the ***spring constant*** for torsional spring and has units of [Torque/angular displacement]=[N-m/rad] in SI units.

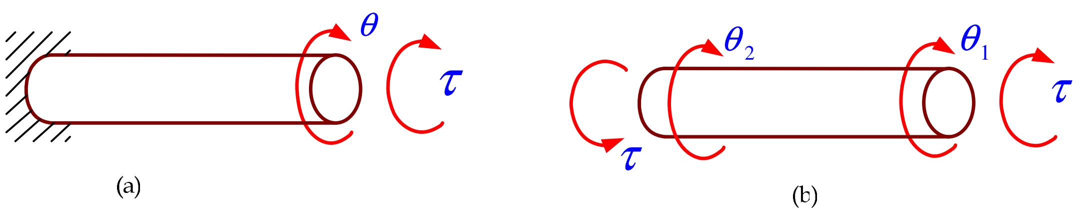


Figure 3-2 (a) A torque τ is applied at one end of torsional spring, and the other end is fixed; (b) a torque τ is applied at one end, and a torque τ, in the opposite direction, is applied at the other end.

At the free end, this torque acts in the torsional spring in the direction opposite to that of the applied torque τ.

For the torsional spring shown in Figure 3-2(b), torques equal in magnitude but opposite in direction, are applied to the ends of the spring. In this case, the torque T acting in the torsional spring is

$$T = k_T \left(\theta_1 - \theta_2 \right) \tag{3-4}$$

At each end, the spring acts in the direction opposite to that of the applied torque at that end.

For linear springs, the spring constant k may be defined as follows

$$\underbrace{\text{spring constant } k}_{\text{for translational spring}} = \frac{\text{change in force}}{\text{change in displacement of spring}} \quad \frac{\text{N}}{\text{m}}$$

$$\underbrace{\text{spring constant } k_T}_{\text{for torsional spring}} = \frac{\text{change in torque}}{\text{change in angular displacement of spring}} \quad \frac{\text{N-m}}{\text{rad}}$$

Spring constants indicate stiffness; a large value of k or k_T corresponds to a hard spring, a small value of k or k_T to a soft spring. The reciprocal of the spring constant k is called ***compliance*** or *mechanical capacitance* C. Thus $C = 1/k$. Compliance or mechanical capacitance indicates the softness of the spring.

PRACTICAL EXAMPLES.

Pictures of various types of real-world springs are found below.

PRACTICAL SPRING VERSUS IDEAL SPRING. Figure 3-3 shows the force displacement characteristic curves for linear and nonlinear springs.

- All practical springs have inertia and damping.
- An ideal spring has neither mass nor damping (internal friction) and will obey the linear force displacement law.

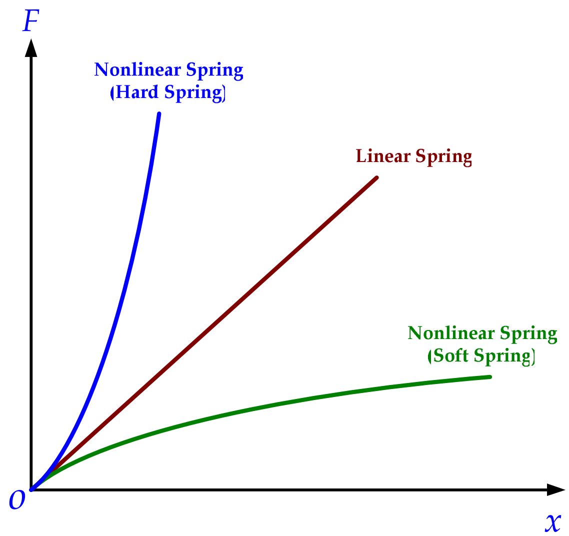


Figure 3-3 Force-displacement characteristic curves for linear and nonlinear springs.

DAMPER ELEMENTS. A damper is a mechanical element that dissipates energy in the form of heat instead of storing it. Figure 3-4(a) shows a schematic diagram of a translational damper, or a dashpot that consists of a piston and an-oil-filled cylinder. Any relative motion between the piston rod and the cylinder is resisted by oil because oil must flow around the piston (or through orifices provided in the piston) from one side to the other.

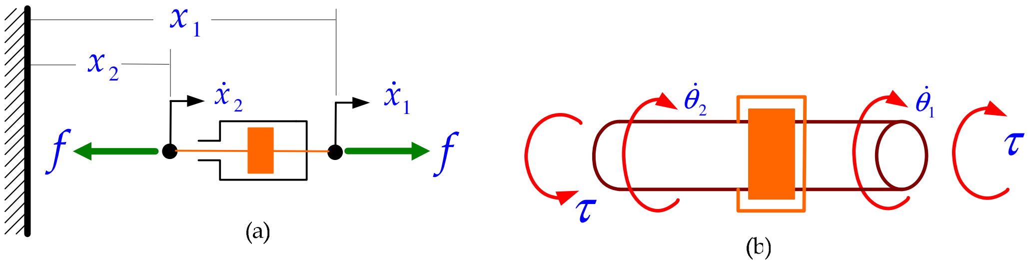


Figure 3-4 (a) Translational damper; (b) torsional (or rotational) damper.

Translational Damper

In Fig 3-4(a), the forces applied at the ends of translation damper are on the same line and are of equal magnitude, but opposite in direction. The velocities of the ends of the damper are $\dot{x}_1$ and $\dot{x}_2$. Velocities $\dot{x}_1$ and $\dot{x}_2$ are taken relative to the same frame of reference.

In the damper, the damping force F that arises in it is proportional to the velocity differences $\dot{x}_1 - \dot{x}_2$ of the ends, or

$$F = b\left(\dot{x}_1 - \dot{x}_2\right) = b\dot{x} \qquad (3\text{-}5)$$

where $\dot{x} = \dot{x}_1 - \dot{x}_2$ and the proportionality constant b relating the damping force F to the velocity difference $\dot{x}$ is called the ***viscous friction coefficient*** or ***viscous friction constant.*** The dimension of b is [force/Velocity] = [N/m-s] in SI units.

Torsional Damper

For the torsional damper shown in Figure 3-4(b), the torques τ applied to the ends of the damper are of equal magnitude, but opposite in direction. The angular velocities of the ends of the damper are $\dot{\theta}_1$ and $\dot{\theta}_2$ and they are taken relative to the same frame of reference. The damping torque T that arises in the damper is proportional to the angular velocity differences $\dot{\theta}_1 - \dot{\theta}_2$ of the ends, or

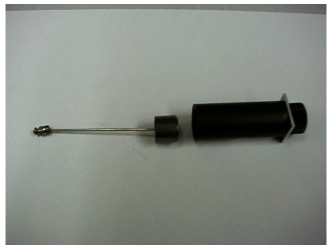
$$T = b_T\left(\dot{\theta}_1 - \dot{\theta}_2\right) = b_T\dot{\theta} \qquad (3\text{-}6)$$

where, analogous to the translation case, $\dot{\theta} = \dot{\theta}_1 - \dot{\theta}_2$ and the proportionality constant b_T relating the damping torque T to the angular velocity difference $\dot{\theta}$ is called the ***viscous friction coefficient*** or ***viscous friction constant.*** The dimension of b is [torque/angular velocity] = [N-m/rad] in SI units.

A damper is an element that provides resistance in mechanical motion, and, as such, its effect on the dynamic behavior of a mechanical system is similar to that of an electrical resistor on the dynamic behavior of an electrical system. Consequently, a damper is often referred to as a ***mechanical resistance element*** and the viscous friction coefficient as the ***mechanical resistance***.

Practical Examples.

Pictures of various examples of real-world dampers are found below.



Practical damper versus ideal damper

- All practical dampers produce inertia and spring effects.
- An ideal damper is massless and springless, dissipates all energy, and obeys the linear force-velocity law (or linear torque-angular velocity law).

Nonlinear friction. Friction that obeys a linear law is called *linear friction*, whereas friction that does not is described as nonlinear. Examples of nonlinear friction include static friction, sliding friction, and square-law friction. Square law-friction occurs when a solid body moves in a fluid medium. Figure 3-5 shows a characteristic curve for square-law friction.

