CHAPTER 2
LAPLACE TRANSFORM

A. BAZOUNE

2.1 INTRODUCTION

Laplace Transform is one of the most important mathematical tools available for modeling and analyzing linear systems.

2.2 COMPLEX NUMBERS, COMPLEX VARIABLES, AND COMPLEX FUNCTIONS

Complex Numbers

Using the notation $j=\sqrt{-1}$, one can express all complex numbers in engineering calculations as

$$z = x + jy$$

where x is the real part and jy is the imaginary part. Notice that both x and y are real and that j is the only imaginary quantity in the expression above.

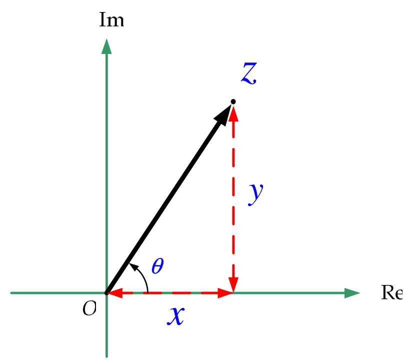


Fig. 2-1 Complex plane representation of a complex number z.

The **Magnitude**, or absolute value, of z is defined as the length of the directed segment shown in Fig. 2-1.

$$\text{Magnitude of } z = |z| = \sqrt{x^2 + y^2}$$

The **angle** of z is the angle that the directed line segment makes with the positive real axis. A ***counterclockwise rotation is defined as the positive direction*** for the measurement of angles.

$$\text{angle of } z = \theta = \tan^{-1}(y/x)$$

A complex number can be written in ***rectangular form*** as:

$$\left.\begin{array}{l} z = x + jy \\ z = |z|(\cos\theta + j\sin\theta) \end{array}\right\} \text{rectangular form}$$

and in ***polar form*** as

$$\left.\begin{array}{l} z = |z|\angle\theta \\ z = |z|e^{j\theta} \end{array}\right\} \text{polar form}$$

In converting complex numbers ***from rectangular to polar from*** , we use

$$z = |z| = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

To convert complex numbers ***from polar to rectangular form***, we employ

$$x = |z|\cos\theta, \qquad y = |z|\sin\theta$$

Complex Conjugate. The ***complex conjugate*** of $z = x + jy$ is defined as

$$\bar{z} = x - jy$$

The complex conjugate of z thus has the ***same real part*** as z and an ***imaginary part that is the negative of the imaginary part of*** z as shown in Fig. 2-2. Notice that

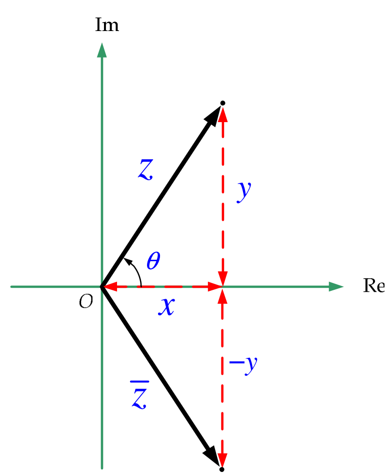


Fig. 2-2 Complex number z and its complex conjugate $\bar{z}$.

$$z = x + jy = |z| \angle \theta = |z|(\cos\theta + j\sin\theta)$$
$$\overline{z} = x - jy = |z| \angle (-\theta) = |z|(\cos\theta - j\sin\theta)$$

Euler's Theorem. The ***power series expansions of*** $\cos\theta$ and $\sin\theta$ are, respectively,

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

and

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

Thus

$$\cos\theta + j\sin\theta = 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \cdots$$

Since

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

it follows that

$$\cos\theta + j\sin\theta = e^{j\theta}$$

Using *the above relation*, one can express the sine and cosine in complex form. Noting that $e^{-j\theta}$ is the complex conjugate of $e^{j\theta}$ and that

$$e^{j\theta} = \cos\theta + j\sin\theta$$
$$e^{-j\theta} = \cos\theta - j\sin\theta$$

By adding the above expressions together, we find that

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

while by subtracting the second expression above from the first one, we obtain

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Complex Algebra.

Equality of complex numbers. Two complex numbers z_1 and z_2 are said to be equal if and only if their real parts are equal and their imaginary parts are equal. So if two complex numbers are written

$$z_1 = x_1 + jy_1, \quad \text{and} \quad z_2 = x_2 + jy_2$$

Then $z_1 = z_2$ if and only if $x_1 = x_2$ and $y_1 = y_2$.

Addition. Two complex numbers z_1 and z_2 in rectangular form can be added by adding the real parts and the imaginary parts separately:

$$z_1 + z_2 = (x_1 + jy_1) + (x_2 + jy_2) = (x_1 + x_2) + j(y_1 + y_2)$$

Subtraction. Subtracting one complex number from another can be considered as adding the negative of the former:

$$z_1 - z_2 = (x_1 + jy_1) - (x_2 + jy_2) = (x_1 - x_2) + j(y_1 - y_2)$$

Multiplication. If a complex number is multiplied by a real number, the result is a complex number whose real and imaginary parts are multiplied by that real number:

$$az = a(x + jy) = ax + jay, \quad (a = \text{ real number})$$

If two complex numbers appear in rectangular form and we want the product in rectangular form, multiplication is accomplished by using the fact that $j^2 = -1$. Thus, if two complex numbers are written

$$z_1 = x_1 + jy_1, \qquad z_2 = x_2 + jy_2$$

Then

$$\begin{aligned} z_1 z_2 &= (x_1 + jy_1)(x_2 + jy_2) = x_1x_2 + jx_1y_2 + jy_1x_2 + j^2y_1y_2 \\ &= (x_1x_2 - y_1y_2) + j(x_1y_2 + y_1x_2) \end{aligned}$$

In polar form, multiplication of two complex numbers can be done easily. The magnitude of the product is the product of the two magnitudes, and the angle of the product is the sum of the two angles. So if two complex numbers are written

$$z_1 = |z_1| \angle \theta_1 , \qquad z_2 = |z_2| \angle \theta_2$$

then

$$z_1 z_2 = |z_1||z_2| \angle (\theta_1 + \theta_2)$$

Multiplication by *j*. It is important to note that multiplication by j is equivalent to a counterclockwise rotation by 90°. For example, if

$$z = x + jy$$

then

$$jz = j(x + jy) = jx + j^2 y = -y + jx$$

or, noting that $j = 1 \angle 90$, if

$$z = |z| \angle \theta$$

then

$$jz = |1| \angle 90 |z| \angle \theta = |z| \angle (\theta + 90)$$

Fig. 2-3 illustrates the multiplication of a complex number z by j.

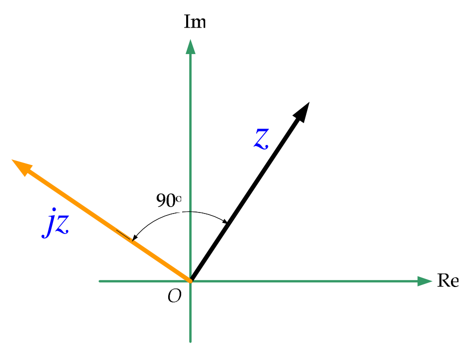


Fig. 2-3 Multiplication of a complex number z by j.

Division. If a complex number $z_1 = |z_1| \angle \theta_1$ is divided by another complex number $z_2 = |z_2| \angle \theta_2$, then

$$\frac{z_1}{z_2} = \frac{|z_1| \angle \theta_1}{|z_2| \angle \theta_2} = \frac{|z_1|}{|z_2|} \angle (\theta_1 - \theta_2)$$

That is, the result consists of the quotient of the magnitudes and the difference of the angles. Division in rectangular form can be done by multiplying the denominator and numerator by the complex conjugate of the denominator. For instance,

$$\frac{z_1}{z_2} = \frac{(x_1 + jy_1)}{(x_2 + jy_2)} = \frac{(x_1 + jy_1)(x_2 - jy_2)}{(x_2 + jy_2)\underbrace{(x_2 - jy_2)}_{\text{complex Conjugate of } z_2}}$$

$$= \frac{(x_1x_2 + y_1y_2) + j(x_2y_1 - x_1y_2)}{(x_2^2 + y_2^2)}$$

$$= \frac{(x_1x_2 + y_1y_2)}{(x_2^2 + y_2^2)} + j\frac{(x_2y_1 - x_1y_2)}{(x_2^2 + y_2^2)}$$

Division by *j*. Division by j is equivalent to a clockwise rotation by 90°. For example, if

$$z = x + jy$$

then

$$\frac{z}{j} = \frac{(x + jy)}{j} = \frac{(x + jy)j}{jj} = \frac{jx - y}{-1} = y - jx$$

or,

$$\frac{z}{j} = \frac{|z|\angle\theta}{1\angle 90^\circ} = |z|\angle(\theta - 90^\circ)$$

Fig. 2-4 illustrates the division of a complex number z by j.

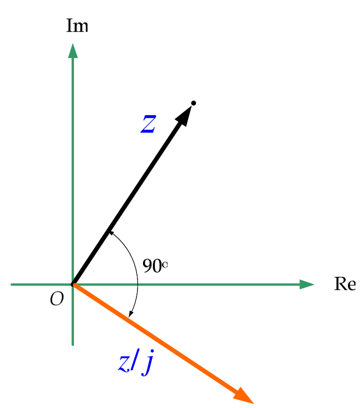


Fig. 2-4 Division of a complex number z by j.

Powers and roots. Multiplying z by n times, we obtain

$$z^n = \left(|z| \angle\theta\right)^n = |z|^n \angle n\theta$$

Extracting the nth root of a complex number is equivalent to raising the number to $1/n$th power.

$$z^{1/n} = \left(|z| \angle\theta\right)^{1/n} = |z|^{1/n} \angle\frac{\theta}{n}$$

For instance, calculate $(8.66 - j5)^3 = ?$

Remember :

The real part $= x = 8.66$
The imaginary part $= y = 5$

The magnitude $= |z| = \sqrt{x^2 + y^2} = \sqrt{8.66^2 + 5^2} = 10$

The angle $= \theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-5}{8.66}\right) = -30^\circ$

Therefore,

$(8.66 - j5)^3 = \left(10 \angle -30^\circ\right)^3 = 1000\angle -90^\circ = 0 - j1000 = -j1000$.

Remarks. It is important to note that

$$|zw| = |z||w|$$
$$|z+w| \neq |z| + |w|$$

Complex Variable. A complex variable has a real part and an imaginary part, both of which are constant. If the real part or the imaginary part (or both) are variables, the complex number is called a ***complex variable.*** In the Laplace transformation, we use the notation s to denote a complex variable; that is,

$$s = \sigma + j\omega$$

where σ is the real part and $j\omega$ is the imaginary part. (Notice that both σ and ω are real.)

Complex Function. A complex function $F(s)$, a function of s has a real part and an imaginary part, or

$$F(s) = F_x + jF_y$$

where F_x and F_y are real quantities.

$$\text{Magnitude of } F(s) = |F(s)| = \sqrt{F_x^2 + F_y^2}$$

$$\text{Angle of } F(s) = \theta = \tan^{-1}\left(F_y / F_x\right)$$

The angle is measured counterclockwise from the positive real axis. The ***complex conjugate*** of $F(s)$ is $\bar{F}(s) = F_x - jF_y$.