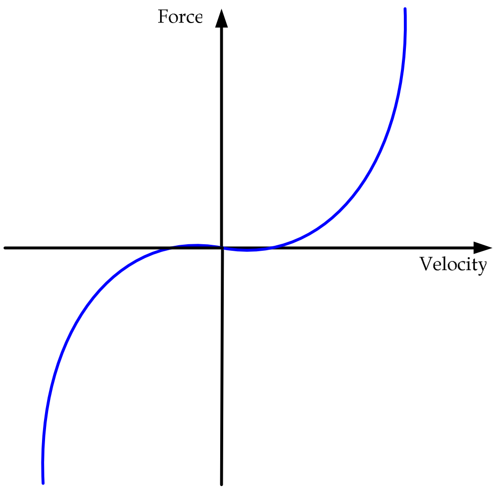


Figure 3-5 Characteristic curve for square-law friction.

3.3 MATHEMATICAL MODELING OF SIMPLE MECHANICAL SYSTEMS

A mathematical model of any mechanical system can be developed by applying Newton's laws to the system.

RIGID BODY. When any real body is accelerated , internal elastic deflections are always present. If these internal deflections are negligibly small relative to the gross motion of the entire body, the body is called ***rigid body***. ***Thus, a rigid body does not deform.***

NEWTON'S LAWS.

NEWTON'S FIRST LAW: (Conservation of Momentum)

The total momentum of a mechanical system is constant in the absence of external forces. Momentum is the product of mass m and velocity v, or $m\,v$, for translational or linear motion. For rotational motion, momentum is the product of moment of inertia J and angular velocity ω, or $J\omega$, and is called angular momentum.

NEWTON'S SECOND LAW:

TRANSLATIONAL MOTION: If a force is acting on rigid body through the center of mass in a given direction, the acceleration of the rigid body in the same direction is directly proportional to the force acting on it and is inversely proportional to the mass of the body. That is,

$$\text{acceleration} = \frac{\text{force}}{\text{mass}}$$

or

$$(\text{mass})(\text{acceleration}) = \text{force}$$

Suppose that forces are acting on a body of mass m. If $\sum F$ is the sum of all forces acting on a mass m through the center of mass in a given direction, then

$$\sum F = m\,a \tag{3-7}$$

where a is the resulting absolute acceleration in that direction. The line of action of the force acting on a body must pass through the center of mass of the body. Otherwise, rotational motion will also be involved.

ROTATIONAL MOTION. For a rigid body in pure rotation about a fixed axis, Newton's second law states that

$$(\text{moment of inertia})(\text{angular acceleration}) = \text{torque}$$

or

$$\sum T = J\,\alpha \qquad (3\text{-}8)$$

where $\sum T$ is the sum of all torques acting about a given axis, J is the moment of inertia of a body about that axis, and α is the angular acceleration of the body.

NEWTON'S THIRD LAW. It is concerned with action and reaction and states that every action is always opposed by an equal reaction.

TORQUE OR MOMENT OF FORCE. Torque, or moment of force, is defined as any cause that tends to produce a change in the rotational motion of a body in which it acts. Torque is the product of a force and the perpendicular distance from a point of rotation to the line of action of the force.

$$[\text{Torque}] = [\text{force} \times \text{distance}] = [\text{N-m}] \text{ in SI units}$$

MOMENTS OF INERTIA. *The moment of inertia* J of a rigid body about an axis is defined by

$$J = \int r^2 dm \qquad (3\text{-}9)$$

Where dm is an element of mass, r is the distance from axis to dm, and integration is performed over the body. In considering moments of inertia, we assume that the rotating body is perfectly rigid. Physically, the moment of inertia of a body is a measure of the resistance of the body to angular acceleration.

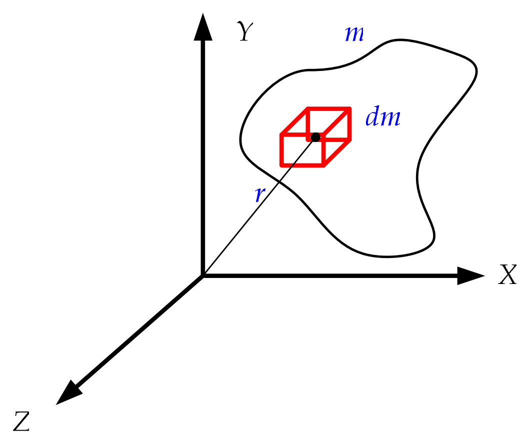


Figure 3-6 Moment of inertia

PARALLEL AXIS THEOREM. Sometimes it is necessary to calculate the moment of inertia of a homogeneous rigid body about an axis other than its geometrical axis.

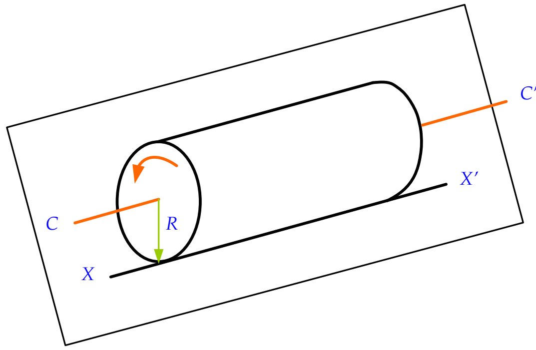


Figure 3-7 Homogeneous cylinder rolling on a flat surface

As an example to that, consider the system shown in Figure 3-7, where a cylinder of mass m and a radius R rolls on a flat surface. The moment of inertia of the cylinder is about axis *CC'* is

$$J_c = \frac{1}{2} m R^2$$

The moment of inertia J_x of the cylinder about axis *xx'* is

$$J_x = J_c + m R^2 = \frac{1}{2} m R^2 + m R^2 = \frac{3}{2} m R^2$$

Forced response and natural response. The behavior determined by a forcing function is called a forced response, and that due initial conditions is called natural response. The period between initiation of a response and the ending is referred to as the transient period. After the response has become negligibly small, conditions are said to have reached a steady state.

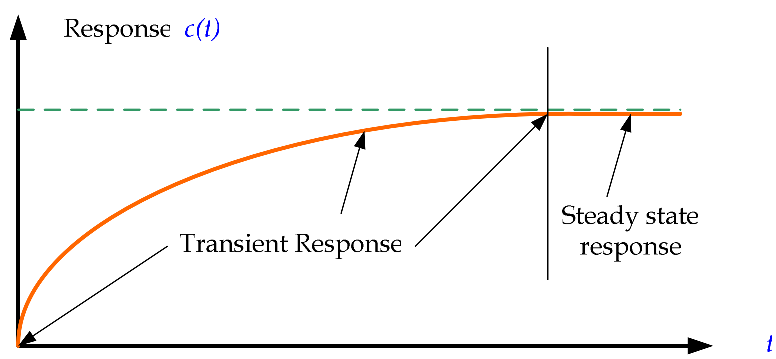


Figure 3-8 Transient and steady state response

Parallel and Series Springs Elements. In many applications, multiple spring elements are used, and in such cases we must obtain the equivalent spring constant of the combined elements.