Complex functions commonly encountered in linear systems analysis are single-valued functions of s and are uniquely determined for a given value of s. Typically, such functions have the form

$$F(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

- Points at which $F(s) = 0$ are called ***zeros***. That is, $s = -z_1, s = -z_2 ,\cdots, s = -z_m$ are ***zeros*** of $F(s)$.
- Points at which $F(s) = \infty$ are called ***poles.*** That is, $s = -p_1, s = -p_2, \cdots, s = -p_n$ are ***poles*** of $F(s)$.
- If the denominator of $F(s)$ involves $k-$multiple factors $(s+p)^k$, then $s = -p$ is called a ***multiple pole of order*** k or ***repeated pole of multiplicity*** k. If $k = 1$, the pole is called a ***simple pole***.

Example

$$G(s) = \frac{K(s+2)(s+10)}{s(s+1)(s+5)(s+15)^2}$$

- Zeros of $G(s)$ are values of s which make $G(s) = 0$, that is $s = -2, s = -10$
- Poles of $G(s)$ are values of s which make $G(s) = \infty$, that is $s = -0,\ s = -1,\ s = -5 \Rightarrow$ Simple and distinct poles

$s = -15,\ s = -15$ $\Rightarrow$ double pole or (pole of multiplicity 2)

Since for large values of s (when $s \rightarrow \infty$)

$$G(s) = \frac{k}{s^3}$$

$G(s)$ possesses a triple zero at $(s = \infty, \infty, \infty)$ (pole of multiplicity 3). If points at infinity are included, $G(s)$ has the same number of poles as zeros.

To summarize: $G(s)$ has five zeros, $(s = -2, -10, \infty, \infty, \infty)$ and five poles, $(s = 0, -1, -5, -15, -15)$

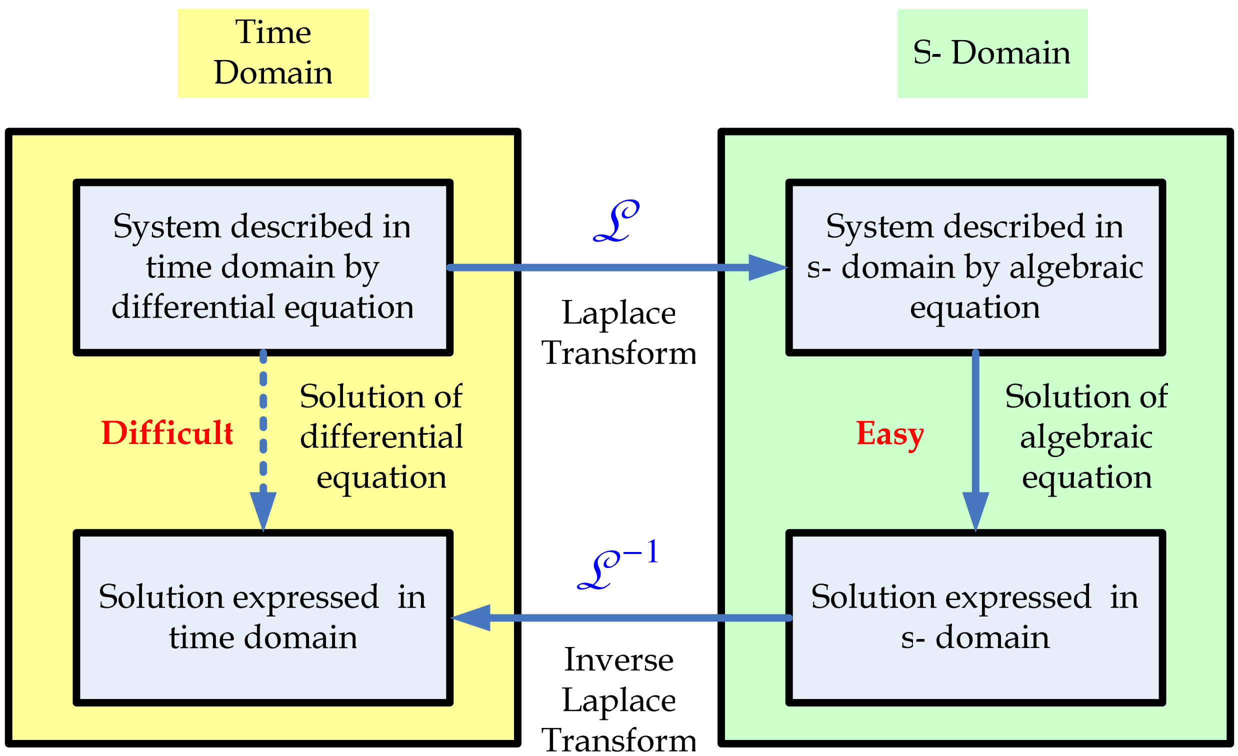
2.3 LAPLACE TRANSFORMATION

What is the Laplace Transform?

It is a solution technique that transforms differential equations in the time domain into algebraic equations in the s-domain.

Why use Laplace Transform?

The **Laplace transform** is a powerful tool formulated to solve a wide variety of **Initial-Value Problems** (IVP). The strategy is to transform the difficult differential equations into simple algebraic problems where solutions can be easily obtained. One then applies the **Inverse Laplace transform** to retrieve the solutions of the original problems. This can be illustrated as follows:



Definition

The Laplace Transform $F(s)$ of $f(t)$ is defined as

$$F(s) = \mathscr{L}\left[f(t)\right] = \int_0^\infty e^{-st} f(t)dt$$

where $f(t) =$ a time function such that $f(t) = 0$ for $t < 0$
$s =$ a complex variable
$F(s) =$ Laplace transform of $f(t)$

Exponential Function: Consider the exponential function shown in Fig. 2-5:

$$f(t) = \begin{cases} 0 & \text{for } t < 0 \\ A e^{-\alpha t} & \text{for } t \ge 0 \end{cases}$$

where A and α are constants. The Laplace transform of $f(t)$ can be obtained as follows

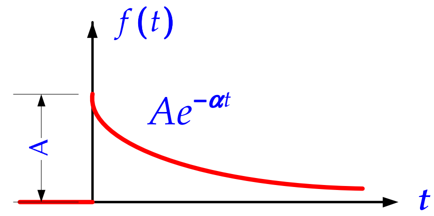


Fig. 2-5. Exponential function

$$F(s) = \mathscr{L}\left[Ae^{-\alpha t}\right] = \int_0^\infty Ae^{-\alpha t} e^{-st} dt = A\int_0^\infty e^{-(\alpha+s)t} dt = \frac{A}{s+\alpha}$$

Step Function: Consider a step function as shown in Fig. 2-6:

$$f(t) = \begin{cases} 0 & \text{for } t < 0 \\ A & \text{for } t > 0 \end{cases}$$

where A is a constant. Notice that this is a special case of the exponential function $Ae^{-\alpha t}$ where $\alpha = 0$. The step function is undefined at $t = 0$. Its Laplace Transform is given by:

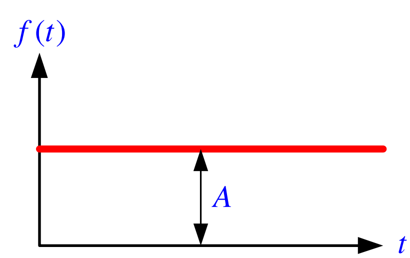


Fig. 2-6. Step function

$$F(s) = \mathscr{L}\left[f(t)\right] = \mathscr{L}[A] = \int_0^\infty A e^{-st} dt = \frac{A}{s}$$

The step function whose height is unity is called a ***unit-step function***. The unit step function that occurs at time $t = t_0$ is frequently written as $1(t - t_0)$. The previous step function whose height is A can be written as $A1(t)$. The Laplace Transform of the ***unit-step function*** that is defined by

$$1(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases}$$

is

$$F(s) = \mathscr{L}\left[1(t)\right] = \int_0^\infty 1e^{-st}dt = \frac{1}{s}$$

Physically, a step function occurring at time $t = t_0$ corresponds to a constant signal suddenly applied to the system at time t equals t_0.

Ramp Function: Consider a ramp function as shown in Fig. 2-7

$$f(t) = \begin{cases} 0 & \text{for } t < 0 \\ At & \text{for } t \geq 0 \end{cases}$$

where A is a constant. The Laplace Transform of the ramp function is:

$$F(s) = \mathscr{L}[At] = A\int_0^\infty te^{-st}dt$$

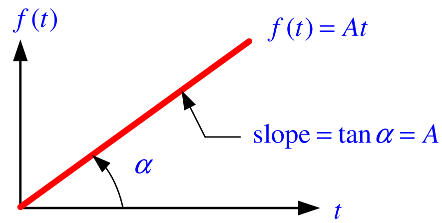


Fig. 2-7. Ramp function

To evaluate the above integral we use the formula for the integration by parts

$$\int_0^\infty u\,dv = uv\,|_0^\infty - \int_0^\infty v\,du$$

where in this case

$$u = At \quad \Rightarrow \quad du = A$$

$$dv = e^{-st}dt \quad \Rightarrow \quad v = -\frac{1}{s}e^{-st}$$

Substituting these expressions into equation the above integral leads to the following expression:

$$F(s) = \mathscr{L}[At] = \int_0^\infty Ate^{-st}dt = -\frac{Ate^{-st}}{s}\,|_0^\infty + \int_0^\infty \frac{Ae^{-st}}{s}dt$$

$$= (0-0) + \frac{\left(-Ae^{-st}\right)}{s^2}\,|_0^\infty = 0 - \frac{(-A)}{s^2} = \frac{A}{s^2}$$

Sinusoidal Function: The Laplace Transform of the sinusoidal function

$$f(t)=\begin{cases}0 & \text{for } t<0\\ A\sin(\omega t) & \text{for } t\geq 0\end{cases}$$

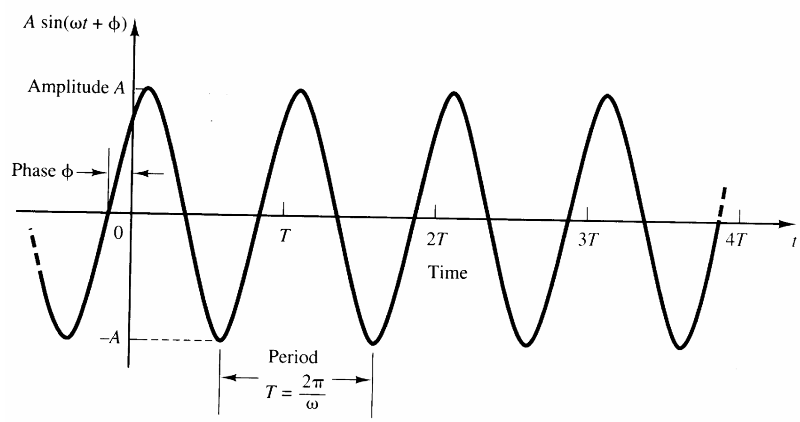


Fig. 2-8. Sinusoidal function

where A and ω are constants, is obtained as follows. Noting that

$$e^{j\omega t}=\cos\omega t+j\sin\omega t \quad \text{and} \quad e^{-j\omega t}=\cos\omega t-j\sin\omega t$$