Parallel Springs. For the springs in parallel, Figure 3-9, the equivalent spring constant k_{eq} is obtained from the relation

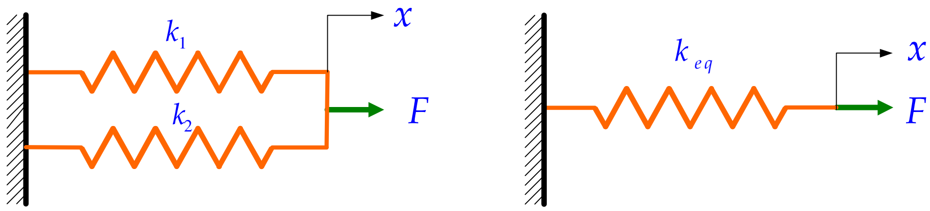


Figure 3-9 Parallel spring elements

$$F = k_1\, x + k_2\, x = (k_1 + k_2)\, x = k_{eq}\, x$$

where

$$k_{eq} = k_1 + k_2 \quad (\text{for parallel springs}) \tag{3-9}$$

This formula can be extended to n springs connected side-by-side as follows:

$$k_{eq} = \sum_{i=1}^{n} k_i \quad (\text{for parallel springs}) \tag{3-10}$$

Series Springs. For the springs in series, Figure 3-10, the force in each spring is the same. Thus

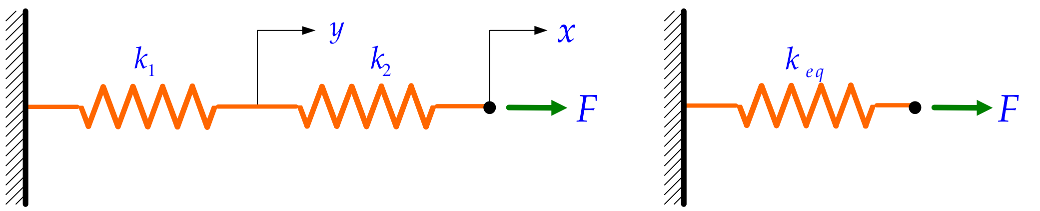


Figure 3-10 Series spring elements

$$F = k_1\, y, \qquad F = k_2\, (x - y)$$

Eliminating from these two equations yields

$$F = k_2 \left(x - \frac{F}{k_1} \right)$$

or

$$F = k_2 x - \frac{k_2}{k_1} F \quad \Rightarrow \quad k_2 x = F + \frac{k_2}{k_1} F = \left(\frac{k_1 + k_2}{k_1} \right) F$$

or

$$x = \left(\frac{k_1 + k_2}{k_2 k_1}\right) F \quad \Leftrightarrow \quad F = \left(\frac{k_2 k_1}{k_1 + k_2}\right) x = \underbrace{\left(\frac{1}{\frac{1}{k_1} + \frac{1}{k_2}}\right)}_{k_{eq}} x$$

where

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} \quad \text{(for series springs)} \tag{3-11}$$

which can be extended to the case of n springs connected end-to-end as follows

$$\frac{1}{k_{eq}} = \sum_{i=1}^{n} \frac{1}{k_i} \quad \text{(for series springs)} \tag{3-12}$$

FREE VIBRATION WITHOUT DAMPING.

Consider the mass spring system shown in Figure 3-11. The equation of motion can be given by

$$m\,\ddot{x} + k\,x = 0$$

or

$$\ddot{x} + \frac{k}{m} x = 0 \quad \Rightarrow \quad \ddot{x} + \omega_n^2\, x = 0$$

where

$$\omega_n = \sqrt{\frac{k}{m}}$$

is the natural frequency of the system and is expressed in rad/s.
Taking LT of both sides of the above equation where $x(0) = x_\circ$ and $\dot{x}(0) = \dot{x}_\circ$ gives

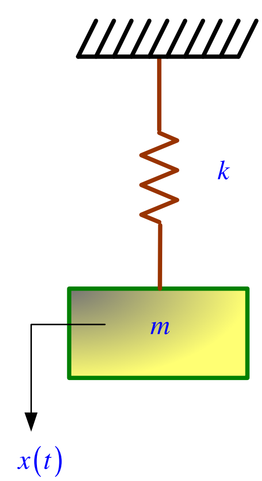


Figure 3-11 Mass Spring System

$$\underbrace{s^2 X(s) - s x(0) - \dot{x}(0)}_{L[\ddot{x}]} + \omega_n^2 X(s) = 0$$

rearrange to get

$$X(s) = \frac{s x_\circ + \dot{x}_\circ}{s^2 + \omega_n^2}, \quad \Rightarrow \text{Remember poles are } \underbrace{s = \pm j\omega_n}_{\text{complex conjugates}}$$

$$X(s) = \frac{\dot{x}_\circ}{\omega_n}\frac{\omega_n}{s^2+\omega_n^2} + x_\circ \frac{s}{s^2+\omega_n^2}$$

and the response $x(t)$ is given by

$$x(t) = \frac{\dot{x}_\circ}{\omega_n}\sin(\omega_n t) + x_\circ \cos(\omega_n t)$$

It is clear that the response $x(t)$ consists of a sine and cosine terms and depends on the values of the initial conditions $x_\circ$ and $\dot{x}_\circ$. Periodic motion such that described by the above equation is called ***simple harmonic motion***.

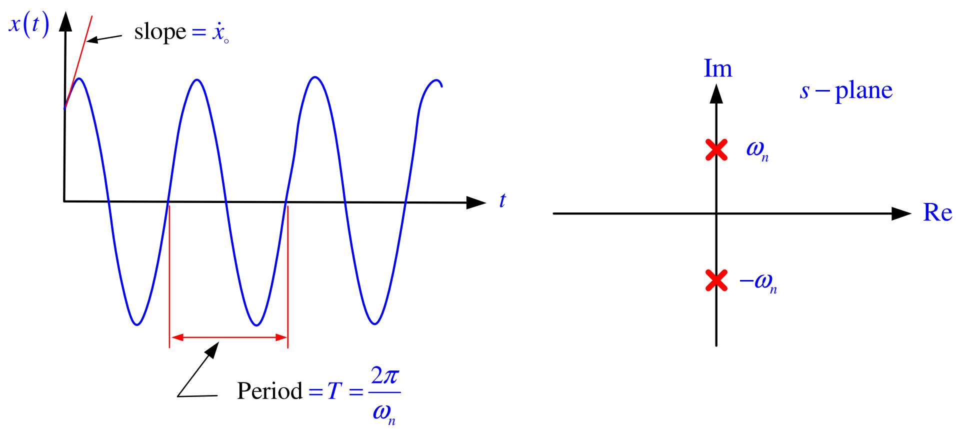


Figure 3-12 Free response of a simple harmonic motion and pole location on the s-plane

if $\dot{x}(0) = \dot{x}_\circ = 0,$

$$x(t) = x_\circ \cos(\omega_n t)$$

The period T is the time required for a periodic motion to repeat itself. In the present case,

$$\text{Period } T = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{\frac{k}{m}}} \text{ seconds}$$

The frequency f of a periodic motion is the number of cycles per second (cps), and the standard unit of frequency is the Hertz (Hz); that is 1 Hz is 1 cps. In the present case,

$$\text{Frequency } f = \frac{1}{T} = \frac{\sqrt{\frac{k}{m}}}{2\pi} \text{ Hz}$$

The undamped natural frequency ω_n is the frequency in the free vibration of a system having no damping. If the natural frequency is measured in Hz or in cps, it is denoted by f_n. If it is measured in rad/sec, it is denoted by ω_n. In the present system,

$$\omega_n = 2\pi f_n = \sqrt{\frac{k}{m}} \text{ rad/sec}$$

ROTATIONAL SYSTEM. Rotor mounted in bearings is shown in figure 3-13 below. The moment of inertia of the rotor about the axis of rotation is J. Friction in the bearings is viscous friction and that no external torque is applied to the rotor.

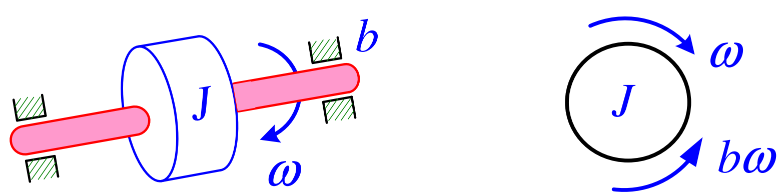


Figure 3-13 Rotor mounted in bearings and its FBD.