$\sin\omega t$ and $\cos\omega t$ can be written as

$$\sin\omega t=\frac{1}{2j}\left(e^{j\omega t}-e^{-j\omega t}\right) \quad \text{and} \quad \cos\omega t=\frac{1}{2}\left(e^{j\omega t}+e^{-j\omega t}\right)$$

Hence

$$\mathscr{L}[A\sin\omega t]=\frac{A}{2j}\int_0^\infty\left(e^{j\omega t}-e^{-j\omega t}\right)e^{-st}dt=\frac{A}{2j}\frac{1}{s-j\omega}-\frac{A}{2j}\frac{1}{s+j\omega}$$

$$=\frac{A}{2j}\left(\frac{s+j\omega-s+j\omega}{s^2+\omega^2}\right)=\frac{A}{2j}\left(\frac{2j\omega}{s^2+\omega^2}\right)=\frac{A\omega}{s^2+\omega^2}$$

Similarly

$$\mathscr{L}[A\cos\omega t]=\frac{A}{2}\int_0^\infty\left(e^{j\omega t}+e^{-j\omega t}\right)e^{-st}dt=\frac{A}{2}\frac{1}{s-j\omega}+\frac{A}{2}\frac{1}{s+j\omega}$$

$$=\frac{A}{2}\left(\frac{s+j\omega+s-j\omega}{s^2+\omega^2}\right)=\frac{A}{2}\left(\frac{2s}{s^2+\omega^2}\right)=\frac{As}{s^2+\omega^2}$$

Remark: The Laplace Transform of any Laplace transformable function $f(t)$ can be obtained by multiplying $f(t)$ by e^{-st} and then integrating the product from 0 to ∞. Once we now the method of obtaining the Laplace Transform, however, it is not necessary to derive the Laplace transform of $f(t)$ each time. Laplace Transform Tables can conveniently be used to find the transform of a given function $f(t)$. Refer to Table 2.1 of the Textbook. Notice that the Laplace Transforms provided in Tables in general are valid for $0 \le t < \infty$.

Translated Functions: Let us obtain the Laplace transform of the translated function $f(t-\alpha)1(t-\alpha)$ where $\alpha \ge 0.$ This function is zero for $t < \alpha.$

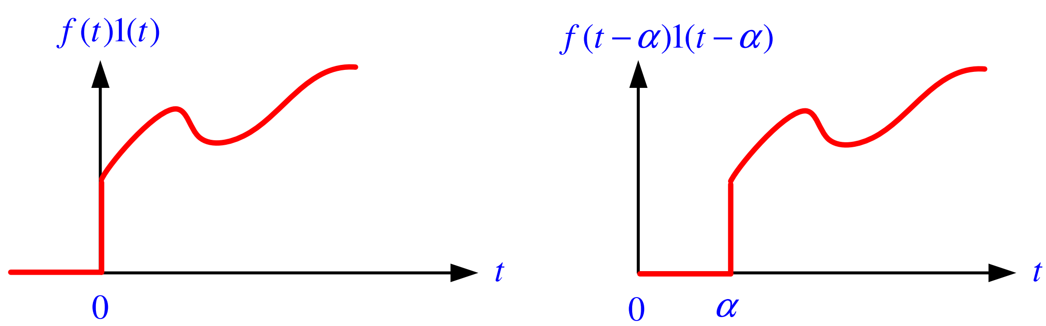


Fig. 2-9 Function $f(t)1(t)$ and translated function $f(t-\alpha)1(t-\alpha)$

By definition, the Laplace transform of $f(t-\alpha)1(t-\alpha)$ is

$$\mathscr{L}\left[f(t-\alpha)1(t-\alpha)\right] = \int_0^\infty f(t-\alpha)1(t-\alpha)e^{-st}dt$$

Let $\tau = t-\alpha$, then

$$t=0 \quad \Rightarrow \quad \tau = -\alpha \quad , \quad t \to \infty \quad \Rightarrow \quad \tau \to \infty \quad \text{and} \quad dt = d\tau$$

and

$$\int_0^\infty f(t-\alpha)1(t-\alpha)e^{-st}dt = \int_{-\alpha}^\infty f(\tau)1(\tau)e^{-s(\tau+\alpha)}d\tau$$

Noting that $f(\tau)1(\tau)=0$ for $\tau < 0,$ we can change the lower limit from $-\alpha$ to $0.$ Thus

$$\int_{-\alpha}^{\infty} f(\tau)1(\tau)e^{-s(\tau+\alpha)}d\tau = \int_{0}^{\infty} f(\tau)1(\tau)e^{-s(\tau+\alpha)}d\tau = \int_{0}^{\infty} f(\tau)e^{-s\tau}e^{-\alpha s}d\tau$$

$$= e^{-\alpha s}\int_{0}^{\infty} f(\tau)e^{-s\tau}d\tau = e^{-\alpha s}F(s)$$

where

$$F(s) = \mathscr{L}\left[f(t)\right] = \int_{0}^{\infty} f(t)e^{-st}dt$$

and also

$$\mathscr{L}\left[f(t-\alpha)1(t-\alpha)\right] = e^{-\alpha s}F(s) \qquad \alpha \ge 0$$

Pulse Function: Consider the pulse function shown in Fig. 2-10

$$f(t) = \begin{cases} \dfrac{A}{t_0} & \text{for } 0 < t < t_0 \\ 0 & \text{for } t < 0,\ t_0 < t \end{cases}$$

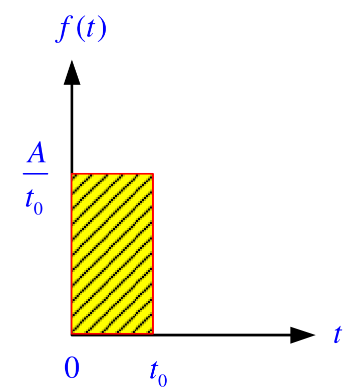
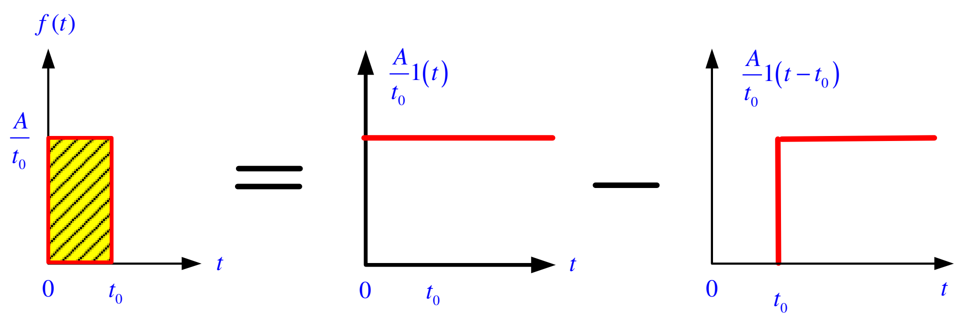


Fig. 2-10 A pulse function

where A and t_0 are constants. The pulse function may be considered as a step function of height A/t_0 that begins at $t=0$ and that is superimposed by a negative step function of height A/t_0 beginning at $t=t_0$; that is ,

$$f(t) = \frac{A}{t_0}1(t) - \frac{A}{t_0}1(t-t_0)$$



Then the Laplace Transform of $f(t)$ is obtained as

$$\mathscr{L}\left[f(t)\right] = \mathscr{L}\left[\frac{A}{t_0}1(t)\right] - \mathscr{L}\left[\frac{A}{t_0}1(t-t_0)\right] = \frac{A}{t_0}\left(\frac{1}{s}\right) - \frac{A}{t_0}\left(\frac{1}{s}\right)e^{-st_0} = \frac{A}{t_0 s}\left(1-e^{-st_0}\right)$$

Impulse Function: The impulse function is a special limiting case of the pulse function. Consider the impulse function

$$f(t) = \begin{cases} \lim\limits_{t_0 \to 0} \dfrac{A}{t_0} & \text{for } \ 0 < t < t_0 \\ 0 & \text{for } \ t < 0, \ t_0 < t \end{cases}$$

Fig. 2-11 depicts the impulse function shown in Fig. 2-10 as t_0 approaches $0.$

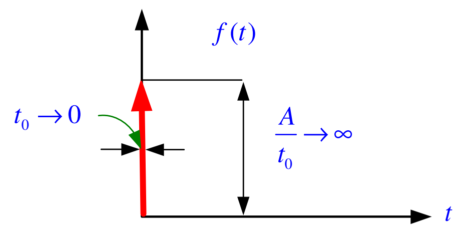


Fig. 2-11. Impulse function

Since the height of the impulse function A/t_0 and the duration is t_0, the area under the impulse is equal to $A.$ As the duration t_0 approaches $0,$ the height A/t_0 approaches infinity, but the area under the impulse remains equal to $A.$ Notice that the magnitude of the impulse is measured by its area. Referring to the transformed equation previously derived for the pulse function, i.e.,

$$\mathcal{L}\left[f(t)\right] = \frac{A}{t_0 s}\left(1 - e^{-s t_0}\right)$$

the Laplace Transform of the impulse function is shown to be

$$\mathcal{L}\left[f(t)\right] = \lim_{t_0 \to 0}\left[\frac{A\left(1 - e^{-s t_0}\right)}{t_0 s}\right] = \lim_{t_0 \to 0} \frac{\dfrac{d}{dt_0}\left[A\left(1 - e^{-s t_0}\right)\right]}{\dfrac{d}{dt_0}\left(t_0 s\right)} = \lim_{t_0 \to 0} \frac{\left[A\left(s e^{-s t_0}\right)\right]}{(s)} = A$$

Thus the Laplace Transform of the impulse function is equal to the area under the impulse. The impulse function whose area is equal to unity is called the ***unit impulse function*** or the ***Dirac delta function***. The unit impulse function occurring at $t = t_0$ is usually denoted by

$$\begin{cases} \delta(t - t_0) = 0 & \text{for } \quad t \neq t_0 \\ \delta(t - t_0) = \infty & \text{for } \quad t = t_0 \\ \int\limits_{-\infty}^{\infty} \delta(t - t_0) = 1 & \end{cases}$$

Relationships among Singular Functions

The ramp, step, and impulse functions represent a family of functions, which as shown in Fig. 2-12 are related by successive integrations.

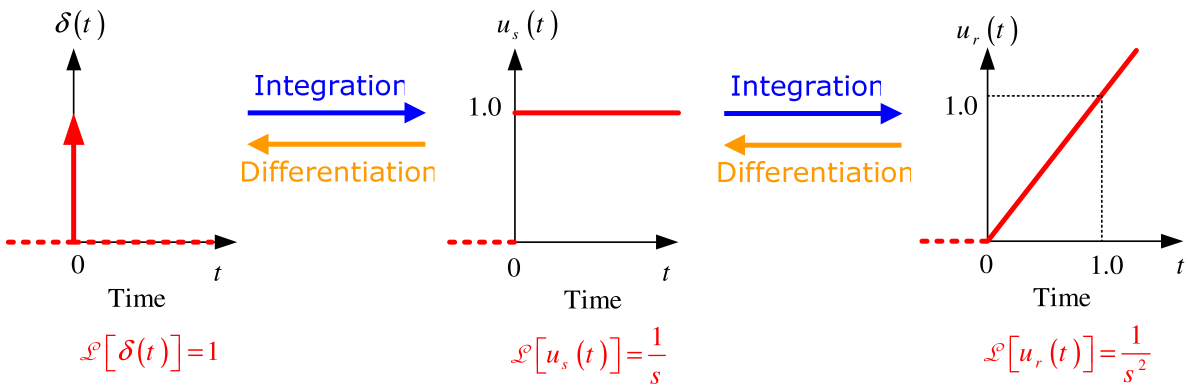


Fig. 2-12 The relationship between singularity functions.

Time Shifting of Singularity Functions

The singularity functions may be used to describe transient inputs that take place at a time other than $t=0$. The discontinuity associated with each function occurs when the function argument is zero; therefore, a step that occurs at time t_0 may be written as $u_s(t-t_0)$ since $t-t_0=0$ at $t=t_0$. This property may be used to synthesize a transient function from a sum of singularity functions; for example, Fig. 2-13 shows the function $u(t)=u_s(t)-2u_s(t-t_0)+u_r(t-2)-u_r(t-3)$.

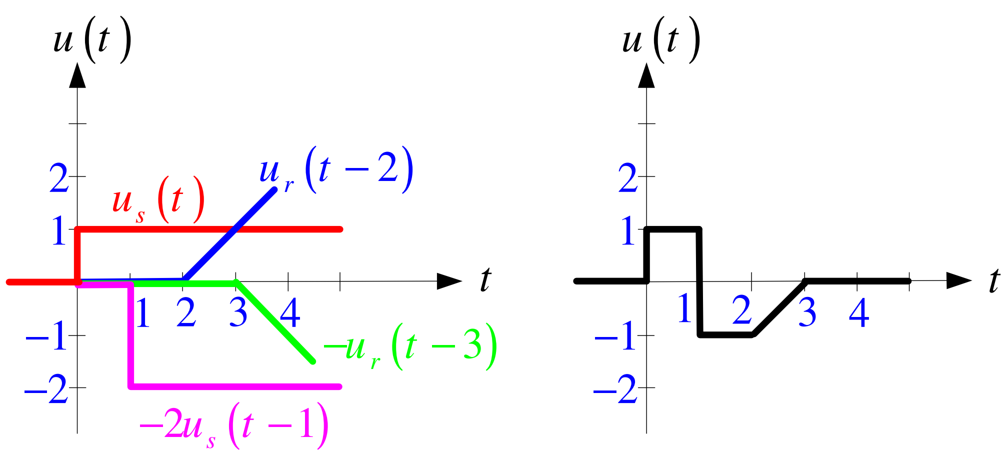


Fig. 2-13 A transient function $u(t)=u_s(t)-2u_s(t-1)+u_r(t-2)-u_r(t-3)$ synthesized from unit singularity functions

Multiplication of $f(t)$ by $e^{-\alpha t}$

If $f(t)$ is Laplace transformable, its Laplace transform being $F(s)$, then the Laplace transform of $e^{-\alpha t} f(t)$ is

$$\mathcal{L}\left[e^{-\alpha t} f(t)\right]=\int_0^\infty e^{-\alpha t} f(t) e^{-st} dt=\int_0^\infty f(t) e^{-(s+\alpha)t}\, dt=F(s+\alpha)$$

EXAMPLE Given Laplace transforms of

$$\mathcal{L}\left[\sin\omega t\right]=\frac{\omega}{s^2+\omega^2}=F(s) \text{ and } \mathcal{L}\left[\cos\omega t\right]=\frac{s}{s^2+\omega^2}=G(s)$$

Find Laplace transform of $e^{-\alpha t}\sin\omega t$ and $e^{-\alpha t}\cos\omega t$

SOLUTION

$$\mathcal{L}\left[e^{-\alpha t}\sin\omega t\right]=F(s+\alpha)=\frac{\omega}{(s+\alpha)^2+\omega^2}$$

and

$$\mathcal{L}\left[e^{-\alpha t}\cos\omega t\right]=G(s+\alpha)=\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$$

Laplace Transform Theorems

Differentiation Theorem

$$\mathcal{L}\left[\frac{d}{dt}f(t)\right]=sF(s)-f(0)$$

$$\mathcal{L}\left[\frac{d^2}{dt^2}f(t)\right]=s^2F(s)-sf(0)-\dot{f}(0)$$

Similarly for the *n*th derivative of $f(t)$, we obtain

$$\mathcal{L}\left[\frac{d^n}{dt^n}f(t)\right]=s^nF(s)-s^{n-1}f(0)-s^{n-2}\dot{f}(0)-\cdots-\overset{(n-1)}{f}(0)$$

In the above, the following quantities $f(0), \dot{f}(0), \cdots, \overset{(n-1)}{f}(0)$ represent the values of $f(t), df(t)/dt, \cdots, d^{n-1}f(t)/dt^{n-1}$, respectively, evaluated at $t=0$.

EXAMPLE Given

$$f(t) = 3\ddot{x} + 5\dot{x} + 7x$$

$$x(0) = 2, \dot{x}(0) = 4$$

Find the Laplace transform F(s) of *f(t)*.

SOLUTION Using the **Differentiation Theorem** on the first two terms leads to:

$$3\mathscr{L}[\ddot{x}] = 3\mathscr{L}\left[\frac{d^2}{dt^2} f(t)\right] = 3\left[s^2 X(s) - sx(0) - \dot{x}(0)\right]$$

$$5\mathscr{L}[\dot{x}] = 5\mathscr{L}\left[\frac{d}{dt} f(t)\right] = 5\left[sX(s) - x(0)\right]$$

Using the **definition** of the Laplace transform on the remaining term gives:

$$7\mathscr{L}[x] = 7\left[X(s)\right]$$

From these results, the **Laplace transform** F(s) of the given equation can be expressed as:

$$3\left[s^2 X(s) - sx(0) - \dot{x}(0)\right] + 5\left[sX(s) - x(0)\right] + 7\left[X(s)\right] = F(s)$$

Rearranging this expression by factoring leads to:

$$[3s^2 + 5s + 7]X(s) - [3s + 5]x(0) - [3]\dot{x}(0) = F(s)$$

Solving this expression for X(s) gives the following **answer**:

$$X(s) = \underbrace{\frac{F(s)}{3s^2 + 5s + 7}}_{\text{response due to force}} + \underbrace{\frac{[3s+5](2) + [3](4)}{3s^2 + 5s + 7}}_{\text{response due to initial conditions}}$$

Final Value Theorem (FVT)

$$\lim_{t\to\infty}[f(t)] = \lim_{s\to 0}[sF(s)]$$

EXAMPLE Given:

$$X(s) = \frac{4(s+5)}{s(s+2)(s+8)}$$

Find the final value of *x(t).*

SOLUTION To solve this problem, use the **Final Value Theorem (FVT)**

$$\lim_{t\to\infty}[f(t)] = \lim_{s\to 0}[sF(s)]$$

Substituting the given expression into this equation leads to the solution:

$$\lim_{s\to 0} s\left(\frac{4(s+5)}{s(s+2)(s+8)}\right) = \frac{20}{16} = \frac{5}{4}$$

Initial Value Theorem (IVT)

$$f(0^+) = \lim_{s\to\infty}[sF(s)]$$

EXAMPLE Given:

$$X(s) = \frac{5(s+3)(s+4)}{s(s+2)(s+6)}$$

Find the initial value of *x(t)*, i.e. find *x(0).*

SOLUTION To solve this problem, use the Initial Value Theorem (IVT):

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

Substituting the given expression into this equation leads to the solution:

$$\lim_{s\to\infty} s\left(\frac{5(s+3)(s+4)}{s(s+2)(s+6)}\right)=5$$

For this example, as "s" goes to infinity, all terms involving "s" in the numerator and denominator **cancel**. (Note that if there was one extra "s" term in the denominator than there was in the numerator, this extra term would **not** be cancelled out and the entire expression would go to zero.)

Integration Theorem

$$\mathscr{L}\left[\int_0^t f(t)\,dt\right]=\frac{F(s)}{s}$$

EXAMPLE Given:

$$\mathscr{L}[\int_0^t At\,dt]$$

Find Laplace transform of the given expression. *(<u>Hint</u>: Let f(t) = At)*

SOLUTION To solve this problem, use the **integration theorem**:

$$\mathscr{L}[\int_0^t f(t)dt]=\mathscr{L}[\int_0^t At\,dt]=\frac{F(s)}{s}$$

Substituting the result for F(s) obtained in the previous example leads to the solution:

$$\mathscr{L}[\int_0^t f(t)dt]=\frac{\left(\frac{A}{s^2}\right)}{s}=\frac{A}{s^3}$$

Use of MATLAB

EXAMPLE Use MATLAB to find Laplace Transform of $f(t)=t\cos(3t)$

MATLAB SOLUTION

```
>>	syms t s
>>	f = t*cos(3*t);
```

Then the Laplace transform of *f* is given by

```
>>	F = laplace(f)
	F =1/(s^2+9)*cos(2*atan(3/s))
>>	F = simplify(expand(F))
	F =(s^2-9)/(s^2+9)^2
```

Thus we obtain the Laplace transform

$$F(s)=\frac{s^2-9}{\left(s^2+9\right)^2}$$

EXAMPLE

Use MATLAB to find Laplace Transform of

$$f(t)=3t-5\cos(2t)$$

MATLAB SOLUTION

```
>>	syms t s
>>	f=3*t-5*sin(2*t);
```

Then the Laplace transform of *f* is given by

```
>>	F = laplace(f)
F=3/s^2-10/(s^2+4)
>>	F = simplify(expand(F))
F =-(7*s^2-12)/s^2/(s^2+4)
```

Thus we obtain the Laplace transform

$$F(s)=\frac{-7s^2+12}{s^2\left(s^2+4\right)}$$

2.4 INVERSE LAPLACE TRANSFORMATION

The inverse Laplace transformation refers to the process of finding the time function $f(t)$ from the corresponding Laplace transform $F(s)$; i.e.,

$$f(t)=\mathscr{L}^{-1}\left[F(s)\right]$$

Several methods are available for finding the inverse Laplace transforms

1. Use Tables of Laplace transforms
2. Use partial-fraction expansion method.

Partial-Fraction Expansion for Finding Inverse Laplace Transforms

If $F(s)$, the Laplace transform of $f(t)$, is broken up into components

$$F(s) = F_1(s) + F_2(s) + F_3(s) + \cdots + F_n(s)$$

and if the inverse Laplace transform of $F_1(s)$, $F_2(s)$, $F_3(s)$, $\cdots, F_n(s)$, are readily available then

$$\begin{aligned}\mathscr{L}^{-1}[F(s)] &= \mathscr{L}^{-1}\left[F_1(s)\right] + \mathscr{L}^{-1}\left[F_2(s)\right] + \cdots + \mathscr{L}^{-1}\left[F_n(s)\right] \\ &= f_1(t) + f_2(t) + \cdots + f_n(t)\end{aligned}$$

where $f_1(t)$, $f_2(t)$, $\cdots$, $f_n(t)$ are the inverse Laplace transform of $F_1(s)$, $F_2(s)$, $\cdots, F_n(s)$, respectively. $F(s)$ frequently occurs in the form

$$F(s) = \frac{N(s)}{D(s)}, \quad m = \deg N(s) < \deg D(s) = n$$

where $N(s)$ and $D(s)$ are polynomials in s and the degree of $D(s)$ is not higher than the degree of $N(s)$. Notice that applying the partial-fraction expansion technique in the search for the inverse Laplace transform of $F(s) = N(s)/D(s)$ requires that the poles of $D(s)$ (roots of the denominator) must be known in advance. Consider $F(s)$ written in the factored form

$$F(s) = \frac{N(s)}{D(s)} = \frac{K(s+z_1)(s+z_2)\cdots(s+z_n)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

where $p_1, p_2, \ldots, p_n$ and $z_1, z_2, \ldots, z_n$ are either real or complex quantities, but for each p_i or z_i there will occur the complex conjugate of p_i or z_i, respectively. Here the highest power of s in $N(s)$ is assumed to be higher than that in $D(s)$.