Apply Newton's second law for a system in rotation

$$\sum M = J\ddot{\theta} = J\dot{\omega}$$

$$J\dot{\omega} + b\omega = 0 \quad \Rightarrow \quad \dot{\omega} + (b/J)\omega = 0$$

or

$$\dot{\omega} + \frac{1}{(J/b)}\omega = 0$$

Define the time constant $\tau = (J/b)$, the previous equation can be written in the form

$$\dot{\omega} + \frac{1}{\tau}\omega = 0, \quad \omega(0) = \omega_\circ$$

which represents the equation of motion as well as the mathematical model of the system shown. It represents a ***first order system***. To find the response $\omega(t)$, take LT of both sides of the previous equation.

$$\left[\underbrace{s\Omega(s)-\omega(0)}_{L[\dot{\omega}]}\right]+\frac{1}{\tau}\left[\underbrace{\Omega(s)}_{L[\omega]}\right]=0$$

$$\left(s+\frac{1}{\tau}\right)\Omega(s)=\omega_\circ \qquad \Rightarrow \qquad \Omega(s)=\frac{\omega_\circ}{s+(1/\tau)}$$

where the denominator $s+(1/\tau)$ is known as the characteristic polynomial and the equation $s+(1/\tau)=0$ is called the characteristic equation.

Taking inverse LT of the above equation will give the expression of $\omega(t)$

$$\omega(t)=\omega_\circ\, e^{-(b/J)t}=\omega_\circ\, e^{-(1/\tau)t}=\omega_\circ\, e^{-\alpha t}$$

It is clear that the angular velocity decreases exponentially as shown in the figure below. Since $\lim_{t\to\infty} e^{-(t/\tau)}=0;$ then for such decaying system, it is convenient to depict the response in terms of a time constant.

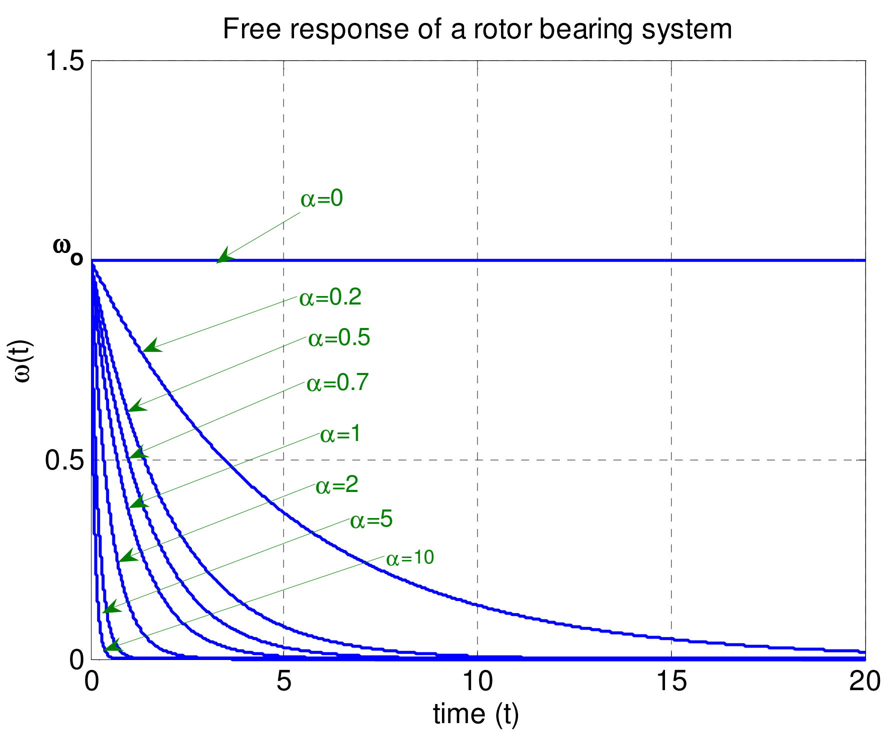


Figure 3-14 Graph of $\omega_\circ\, e^{-\alpha t}$ for ranges of α.

A **time constant** is that value of time that makes the exponent equal to -1. For this system, time constant $\tau = J/b$. When $t = \tau$, the exponent factor is

$$e^{-(t/\tau)} = e^{-(\tau/\tau)} = e^{-1} = 0.368 = 36.8\ \%$$

This means that when **time constant** $= \tau$, the time response is reduced to $36.8\ \%$ of its final value. We also have

$$\tau = J/b = \text{time constant}$$
$$\omega(\tau) = 0.37\ \omega_\circ$$
$$\omega(4\tau) = 0.02\ \omega_\circ$$

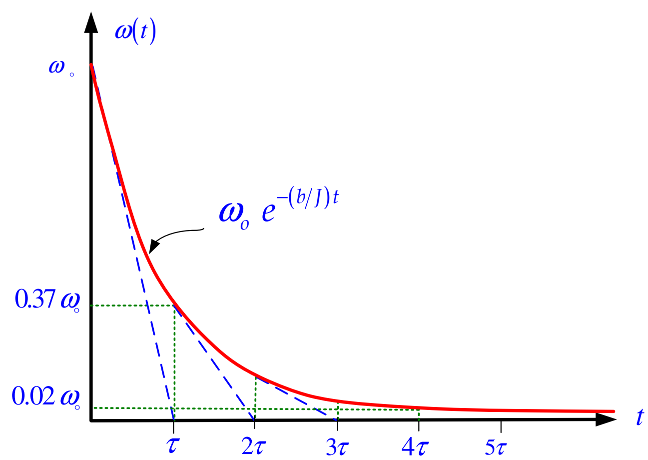


Figure 3-15 Curve of angular velocity ω versus time t for the rotor shown in Figure 3-13.

http://www.sciences.univ-nantes.fr/physique/perso/gtulloue/equadiff/equadiff.html

Spring-Mass-Damper System. Consider the simple mechanical system shown involving viscous damping. Obtain the mathematical model of the system shown.

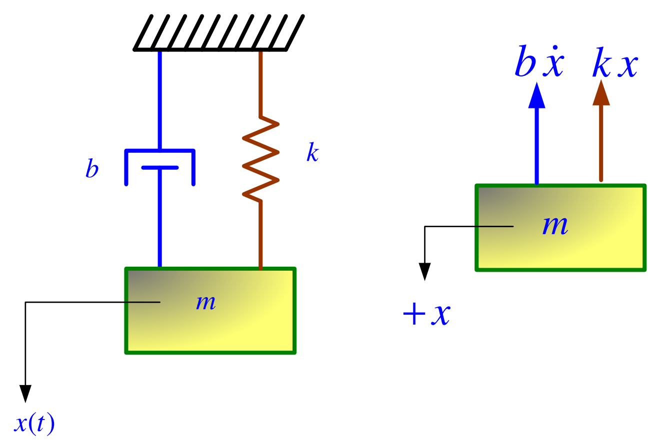


Figure 3-16 Mass -Spring –Damper System and the FBD.

i) The FBD is shown in the figure 3-16.
ii) Apply Newton's second law of motion to a system in translation:

$$\sum F = m\ddot{x}$$
$$-b\dot{x} - kx = m\ddot{x}$$

or

$$m\ddot{x} + b\dot{x} + kx = 0 \quad \Rightarrow \quad \text{Free Vibration of a second order system}$$

If in SI units $m = 0.1\,\text{kg}$, $b = 0.4\,\text{N/m-s}$, and $k = 4\,\text{N/m}$, the above differential equation becomes

$$0.1\ddot{x} + 0.4\dot{x} + 4x = 0 \quad \Rightarrow \quad \ddot{x} + 4\dot{x} + 40x = 0$$

To obtain the free response $x(t)$, assume $x(0) = x_\circ$ and $\dot{x}(0) = 0$. By writing the Laplace transform of $x(t)$ as $L[x(t)] = X(s)$, we obtain Laplace transform of both sides of the given equation

$$\underbrace{\left[s^2X(s) - s\,x(0) - \dot{x}(0)\right]}_{L[\ddot{x}(t)]} + 4\underbrace{\left[sX(s) - x(0)\right]}_{L[\dot{x}(t)]} + 40\underbrace{\left[X(s)\right]}_{L[x(t)]} = 0$$

Substitute in the transformed equation $x(0) = x_\circ$ and $\dot{x}(0) = 0$, and rearrange, we obtain

$$\left[s^2 + 4s + 40\right]X(s) = \left[sx_\circ + 4x_\circ\right]$$

or solving for $X(s)$ yields

$$X(s) = \frac{(sx_\circ + 4x_\circ)}{s^2 + 4s + 40} = \frac{(s+4)}{\underbrace{s^2 + 4s + 40}_{\text{Characteristic polynomial}}}x_\circ$$