Notice that if the degree of the numerator is greater than (or equal to) that of the denominator, then polynomial division must be performed so that the remainder polynomial is of a lower degree than $D(s)$. For instance,

$$\frac{s^3+2}{s^2+1} = s + \frac{-s+2}{s^2+1}, \quad \deg(-s+2) < \deg(s^2+1)$$

Case I. Distinct Real Poles

In this case, $F(s)$ can be expanded into a sum of partial fractions

$$F(s) = \frac{N(s)}{D(s)} = \frac{r_1}{(s+p_1)} + \frac{r_2}{(s+p_2)} + \cdots + \frac{r_n}{(s+p_n)}$$

where $r_k\ (k = 1, 2, \ldots, n)$ are constants. The coefficient r_k is called the residue at the pole at $s = -p_k$. The value of r_k can be found by multiplying both sides of the above equation $(s+p_k)$ and letting $s = -p_k$, which gives

$$\left[(s+p_k)\frac{N(s)}{D(s)}\right]_{s=-p_k} = \left[\frac{r_1(s+p_k)}{(s+p_1)} + \frac{r_2(s+p_k)}{(s+p_2)} + \cdots + \frac{r_k(s+p_k)}{(s+p_k)} + \cdots + \frac{r_n(s+p_k)}{(s+p_n)}\right]_{s=-p_k} = r_k$$

We see that all the expanded terms drop out with the exception of r_k. Thus the residue r_k is found from

$$r_k = \left[(s+p_k)\frac{N(s)}{D(s)}\right]_{s=-p_k} \tag{2.6}$$

Since

$$L^{-1}\left[\frac{r_k}{(s+p_k)}\right] = r_k e^{-p_k t}$$

$f(t)$ is obtained as

$$f(t) = \mathscr{L}^{-1}[F(s)] = r_1 e^{-p_1 t} + r_2 e^{-p_2 t} + \cdots + r_k e^{-p_k t} \qquad t \ge 0$$

EXAMPLE Find the inverse Laplace transform of

$$F(s) = \frac{s+3}{(s+1)(s+2)}$$

SOLUTION The partial fraction expansion of $F(s)$ is

$$F(s) = \frac{s+3}{(s+1)(s+2)} = \frac{r_1}{(s+1)} + \frac{r_2}{(s+2)}$$

where r_1 and r_2 are found by using Equation (2.6)

$$r_1 = \left[\cancel{(s+1)} \frac{(s+3)}{\cancel{(s+1)}(s+2)} \right]_{s=-1} = \left[\frac{(s+3)}{(s+2)} \right]_{s=-1} = \frac{-1+3}{-1+2} = 2$$

$$r_2 = \left[\cancel{(s+2)} \frac{(s+3)}{(s+1)\cancel{(s+2)}} \right]_{s=-2} = \left[\frac{(s+3)}{(s+1)} \right]_{s=-2} = \frac{-2+3}{-2+1} = -1$$

Thus

$$F(s) = \frac{s+3}{(s+1)(s+2)} = \frac{2}{(s+1)} + \frac{-1}{(s+2)}$$

$$f(t) = \mathscr{L}^{-1}[F(s)] = \mathscr{L}^{-1}\left[\frac{2}{(s+1)} \right] + \mathscr{L}^{-1}\left[\frac{-1}{(s+2)} \right] = 2e^{-t} - e^{-2t} \qquad t \geq 0$$

Use of MATLAB: Use MATLAB to find the inverse Laplace Transform of the above example

$$F(s) = \frac{s+3}{(s+1)(s+2)}$$

```
>> syms t s
>> F = (s+3)/((s+1)*(s+2))
```

Then the inverse Laplace transform of $f(t)$ is given by

```
>> F = ilaplace(f)
f =2*exp(-t)-exp(-2*t)
>> pretty(f)
2 exp(-t) - exp(-2 t)
```

Thus we obtain the Laplace transform

$$f(t) = L^{-1}[F(s)] = 2e^{-t} - e^{-2t} \qquad t \geq 0$$

Example Obtain the inverse Laplace transform of

$$F(s) = \frac{s^3 + 5s^2 + 9s + 7}{(s+1)(s+2)}$$

Solution

Since the degree of the numerator is higher than that of the denominator polynomial, we must divide the numerator by the denominator

$$F(s) = s + 2 + \frac{s+3}{(s+1)(s+2)} = s + 2 + \underbrace{\frac{2}{(s+1)} + \frac{-1}{(s+2)}}_{\text{previous example}}$$

Notice that $\mathscr{L}\left[\delta(t)\right] = 1$ and $\mathscr{L}\left[\frac{d\delta(t)}{dt}\right] = s$, so the inverse Laplace transform of $F(s)$ is given by

$$f(t) = \frac{d}{dt}\delta(t) + 2\delta(t) + 2e^{-t} - e^{-2t} \qquad t \geq 0$$

Case II. Complex Conjugate Poles

Consider a function $F(s)$ that involves a quadratic factor $s^2 + as + b$ in the denominator. If this quadratic factor has a pair of complex conjugate poles, then it is better not to factor this term in order to avoid complex numbers. For example, if $F(s)$ is given as

$$F(s) = \frac{p(s)}{s\left(s^2 + as + b\right)}$$

where $a \geq 0$ and $b \geq 0$, and $s^2 + as + b = 0$ has a pair of complex conjugate poles, then expand $F(s)$ into the following partial-fraction expansion form:

$$F(s) = \frac{c}{s} + \frac{ds + e}{s^2 + as + b}$$

Example

Obtain: the inverse Laplace transform of

$$F(s) = \frac{2s + 12}{s^2 + 2s + 5}$$

Solution 1: Use of Complex Numbers

Notice that the poles of the denominator are

$$s_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-2\pm\sqrt{4-4\times1\times5}}{2}=\frac{-2\pm\sqrt{-16}}{2}=-1\pm j2\,.$$

Therefore $F(s)$ can be written as

$$F(s)=\frac{2s+12}{s^2+2s+5}=\frac{2s+12}{(s+1+j2)(s+1-j2)}=\frac{\alpha}{s+1+j2}+\frac{\beta}{s+1-j2}$$

where the constants α and β that can be found as before

$$\alpha=\left.\frac{\cancel{(s+1+j2)}(2s+12)}{\cancel{(s+1+j2)}(s+1-j2)}\right|_{s=-1-j2}=\frac{(2(-1-j2)+12)}{(-1-j2+1-j2)}=\frac{10-j4}{-j4}=1-j\frac{5}{2}$$

$$\beta=\left.\frac{\cancel{(s+1-j2)}(2s+12)}{(s+1+j2)\cancel{(s+1-j2)}}\right|_{s=-1+j2}=\frac{(2(-1+j2)+12)}{(-1+j2+1+j2)}=\frac{10+j4}{+j4}=1+j\frac{5}{2}$$

Notice that β is the complex conjugate of α. Substitute the values of α and β into the expression of $F(s)$

$$F(s)=\frac{2s+12}{s^2+2s+5}=\frac{1-j\frac{5}{2}}{s+1+j2}+\frac{1+j\frac{5}{2}}{s+1-j2}$$

and

$$\begin{aligned}
f(t)=\mathscr{L}^{-1}[F(s)]&=\left(1-j\frac{5}{2}\right)e^{-(1+j2)t}+\left(1+j\frac{5}{2}\right)e^{-(1-j2)t}\\
&=e^{-t}e^{-j2t}-j\frac{5}{2}e^{-t}e^{-j2t}+e^{-t}e^{j2t}+j\frac{5}{2}e^{-t}e^{j2t}\\
&=2e^{-t}\left(\frac{e^{j2t}+e^{-j2t}}{2}\right)+j5e^{-t}\left(\frac{e^{j2t}-e^{-j2t}}{2}\right)\\
&=2e^{-t}\underbrace{\left(\frac{e^{j2t}+e^{-j2t}}{2}\right)}_{\cos(2t)}-5e^{-t}\underbrace{\left(\frac{e^{j2t}-e^{-j2t}}{2j}\right)}_{\sin(2t)}\\
&=2e^{-t}\cos(2t)-5e^{-t}\sin(2t)
\end{aligned}$$

Solution 2: Completing The square

The expression of $F(s)$ can be written in general as

$$F(s)=\frac{cs+d}{s^2+2as+a^2+\omega^2}$$

where a and ω are positive real. It is clear that the denominator in the above expression is a complete square, i.e., it can be written as

$$s^2+2as+a^2+\omega^2=\left(s^2+a^2\right)+\omega^2$$

Let's us write the expression of $F(s)$ into the following form

$$F(s)=\frac{cs+d}{s^2+2as+a^2+\omega^2}=\frac{cs+d}{(s+a)^2+\omega^2}=\frac{c(s+a)+d-ca}{(s+a)^2+\omega^2}$$

$$=\frac{c(s+a)}{(s+a)^2+\omega^2}+\frac{d-ca}{(s+a)^2+\omega^2}=\frac{c(s+a)}{(s+a)^2+\omega^2}+\frac{d-ca}{\omega}\frac{\omega}{(s+a)^2+\omega^2}$$

The inverse Laplace transform is then

$$f(t)=\mathscr{L}^{-1}[F(s)]=c\,\mathscr{L}^{-1}\left[\frac{(s+a)}{(s+a)^2+\omega^2}\right]+\left(\frac{d-ca}{\omega}\right)\mathscr{L}^{-1}\left[\frac{\omega}{(s+a)^2+\omega^2}\right]$$

$$=ce^{-at}\cos\omega t+\left(\frac{d-ca}{\omega}\right)e^{-at}\sin\omega t$$

In our example we have

$$F(s)=\frac{2s+12}{s^2+2s+5}=\frac{2s+12}{\underbrace{s^2+2s+1}_{=(s+1)^2}+\underbrace{4}_{2^2}}=\frac{2s+12}{(s+1)^2+2^2}$$

Thus we have $a=1, \omega=2, c=2, d=12.$ Therefore, substitute into the expression of $f(t)$

$$f(t)=\mathscr{L}^{-1}\left[F(s)\right]=ce^{-at}\cos\omega t+\left(\frac{d-ca}{\omega}\right)e^{-at}\sin\omega t$$

$$=2e^{-t}\cos 2t+\underbrace{\left(\frac{12-2\times 1}{2}\right)}_{=5}e^{-t}\sin 2t$$

$$=2e^{-t}\cos 2t+5e^{-t}\sin 2t=e^{-t}\left(2\cos 2t+5\sin 2t\right)$$

Method 2: Use of Complex numbers

This method is a lengthy process we will see it in a separate problem in the help session.

Case III. Multiple Poles

Consider the following expression of $F(s)=\dfrac{s^2+2s+3}{(s+1)^3}$,

As can be seen $F(s)$ has poles $s=-1,-1,-1.$ Thus we say $F(s)$ has a pole $s=-1$ of multiplicity 3. Hence $F(s)$ can be written in the following form

$$F(s)=\frac{s^2+2s+3}{(s+1)^3}=\frac{B(s)}{A(s)}=\frac{b_3}{(s+1)^3}+\frac{b_2}{(s+1)^2}+\frac{b_1}{(s+1)}$$

where $b_1, b_2,$ and b_3 are determined as follows. By multiplying both sides of the last equation by $(s+1)^3$, we obtain

$$(s+1)^3\frac{B(s)}{A(s)}=b_3+b_2(s+1)+b_1(s+1)^2 \tag{2.7}$$

Then, letting $s=-1,$ Equation (2.7) gives

$$\left[(s+1)^3\frac{B(s)}{A(s)}\right]_{s=-1}=b_3$$

Also differentiation of both sides of Equation (2.7) gives

$$\frac{d}{ds}\left[(s+1)^3\frac{B(s)}{A(s)}\right]=b_2+2b_1(s+1) \tag{2.8}$$

If we let $s=-1,$ in Equation (2.8), then

$$\frac{d}{ds}\left[(s+1)^3\frac{B(s)}{A(s)}\right]_{s=-1}=b_2$$

By differentiating both sides of Equation (2.8) with respect to s, we obtain

$$\frac{d^2}{ds^2}\left[(s+1)^3\frac{B(s)}{A(s)}\right]=2b_1 \quad\Rightarrow\quad b_1=\frac{1}{2!}\frac{d^2}{ds^2}\left[(s+1)^3\frac{B(s)}{A(s)}\right]$$

Therefore

$$b_3=\left[(s+1)^3\frac{B(s)}{A(s)}\right]_{s=-1}=\left[\cancel{(s+1)^3}\frac{s^2+2s+3}{\cancel{(s+1)^3}}\right]_{s=-1}=(-1)^2+2(-1)+3=2$$

$$b_2=\frac{d}{ds}\left[(s+1)^3\frac{B(s)}{A(s)}\right]_{s=-1}=\frac{d}{ds}\left[\cancel{(s+1)^3}\frac{s^2+2s+3}{\cancel{(s+1)^3}}\right]_{s=-1}$$

$$=\frac{d}{ds}\left[s^2+2s+3\right]_{s=-1}=\left[2s+2\right]_{s=-1}=\left[2(-1)+2\right]=0$$

$$b_1=\frac{1}{2!}\frac{d^2}{ds^2}\left[(s+1)^3\frac{B(s)}{A(s)}\right]_{s=-1}=\frac{1}{2!}\frac{d^2}{ds^2}\left[\cancel{(s+1)^3}\frac{s^2+2s+3}{\cancel{(s+1)^3}}\right]_{s=-1}$$

$$=\frac{1}{2!}\frac{d}{ds}\left[2s+2\right]_{s=-1}=\frac{1}{2!}\left[2\right]_{s=-1}=1$$

Therefore

$$F(s)=\frac{2}{(s+1)^3}+\frac{0}{(s+1)^2}+\frac{1}{(s+1)}$$

and

$$f(t)=\mathscr{L}^{-1}\left[F(s)\right]=t^2e^{-t}+0+e^{-t} \qquad t\geq 0$$

2.5 SOLVING LINEAR, TIME INVARIANT DIFFERENTIAL EQUATIONS

The Laplace transform method yields the complete solution (complementary solution and particular solution) of linear, time invariant, differential equations. Classical methods for finding the complete solution of a differential equation require the evaluation of integration constants from the initial conditions. In the case of Laplace transform method, however, this requirement is unnecessary because the initial conditions are automatically included in the Laplace transform of the differential equation.

EXAMPLE Initial Value Problem (IVP)

Solve: $y''+y=t,\quad y(0)=1,\quad y'(0)=1$

SOLUTION

By writing the Laplace transform of $y(t)$ as $\mathscr{L}[y(t)]=Y(s)$, we obtain

$$\mathscr{L}[\dot{y}(t)]=sY(s)-y(0)$$
$$\mathscr{L}[\ddot{y}(t)]=s^2Y(s)-s\,y(0)-\dot{y}(0)$$

For zero initial conditions, i.e., $y(0)=\dot{y}(0)=0$, the above transforms become

$$\mathscr{L}[\dot{y}(t)]=sY(s)$$
$$\mathscr{L}[\ddot{y}(t)]=s^2Y(s)$$

Step 1: Take Laplace Transform (LT) of both sides of the above equation:

$$\underbrace{s^2Y(s)-sy(0)-y'(0)}_{\mathscr{L}[y'']}+\underbrace{Y(s)}_{\mathscr{L}[y]}=\underbrace{1/s^2}_{\mathscr{L}[t]}$$

or

$$\left(s^2+1\right)Y(s)=\frac{1}{s^2}+s+1$$

Step 2: Solving for $Y(s)$ gives

$$Y(s)=\frac{1}{s^2\left(s^2+1\right)}+\frac{s}{\left(s^2+1\right)}+\frac{1}{\left(s^2+1\right)}$$

Let

$$Q(s)=\frac{1}{s^2\left(s^2+1\right)}=\frac{1}{s^2}-\frac{1}{\left(s^2+1\right)}$$

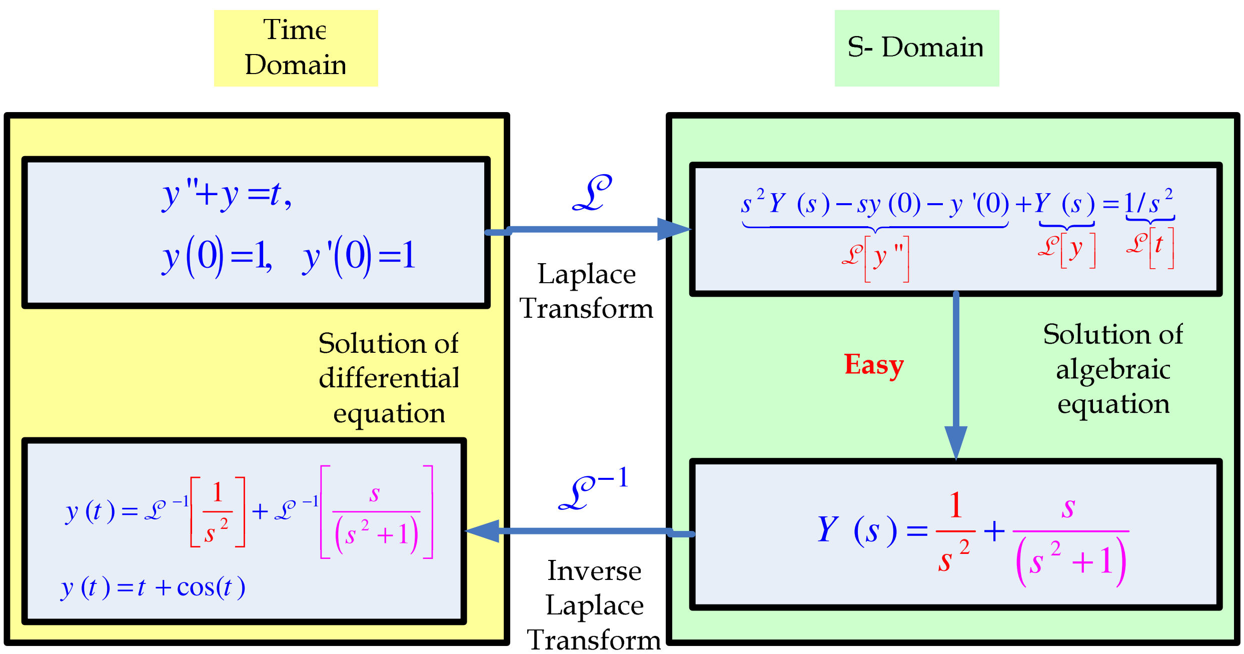
Therefore

$$Y(s)=\frac{1}{s^2}-\frac{1}{(s^2+1)}+\frac{s}{(s^2+1)}+\frac{1}{(s^2+1)}=\frac{1}{s^2}+\frac{s}{(s^2+1)}$$

Step 3: Solving

$$y(t)=\mathscr{L}^{-1}\left[\frac{1}{s^2}\right]+\mathscr{L}^{-1}\left[\frac{s}{(s^2+1)}\right]=t+\cos(t)$$

The diagram below summarizes the approach



EXAMPLE Find the solution $x(t)$ of the following Initial Value Problem (IVP)

$$\ddot{x}+3\dot{x}+2x=0, \qquad x(0)=a, \quad \dot{x}(0)=b$$

where a and b are constants.