Which can be written as

$$G(s) = \frac{X(s)}{x_\circ} = \frac{(s+4)}{\underbrace{s^2 + 4s + 40}_{\text{Characteristic polynomial}}}$$

where $G(s)$ is referred to as the ***transfer function*** that gives the relationship between the input $x_\circ$ and the output $X(s)$. $G(s)$ can be shown graphically as:

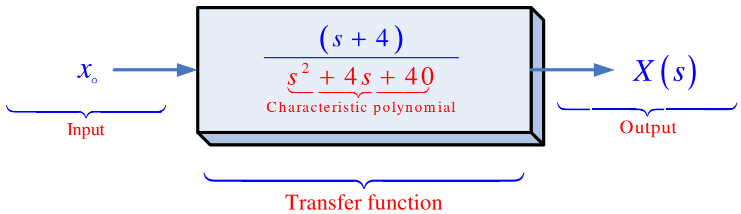


Figure 3-17 Transfer function between input and output.

iii) It is clear that the characteristic equation of the system is $s^2+4s+40=0$ and has complex conjugates roots.

$$s^2+4s+40=\underbrace{s^2+4s+4}_{(s+2)^2}+36=(s+2)^2+6^2=0$$

The roots of the above equation are therefore complex conjugate poles given by

$$s_1=-2+j6 \quad \text{and} \quad s_2=-2-j6$$

iv) The expression of $X(s)$ can be written now as:

$$X(s)=\frac{(s+4)}{s^2+4s+40}x_\circ=\frac{(s+2+2)}{(s+2)^2+6^2}x_\circ=\frac{(s+2)}{(s+2)^2+6^2}x_\circ+\frac{2}{(s+2)^2+6^2}x_\circ$$

$$=\frac{(s+2)}{(s+2)^2+6^2}x_\circ+\frac{1}{3}\frac{6}{(s+2)^2+6^2}x_\circ$$

v) Solving for $x(t)=L^{-1}\left[X(s)\right]$ yields

$$x(t)=x_\circ\left(e^{-2t}\cos 6t+\frac{1}{3}e^{-2t}\sin 6t\right)$$

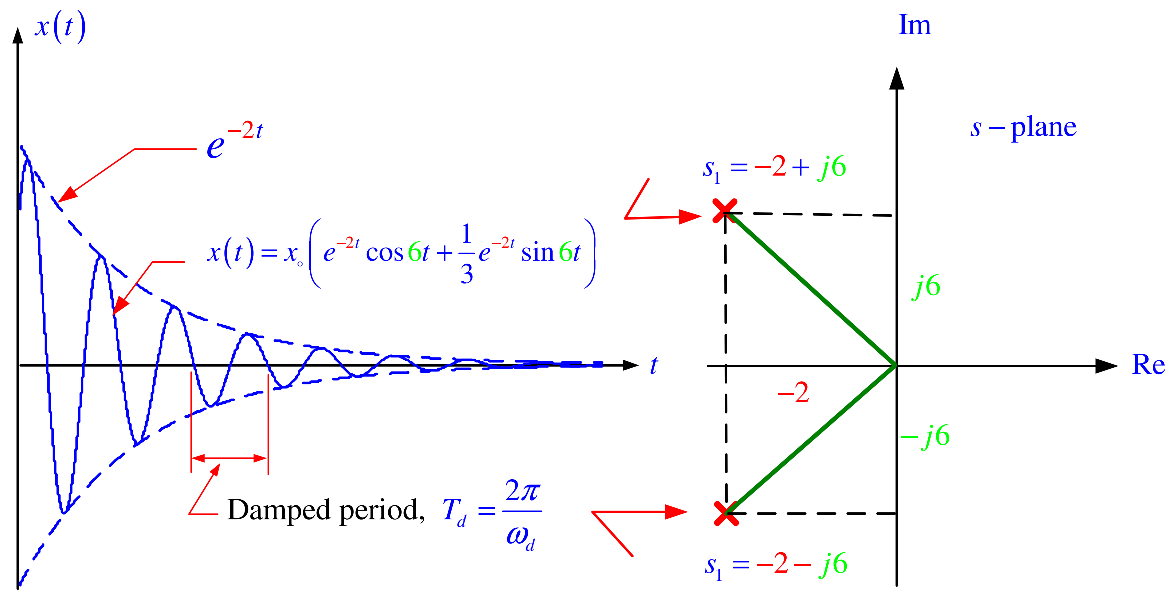
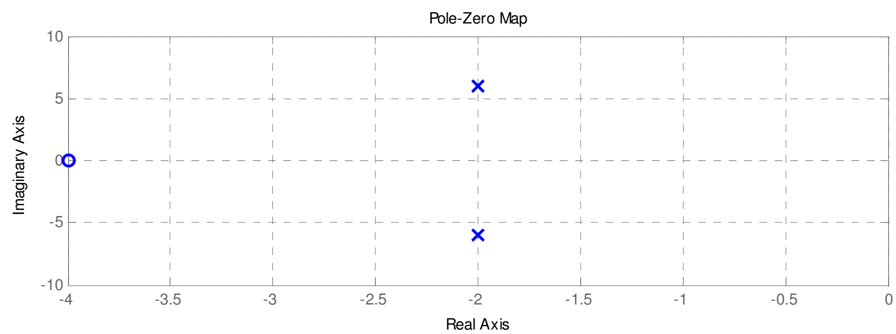


Figure 3-18 Free Vibration of the mass-spring-damper system described by $\ddot{x}+4\dot{x}+40\,x=0$ with initial conditions $x(0)=x_{\circ}$ and $\dot{x}(0)=0,$.



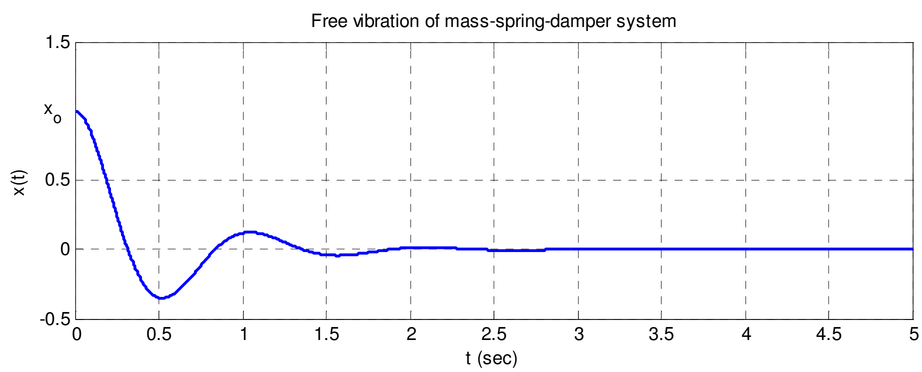


Figure 3-18 Free Vibration of the mass-spring-damper system described by $\ddot{x}+4\dot{x}+40\,x=0$ with initial conditions $x(0)=x_{\circ}$ and $\dot{x}(0)=0,$.