SOLUTION

Step 1: Take LT of both sides of the given equation to obtain an algebraic equation *X(s)*

By writing the Laplace transform of $x(t)$ as $\mathscr{L}[x(t)]=X(s)$, we obtain

$$\mathscr{L}\left[\dot{x}(t)\right]=sX(s)-x(0)$$
$$\mathscr{L}\left[\ddot{x}(t)\right]=s^2X(s)-s\,x(0)-\dot{x}(0)$$

Take Laplace transform of both sides of the given equation

$$\underbrace{\left[s^2X(s)-s\,x(0)-\dot{x}(0)\right]}_{\mathscr{L}\left[\ddot{x}(t)\right]}+3\underbrace{\left[sX(s)-x(0)\right]}_{\mathscr{L}\left[\dot{x}(t)\right]}+2\underbrace{\left[X(s)\right]}_{\mathscr{L}\left[x(t)\right]}=0$$

By substituting the initial conditions $x(0)=a,\quad \dot{x}(0)=b$ into the last equations, one obtains

$$\left[s^2X(s)-a\,s-b\right]+3\left[sX(s)-a\right]+2\left[X(s)\right]=0$$

or

$$\left[s^2+3s+2\right]X(s)=as+b+3a$$

Step 2: Solving for $X(s)$, we obtain

$$X(s)=\frac{as+b+3a}{\underbrace{s^2+3s+2}_{=(s+1)(s+2)}}=\frac{K_1}{s+1}+\frac{K_2}{s+2}$$

since $X(s)$ has distinct poles, K_1 and K_2 can be found easily

$$K_1=\left.\frac{\cancel{(s+1)}(as+b+3a)}{\cancel{(s+1)}(s+2)}\right|_{s=-1}=\left.\frac{(a(-1)+b+3a)}{(-1+2)}\right|=2a+b$$
$$K_2=\left.\frac{\cancel{(s+2)}(as+b+3a)}{\cancel{(s+2)}(s+1)}\right|_{s=-2}=\left.\frac{(a(-2)+b+3a)}{(-2+1)}\right|=-(a+b)$$

Thus

$$X(s)=\frac{2a+b}{s+1}-\frac{a+b}{s+2}$$

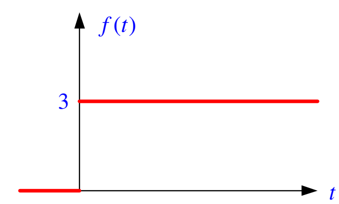
Step 3: Take $\mathscr{L}^{-1}\left[X(s)\right]$ to obtain the time domain solution $x(t)$

$$x(t) = \mathscr{L}^{-1}[X(s)] = \mathscr{L}^{-1}\left[\frac{2a+b}{s+1}\right] - \mathscr{L}^{-1}\left[\frac{a+b}{s+2}\right]$$
$$= (2a+b)e^{-t} - (a+b)e^{-2t} \qquad t \geq 0$$

Example Find the solution $x(t)$ of the following Initial Value Problem (IVP)

$$\ddot{x} + 2\dot{x} + 5x = f(t), \qquad x(0) = 0, \quad \dot{x}(0) = 0$$

where $f(t)$ is a function given by its graph.



Solution

Step 1: Find the explicit expression of $f(t)$

The function $f(t)$ given on the RHS of the IVP is a step function $u(t)$ with height equals to 3. and $\mathscr{L}[u(t)] = \mathscr{L}[3] = 3/s$

Step 2: Take LT of both sides of the given equation to obtain an algebraic equation $X(s)$

Remember that we have zero initial conditions. For zero initial conditions, i.e., $x(0) = \dot{x}(0) = 0$, the above transforms become

$$\mathscr{L}[\dot{x}(t)] = sX(s)$$
$$\mathscr{L}[\ddot{x}(t)] = s^2X(s)$$

Take Laplace transform of both sides of the given equation

$$s^2X(s) + 2sX(s) + 5X(s) = \frac{3}{s}$$

Solving for $X(s)$, we obtain

$$X(s) = \frac{3}{s(s^2+2s+5)} = \frac{K_1}{s} + \frac{cs+d}{s^2+2s+5}$$

where K_1, K_2 and K_3 are found as follows

$$3 = K_1\left(s^2+2s+5\right)+\left(cs+d\right)s$$

or

$$3 = \left(K_1+c\right)s^2+\left(2K_1+d\right)s+5K_1$$

By comparing coefficients of the s^2, s and s^0, terms on both sides of this last equation respectively, we obtain

$$s^0 \text{ terms:}\quad 5K_1 = 3 \quad\Rightarrow\quad K_1 = 3/5$$

$$s^1 \text{ terms:}\quad 2K_1+d = 0 \quad\Rightarrow\quad d = -2K_1 = -2(3/5) = -6/5$$

$$s^2 \text{ terms:}\quad K_1+c = 0 \quad\Rightarrow\quad c = -K_1 = -3/5$$

Thus

$$X(s) = \frac{3}{s\left(s^2+2s+5\right)} = \frac{(3/5)}{s}+\frac{(-3/5)s+(-6/5)}{\underbrace{s^2+2s+5}_{=s^2+2s+1+4}} = \frac{(3/5)}{s}+\frac{(-3/5)s+(-6/5)}{(s+1)^2+2^2}$$

$$x(t) = \mathscr{L}^{-1}\left[X(s)\right] = \mathscr{L}^{-1}\left[\frac{(3/5)}{s}\right]+\mathscr{L}^{-1}\left[\frac{(-3/5)s+(-6/5)}{\underbrace{s^2+2s+5}_{=s^2+2s+1+4}}\right]$$

$$= \mathscr{L}^{-1}\left[\frac{(3/5)}{s}\right]+\mathscr{L}^{-1}\left[\underbrace{\frac{(-3/5)s+(-6/5)}{(s+1)^2+2^2}}_{G(s)}\right]$$

The second term of the last equation can be written according to the completing the square rule. Thus we have $a=1, \omega=2, c=-3/5, d=-6/5$. Therefore substitute into the expression of $f(t)$

$$G(t) = \mathscr{L}^{-1}[G(s)] = ce^{-at}\cos\omega t + \left(\frac{d - ca}{\omega}\right)e^{-at}\sin\omega t$$

$$= -\frac{3}{5}e^{-t}\cos 2t + \underbrace{\left(\frac{-\frac{3}{5} - \left(-\frac{6}{5}\right)}{2}\right)}_{=\frac{3}{5}} e^{-t}\sin 2t = -\frac{3}{5}e^{-t}\cos 2t - \frac{3}{10}e^{-t}\sin 2t$$

Hence

$$x(t) = \mathscr{L}^{-1}[X(s)] = \mathscr{L}^{-1}\left[\frac{(3/5)}{s}\right] + \mathscr{L}^{-1}\left[\frac{(-3/5)s + (-6/5)}{\underbrace{s^2 + 2s + 5}_{=s^2+2s+1+4}}\right]$$

$$= \frac{3}{5} - \frac{3}{5}e^{-t}\cos 2t - \frac{3}{10}e^{-t}\sin 2t$$

Chapter 2
Some Input Examples

A. Bazoune

1. Examples of Initial Conditions

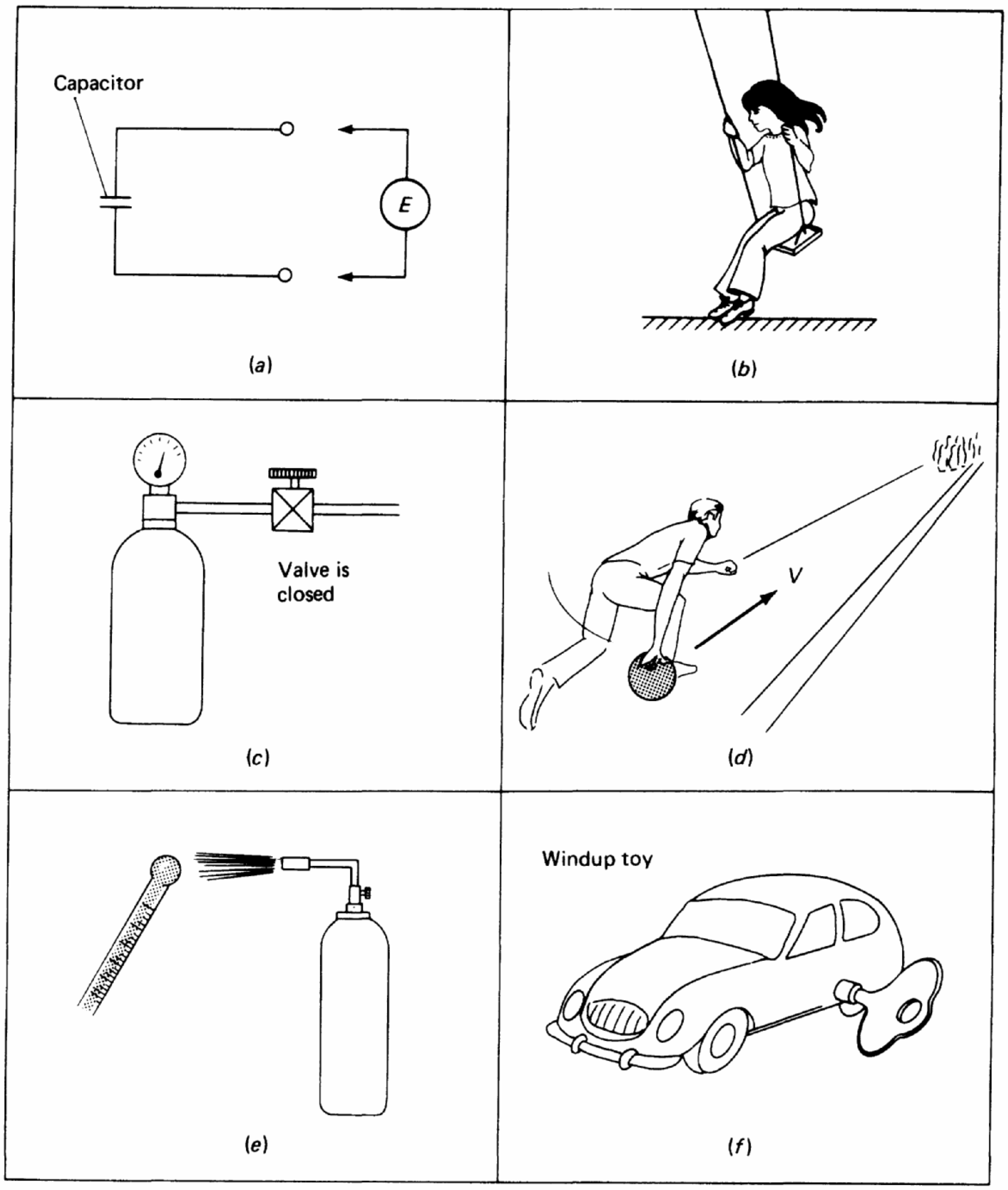


Figure 3.1.1-1 Some examples of initial conditions. (a) Initial charge. (b) Initial displacement. (c) Initial pressure. (d) Initial velocity. (e) Initial temperature. (f) Initial angular displacement.