Remark: The expression of $a\cos(\omega t)+b\sin(\omega t)$ can be written in terms of $\cos(\omega t)$ or $\sin(\omega t)$, that is

$$a\cos(\omega t)+b\sin(\omega t)=\sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\cos(\omega t)+\frac{b}{\sqrt{a^2+b^2}}\cos(\omega t)\right)$$

Define ψ such that

$$\cos\psi=\frac{a}{\sqrt{a^2+b^2}} \quad \text{and} \quad \sin\psi=\frac{b}{\sqrt{a^2+b^2}} \quad \Rightarrow \quad \psi=\tan^{-1}\left(\frac{b}{a}\right)$$

Therefore

$$a\cos(\omega t)+b\sin(\omega t)=\sqrt{a^2+b^2}\left(\cos\psi\cos(\omega t)+\sin\psi\sin(\omega t)\right)$$

Using the identity

$$\cos(\omega t-\psi)=\cos(\omega t)\cos\psi+\sin(\omega t)\sin\psi$$

Therfore,

$$a\cos(\omega t)+b\sin(\omega t)=\sqrt{a^2+b^2}\left(\cos\omega t-\psi\right)$$

or

$$a\cos(\omega t)+b\sin(\omega t)=\sqrt{a^2+b^2}\left(\sin\omega t+\varphi\right)$$

Where

$$\sin\varphi=\frac{a}{\sqrt{a^2+b^2}} \quad \text{and} \quad \sin\varphi=\frac{b}{\sqrt{a^2+b^2}} \quad \Rightarrow \quad \varphi=\tan^{-1}\left(\frac{a}{b}\right)$$

Therefore,

$$x(t)=\frac{\sqrt{10}}{3}x_{\circ}e^{-2t}\left(\sin 6t+71.56^{\circ}\right)$$

3.4 WORK ENERGY, AND POWER

Work. The ***work*** done in a mathematical system is the product of a force and a distance (or a torque and the angular displacement) through which the force is exerted with both force and distance measured in the same direction.

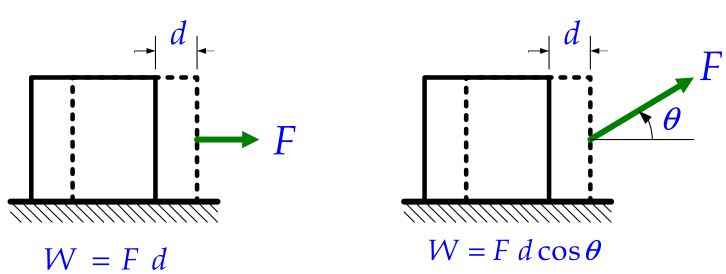


Figure 3-19 Work done by a force

The units of work in SI units are :

$$[\text{work}] = [\text{force} \times \text{distance}] = [\text{N-m}] = [\text{Joule}] = [\text{J}] \ .$$

The work done by a spring is given by:

$$W = \int_0^x \underbrace{k\,x}_{F}\,\mathrm{d}x = \frac{1}{2} k\,x^2$$

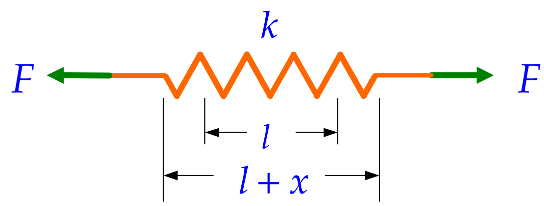


Figure 3-19 Work done by a spring.

ENERGY. ***Energy*** can be defined as ***ability to do work***. Energy can be found in many different forms and can be converted from one form to another. For instance, an electric motor converts electrical energy into mechanical energy, a battery converts chemical energy into electrical energy, and so forth.

According to the law of conservation of energy, energy can be neither created nor destroyed. This means that the increase in the total energy within a system is equal to the net energy input to the system. So if there is no energy input, there is no change in the total energy of the system.

POTENTIAL ENERGY. The ***Energy*** that a body possesses because of its position is called ***potential energy.***

- In mechanical systems, only mass and spring can store potential energy.
- The change in the potential energy stored in a system equals the work required to change the system's configuration.
- Potential energy is always measured with reference to some chosen level and is relative to that level.

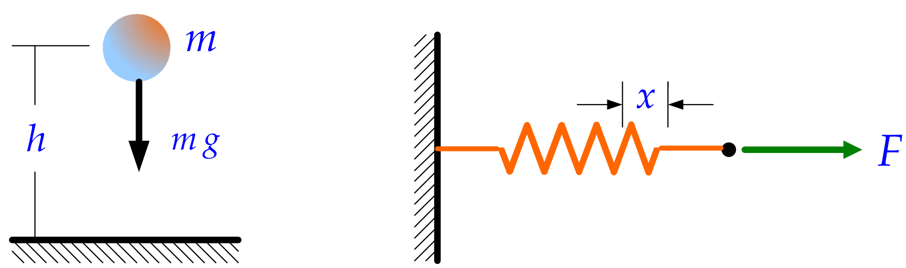


Figure 3-20 Potential energy

Refer to Figure 3-20, the potential energy, U of a mass m is given by:

$$U = \int_0^x mg\,\mathrm{d}x = mgh$$

For a translational spring, the potential energy U (sometimes called strain energy which is potential energy that is due to elastic deformations) is:

$$U = \int_0^x F\,\mathrm{d}x = \int_0^x k\,x\,\mathrm{d}x = \frac{1}{2}k\,x^2$$

If the initial and final values of x are x_1 and x_2, respectively, then

$$\text{Change in potential energy } \Delta U = \int_{x_1}^{x_2} F\,\mathrm{d}x = \int_{x_1}^{x_2} k\,x\,\mathrm{d}x = \frac{1}{2}k\,x_2^2 - \frac{1}{2}k\,x_1^2$$

Similarly, for a torsional spring

$$\text{Change in potential energy } \Delta U = \int_{\theta_1}^{\theta_2} T\,\mathrm{d}\theta = \int_{\theta_1}^{\theta_2} k_T\,\theta\,\mathrm{d}x = \frac{1}{2}k_T\,\theta_2^2 - \frac{1}{2}k_T\,\theta_1^2$$

KINETIC ENERGY. Only inertia elements can store kinetic energy in mechanical systems.

$$T = \text{Kinetic energy} = \begin{cases} \frac{1}{2}mv^2 & \text{(Translation)} \\ \frac{1}{2}J\dot{\theta}^2 & \text{(Rotation)} \end{cases}$$

The change in kinetic energy of the mass is equal to the work done on it by an applied force as the mass accelerates or decelerates. Thus, the change in kinetic energy T of a mass m moving in a straight line is

$$\text{Change in kinetic energy } \Delta T = \Delta W = \int_{x_1}^{x_2} F\,\mathrm{d}x = \int_{t_1}^{t_2} F\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t$$

$$= \int_{t_1}^{t_2} F\,v\,\mathrm{d}t = \int_{t_1}^{t_2} m\dot{v}\,v\,\mathrm{d}t = \int_{v_1}^{v_2} m\,v\,\mathrm{d}v$$

$$= \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

The change in kinetic energy of a moment of inertia in pure rotation at angular velocity $\dot{\theta}$ is

$$\text{Change in kinetic energy } \Delta T = \frac{1}{2}J\dot{\theta}_2^2 - \frac{1}{2}J\dot{\theta}_1^2$$

DISSIPATED ENERGY. Consider the damper shown in Figure 3-21 in which one end is fixed and the other end is moved from x_1 to x_2. The dissipated energy ΔW in the damper is equal to the net work done on it:

$$\Delta W = \int_{x_1}^{x_2} F\,\mathrm{d}x = \int_{x_1}^{x_2} \underbrace{b\,\dot{x}}_{F}\,\mathrm{d}x = b\int_{t_1}^{t_2} \dot{x}\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t = b\int_{t_1}^{t_2} \dot{x}^2\,\mathrm{d}t$$

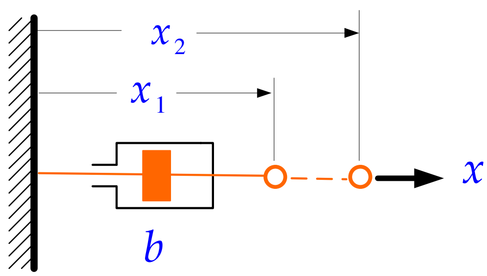


Figure 3-20 Damper.

POWER. Power is the time rate of doing work. That is,

$$\text{Power} = P = \frac{\mathrm{d}W}{\mathrm{d}t}$$

where $\mathrm{d}W$ denotes work done during time interval $\mathrm{d}t$.
In SI units, the work done is measured in Newton-meters and the time in seconds. The unit of power is :

$$[\text{Power}] = \left[\frac{\text{N-m}}{\text{s}}\right] = \left[\frac{\text{Joule}}{\text{s}}\right] = [\text{Watt}] = \text{W} \;.$$

PASSSIVE ELEMENTS. Non-energy producing element. They can only store energy, not generate it such as springs and masses.

ACTIVE ELEMENTS. Energy producing elements such as external forces and torques.

ENERGY METHOD FOR DERIVING EQUATIONS OF MOTION. Equations of motion are derive from the fact that the total energy of a system remains the same if no energy enters or leaves the system.

Conservative Systems.

Systems that do not involve friction (damping) are called *conservative systems*.

$$\underbrace{\Delta(T+U)}_{\text{Change in the total energy}} = \underbrace{\Delta W}_{\text{Net work done on the system by external forces}}$$

If no external energy enters the system ($\Delta W = 0$, no work done by external forces) then

$$\Delta(T+U)=0$$

or

$$(T+U)=\text{constant}$$

Conservation of energy only for conservative systems (No friction or damping)

An Energy Method for Determining Natural Frequencies.

The natural frequency of a conservative system can be obtained from a consideration of the kinetic energy and the potential energy of the system. Let us assume that we choose the datum line so that the potential energy at the equilibrium state is zero. Then in such a conservative system, the maximum kinetic energy equals the maximum potential energy , or

$$T_{max} = U_{max}$$

Solved Problems.

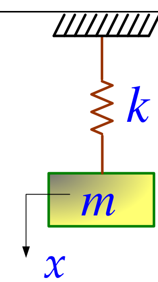
Example 3-5 Page 80 (Textbook)

Consider the system shown in the Figure shown. The displacement x is measured from the equilibrium position.

The Kinetic energy is: $T = \frac{1}{2} m \dot{x}^2$

The potential energy is: $U = \frac{1}{2} k x^2$

The total energy of the system is $T + U = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2$



The change in the total energy is

$$\frac{\mathrm{d}}{\mathrm{d}t}(T+U)=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m\dot{x}^2+\frac{1}{2}kx^2\right)=0$$

$$=\frac{1}{\not{2}}\times\not{2}\times m\dot{x}\ddot{x}+\frac{1}{\not{2}}\times\not{2}\times kx\dot{x}=0$$

$$=\dot{x}\left(m\ddot{x}+kx\right)=0$$

Since $\dot{x}$ is not zero then we should have

$$m\ddot{x}+kx=0$$

or

$$\ddot{x}+\frac{k}{m}x=0 \quad\Rightarrow\quad \ddot{x}+\omega_n^2\,x=0$$

where

$$\omega_n=\sqrt{\frac{k}{m}}$$

is the natural frequency of the system and is expressed in rad/s. Another way of finding the natural frequency of the system is to assume a displacement of the form

$$x=A\sin\omega_n t$$

Where A is the amplitude of vibration. Consequently,

$$T=\frac{1}{2}m\dot{x}^2=\frac{1}{2}m\,A^2\omega^2\left(\cos\omega_n t\right)^2$$

$$U=\frac{1}{2}kx^2=\frac{1}{2}k\,A^2\left(\sin\omega_n t\right)^2$$

Hence the maximum values of T and U are given by

$$T_{\max}=\frac{1}{2}m\,A^2\omega_n^2, \qquad U_{\max}=\frac{1}{2}k\,A^2$$

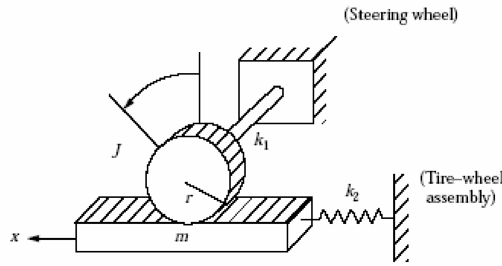
Since $T_{\max}=U_{\max}$, we have

$$\frac{1}{2}m\,A^2\omega_n^2=\frac{1}{2}k\,A^2$$

From which

$$\omega_n=\sqrt{\frac{k}{m}}$$

1.49 Use the energy method to calculate the equation of motion and natural frequency of an airplane's steering mechanism for the nose wheel of its landing gear. The mechanism is modeled as the single-degree-of-freedom system illustrated in Figure P1.49.



The steering wheel and tire assembly are modeled as being fixed at ground for this calculation. The steering rod gear system is modeled as a linear spring and mass system (m, k_2) oscillating in the x direction. The shaft-gear mechanism is modeled as the disk of inertia J and torsional stiffness k_2. The gear J turns through the angle θ such that the disk does not slip on the mass. Obtain an equation in the linear motion x.

Solution:

From kinematics: $x = r\theta, \Rightarrow \dot{x} = r\dot{\theta}$

Kinetic energy: $T = \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}m\dot{x}^2$

Potential energy: $U = \frac{1}{2}k_2x^2 + \frac{1}{2}k_1\theta^2$

Substitute $\theta = \frac{x}{r}: T + U = \frac{1}{2}\frac{J}{r^2}\dot{x}^2 + \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k_2x^2 + \frac{1}{2}\frac{k_1}{r^2}x^2$

Derivative: $\frac{d(T+U)}{dt} = 0$

$$\frac{J}{r^2}\dot{x}\ddot{x} + m\dot{x}\ddot{x} + k_2x\dot{x} + \frac{k_1}{r^2}x\dot{x} = 0$$

$$\left[\left(\frac{J}{r^2} + m\right)\ddot{x} + \left(k_2 + \frac{k_1}{r^2}\right)x\right]\dot{x} = 0$$

Equation of motion: $\left(\frac{J}{r^2} + m\right)\ddot{x} + \left(k_2 + \frac{k_1}{r^2}\right)x = 0$

Natural frequency: $\omega_n = \sqrt{\dfrac{k_2 + \frac{k_1}{r^2}}{\frac{J}{r^2} + m}}$

1.50 A control pedal of an aircraft can be modeled as the single-degree-of-freedom system of Figure P1.50. Consider the lever as a massless shaft and the pedal as a lumped mass at the end of the shaft. Use the energy method to determine the equation of motion in θ and calculate the natural frequency of the system. Assume the spring to be unstretched at θ = 0.

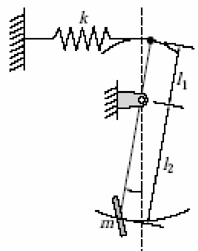


Figure P1.50

Solution: In the figure let the mass at θ = 0 be the lowest point for potential energy. Then, the height of the mass m is $(1-\cos\theta)\ell_2$.

Kinematic relation: $x = \ell_1\theta$

Kinetic Energy: $T = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\ell_2^2\dot{\theta}^2$

Potential Energy: $U = \frac{1}{2}k(\ell_1\theta)^2 + mg\ell_2(1-\cos\theta)$

Taking the derivative of the total energy yields:

$$\frac{d}{dt}(T+U) = m\ell_2^2\dot{\theta}\ddot{\theta} + k(\ell_1^2\theta)\dot{\theta} + mg\ell_2(\sin\theta)\dot{\theta} = 0$$

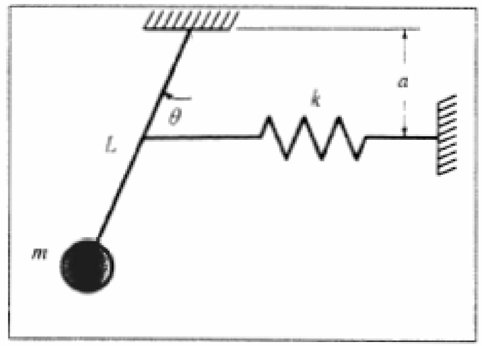
Rearranging, dividing by $d\theta/dt$ and approximating sinθ with θ yields:

$$m\ell_2^2\ddot{\theta} + (k\ell_1^2 + mg\ell_2)\theta = 0$$

$$\Rightarrow \omega_n = \sqrt{\frac{k\ell_1^2 + mg\ell_2}{m\ell_2^2}}$$

Problem # 2: (35 marks)

The figure shows a pendulum which consists of a light rod of length L pivoted to a fixed point at one end and having a mass m to its other end. A spring of stiffness k is attached as shown, at a distance a from the pivot. In the position shown the rod is displaced with a small angle θ from the equilibrium position.



Find the frequency of free oscillations of small amplitude in the plane of the diagram.