2. Examples of Impulse Inputs

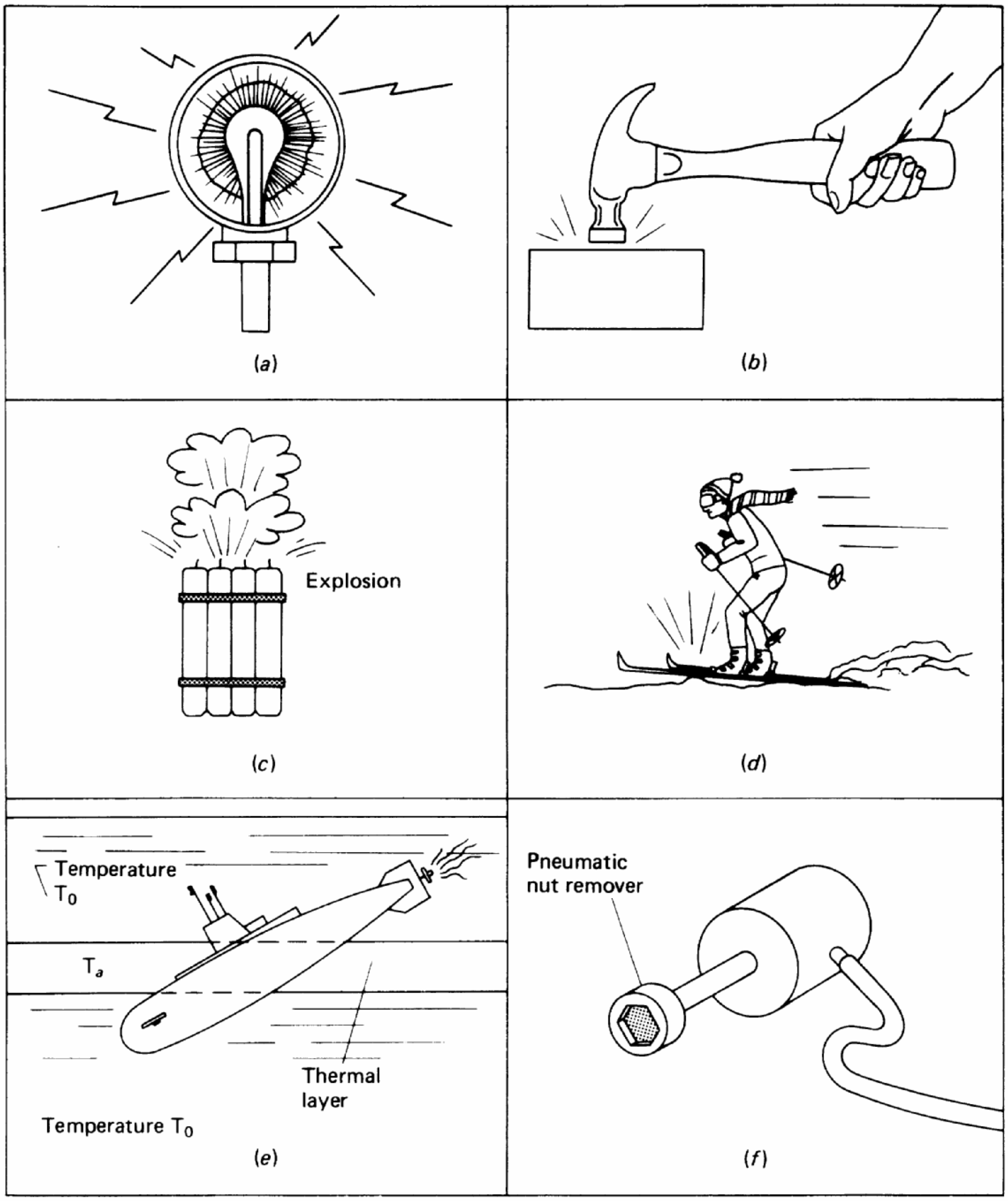


Figure 3.1.2-1 Some examples of impulse inputs. (a) Electronic flash gun. (b) Impulse force. (c) Impulse pressure. (d) Translatory impulse displacement. (e) Thermal shock. (f) Impulse torque.

3. Examples of Step Inputs

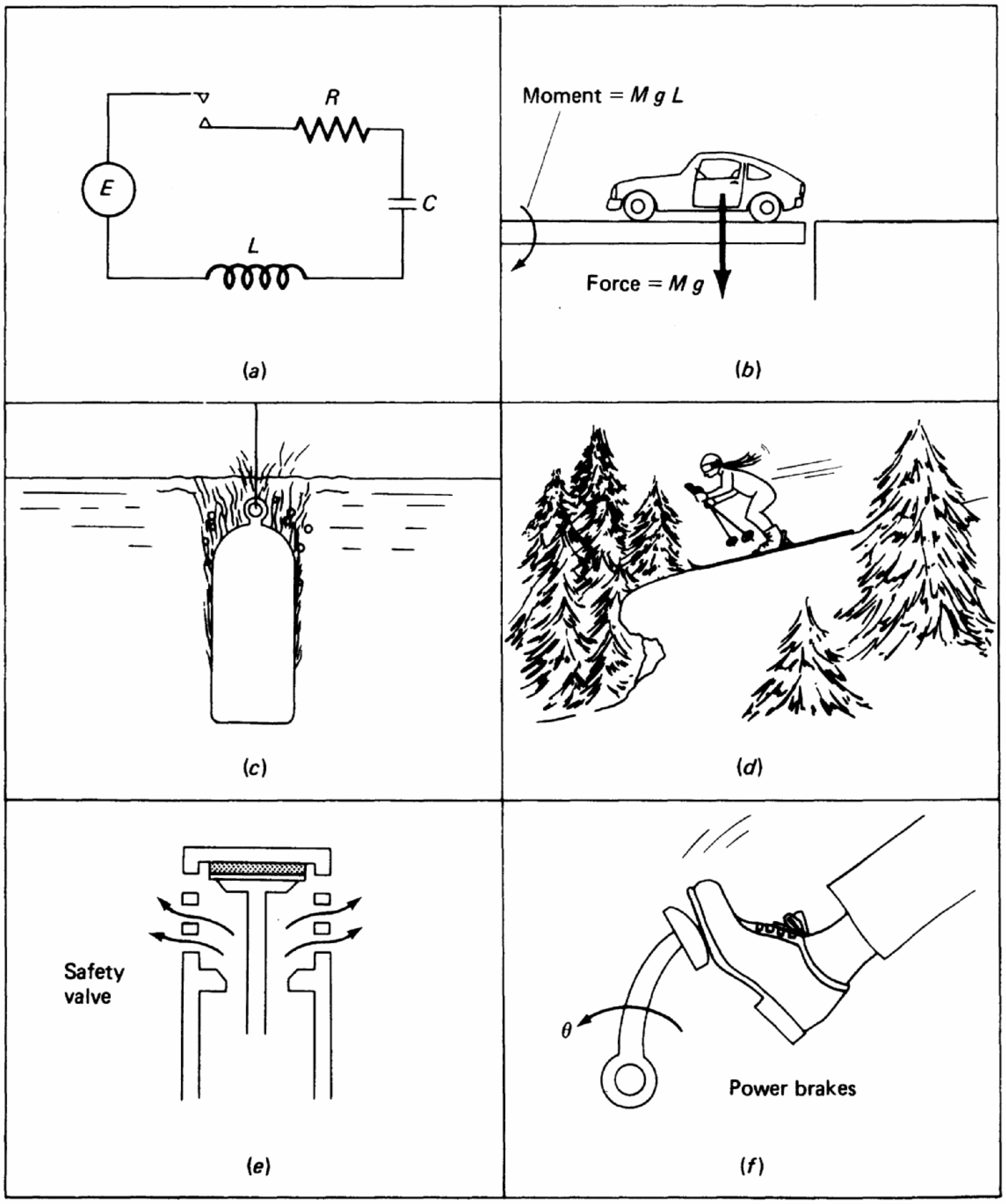


Figure 3.1.3-1 Some examples of step inputs. (a) Closing a switch. (b) Step force and step moment. (c) Quenching hot steel. (d) Translatory step displacement. (e) Step pressure. (f) Rotary step displacement.

4. Examples of Ramp Inputs

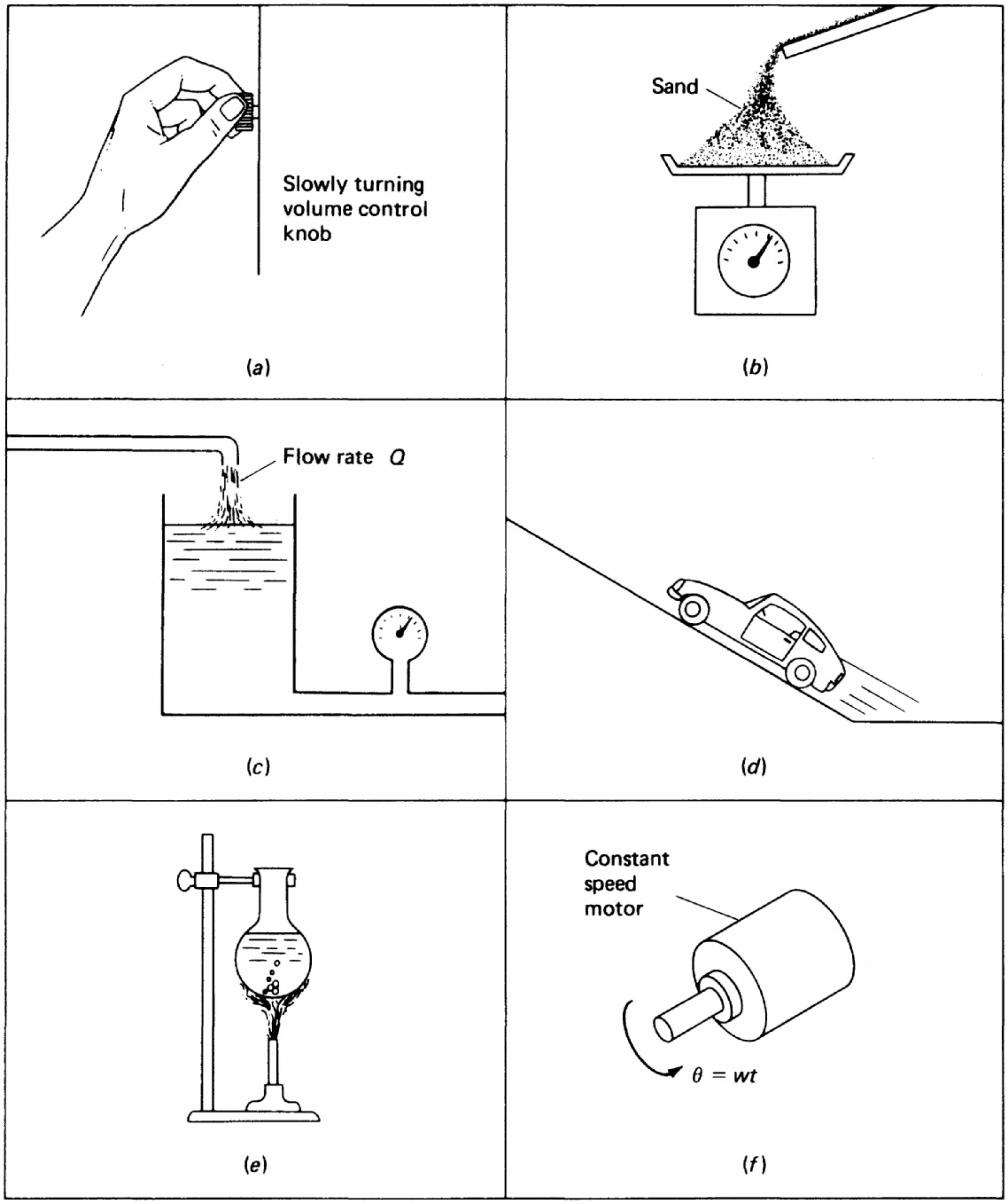


Figure 3.1.4-1 Some examples of ramp inputs. (a) Ramp voltage. (b) Ramp force. (c) Ramp pressure. (d) Translatory ramp displacement. (e) Constant temperature rise. (f) Rotary ramp displacement.