ENERGY METHOD

Kinetic energy $T = \frac{1}{2} m \dot{x}^2 = \frac{1}{2} m \left(L \dot{\theta}\right)^2 = \frac{1}{2} m L^2 \dot{\theta}^2$

Potential Energy $V = \underbrace{\frac{1}{2} k x_1^2}_{\text{Due to elastic deformation}} + \underbrace{m g L(1-\cos\theta)}_{\text{Due to gravitation}}$

Since $x_1 = a\,\theta$ the above expression becomes

$$T + V = \frac{1}{2} m L^2 \dot{\theta}^2 + \frac{1}{2} k a^2 \theta^2 + m g L(1-\cos\theta)$$

$$\frac{d}{dt}(T+V) = \frac{1}{\cancel{2}} \cancel{2} m L^2 \ddot{\theta}\dot{\theta} + \frac{1}{\cancel{2}} \cancel{2} k a^2 \dot{\theta}\theta + \dot{\theta} m g L \sin\theta = 0$$

Since for small oscillations $\sin\theta = \theta$, the above expression can be written as

$$\dot{\theta}\left[m L^2 \ddot{\theta} + \left(k a^2 + m g L\right)\theta\right] = 0$$

since $\dot{\theta} \neq 0$, then

$$\left[m L^2 \ddot{\theta} + \left(k a^2 + m g L\right)\theta\right] = 0$$

or

$$\ddot{\theta} + \frac{\left(k a^2 + m g L\right)}{m L^2}\theta = 0$$

compared to the standard harmonic oscillator

$$\ddot{\theta} + \omega_n^2 \theta = 0$$

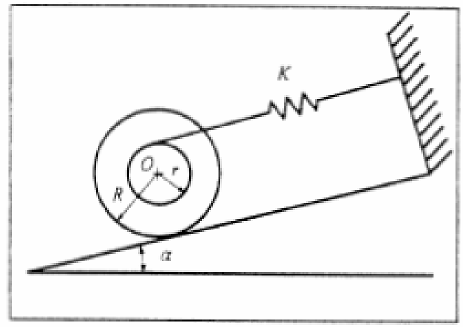
one can write

$$\omega_n = \sqrt{\frac{\left(k a^2 + m g L\right)}{m L^2}}$$

NEWTON'S SECOND LAW METHOD

Refer to the solved problems distributed to you or to problem A.6.7 Page 270 of your textbook

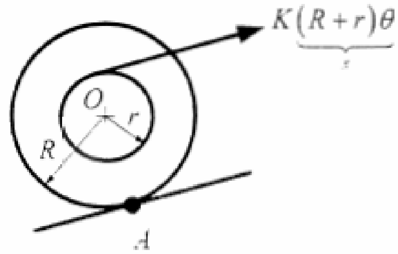
Problem # 3: (35 marks)

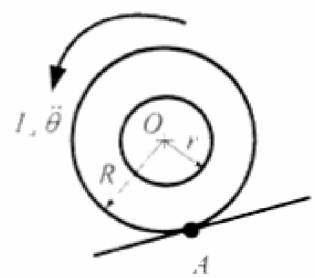


A uniform wheel of radius R can roll without slipping on an inclined plane. Concentric with the wheel, and fixed to it, is a drum of radius r around which is wrapped one end of the string. The other end of the string is fastened to a spring of stiffness k, as shown. Both spring and string are parallel to the plane. The total mass of the wheel/drum assembly is M and its moment of inertia about the axis through the center of the wheel O is J. If the wheel is displaced a small distance from its equilibrium position and released (neglecting damping)

1. **Derive** the equation describing the ensuing motion.
2. **Find** the frequency of the oscillations. Damping is negligible.

NEWTON'S SECOND LAW METHOD





If the wheel is given an anticlockwise rotation θ from the equilibrium position, the spring extension is $(R+r)\theta$ so that the restoring spring force is $K(R+r)\theta$. The rotation is instantaneously about the contact point A, so that taking moment about point O gives

$$\sum T = I_A \ddot{\theta}$$

or

$$-K(R+r)^2\theta = I_A \ddot{\theta}$$

where the moment due to weight cancels with the moment due to initial spring tension. The above expression can be written as:

$$I_A \ddot{\theta} + K(R+r)^2\theta = 0$$

or

$$\ddot{\theta} + \frac{K(R+r)^2}{I_A}\theta = 0$$

comparing this to the standard harmonic oscillator equation

$$\ddot{\theta} + \omega_n^2\theta = 0$$

one can write

$$\omega_n = \sqrt{\frac{K(R+r)^2}{I_A}}$$

knowing that $I_A = I_0 + m\,R^2$, the above expression can be written as

$$\omega_n = \sqrt{\frac{K(R+r)^2}{I_0 + m\,R^2}}$$

ENERGY METHOD

Kinetic energy $T = \frac{1}{2} J_A \dot{\theta}^2$

Potential Energy $V == \frac{1}{2} k(R+r)^2 \theta^2$

Where weight and initial spring tension effects cancel

$$T + V = \frac{1}{2} J_A \dot{\theta}^2 + \frac{1}{2} k(R+r)^2 \theta^2$$

$$\frac{d}{dt}(T+V) = \frac{1}{2} 2 J_A \ddot{\theta}\dot{\theta} + \frac{1}{2} 2k(R+r)^2 \dot{\theta}\theta = 0$$

The above expression can be written as

$$\dot{\theta}\left[J_A \ddot{\theta} + k(R+r)^2 \theta\right] = 0$$

since $\dot{\theta} \neq 0$, then

$$\left[J_A \ddot{\theta} + k(R+r)^2 \theta\right] = 0$$

or

$$\ddot{\theta} + \frac{k(R+r)^2}{J_A}\theta = 0$$

compared to the standard harmonic oscillator

$$\ddot{\theta} + \omega_n^2\,\theta = 0$$

one can write

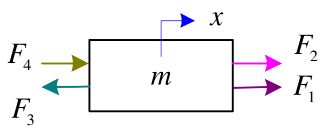
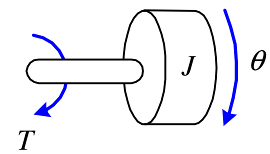
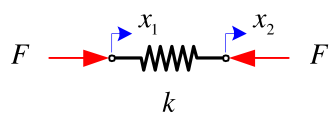
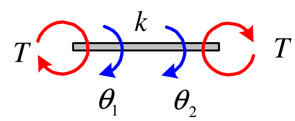
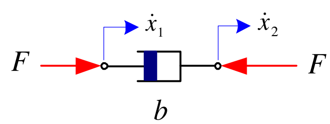
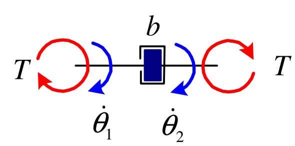
$$\omega_n = \sqrt{\frac{K(R+r)^2}{I_A}}$$

knowing that $I_A = I_0 + m\,R^2$, the above expression can be written as

$$\omega_n = \sqrt{\frac{K(R+r)^2}{I_0 + m\,R^2}}$$

which is similar to the one obtained above.

TABLE 1. SUMMARY OF ELEMENTS INVOLVED IN LINEAR MECHANICAL SYSTEMS

Element	Translation	Rotation
Inertia	 $\sum F = m\,a$	 $\sum T = J\,\alpha$
Spring	 $F = k(x_1 - x_2) = kx$	 $T = k(\theta_1 - \theta_2) = k\theta$
Damper	 $F = b(\dot{x}_1 - \dot{x}_2) = b\dot{x}$	 $T = b(\dot{\theta}_1 - \dot{\theta}_2) = b\dot{\theta}$

PROCEDURE

The motion of mechanical elements can be described in various dimensions as translational, rotational, or combination of both. The equations governing the motion of mechanical systems are often formulated from Newton's law of motion.

1. **Construct** a model for the system containing interconnecting elements.
2. **Draw** the free-body diagram.
3. **Write** equations of motion of all forces acting on the free body diagram. For translational motion, the equation of motion is Equation (1), and for rotational motion, Equation (2) is used.