Chapter 10

Time-Domain Analysis and Design of Control Systems

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10.5 TRANSIENT RESPONSE SPECIFICATIONS

Because systems that stores energy cannot respond instantaneously, they exhibit a transient response when they are subjected to inputs or disturbances. Consequently, the transient response characteristics constitute one of the most important factors in system design.

In many practical cases, the desired performance characteristics of control systems can be given in terms of transient-response specifications. Frequently, such performance characteristics are specified in terms of the transient response to unit-step input, since such an input is easy to generate and is sufficiently drastic. (If the response of a linear system to a step input is known, it is mathematically possible to compute the system's response to any input).

The transient response of a system to a unit step-input depends on initial conditions. For convenience in comparing the transient responses of various systems, it is common practice to use standard initial conditions: The system is at rest initially, with its output and all time derivatives thereof zero. Then the response characteristics can be easily compared.

Transient-Response Specifications. The transient response of a practical control system often exhibits damped oscillations before reaching a steady state. In specifying the transient-response characteristics of a control system to a unit-step input, it is common to name the following:

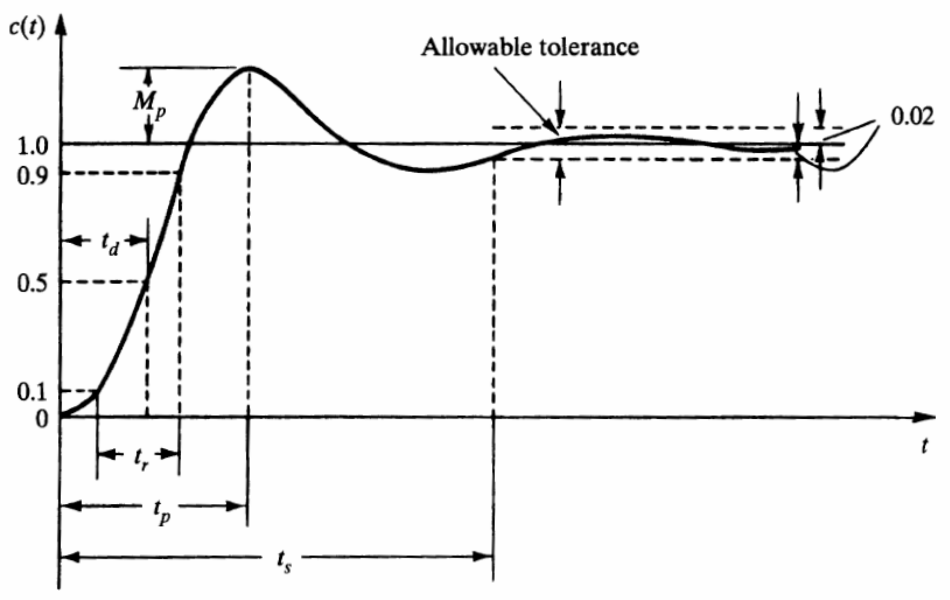


Figure 10–21 Transient-response specifications.

1. Delay time, T_d
2. Rise time, T_r
3. Peak time, T_p
4. Maximum overshoot, M_p
5. Settling time, T_s

These specifications are defined next and are shown in graphically in Figure 10-21.

Delay Time. The delay time T_d is the time needed for the response to reach half of its final value the very first time.

Rise Time. The rise time T_r is the time required for the response to rise from 10% to 90%, 5% to 95%, or 0% to 100% of its final value. For underdamped second order systems, the 0% to 100% rise time is normally used. For overdamped systems, the 10% to 90% rise time is common.

Peak Time. The peak time T_p is the time required for the response to reach the first peak of the overshoot.

Maximum (percent Overshoot). The maximum percent overshoot M_p is the maximum peak value of the response curve [the curve of $c(t)$ versus t], measured from $c(\infty)$. If $c(\infty)=1$, the maximum percent overshoot is $M_p \times 100\%$. If the final steady state value $c(\infty)$ of the response differs from unity, then it is common practice to use the following definition of the ***maximum percent overshoot:***

$$\text{Maximum percent overshoot } = \frac{C(t_p)-C(\infty)}{C(\infty)} \times 100\%$$

Settling Time. The settling time T_s is the time required for the response curve to reach and stay within 2% of the final value. In some cases, 5% instead of 2%, is used as the percentage of the final value. The settling time is the largest time constant of the system.

Comments. If we specify the values of T_d, T_r, T_p, T_s and M_p, the shape of the response curve is virtually fixed as shown in Figure 10.22.

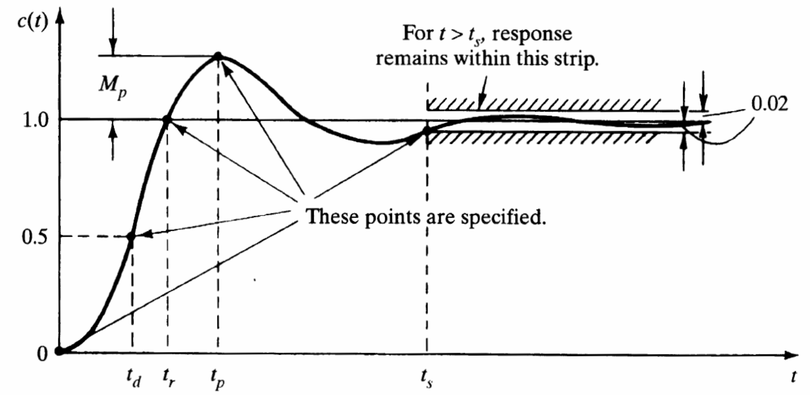


Figure 10-22 Specifications of transient-response curve.

A Few Comments on Transient Response-Specifications.

In addition of requiring a dynamic system to be **stable**, i.e., its response does not increase unbounded with time (a condition that is satisfied for a second order system provided that $\zeta \geq 0$, we also require the response:

- to be fast
- does not excessively overshoot the desired value (i.e., relatively stable) and
- to reach and remain close to the desired reference value in the minimum time possible.

Second-Order Systems and Transient-Response-Specifications.

The response for a unit step input of an underdamped second order system $(0 < \zeta < 1)$ is given by

$$c(t) = 1 - \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_d t - e^{-\zeta\omega_n t} \cos \omega_d t$$
$$= 1 - e^{-\zeta\omega_n t} \left\{ \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t + \cos \omega_d t \right\} \quad (10\text{-}13)$$

or

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left\{ \omega_d t + \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta} \right\} \quad (10\text{-}14)$$

A family of curves $c(t)$ plotted against t with various values of ζ is shown in Figure 10-24.

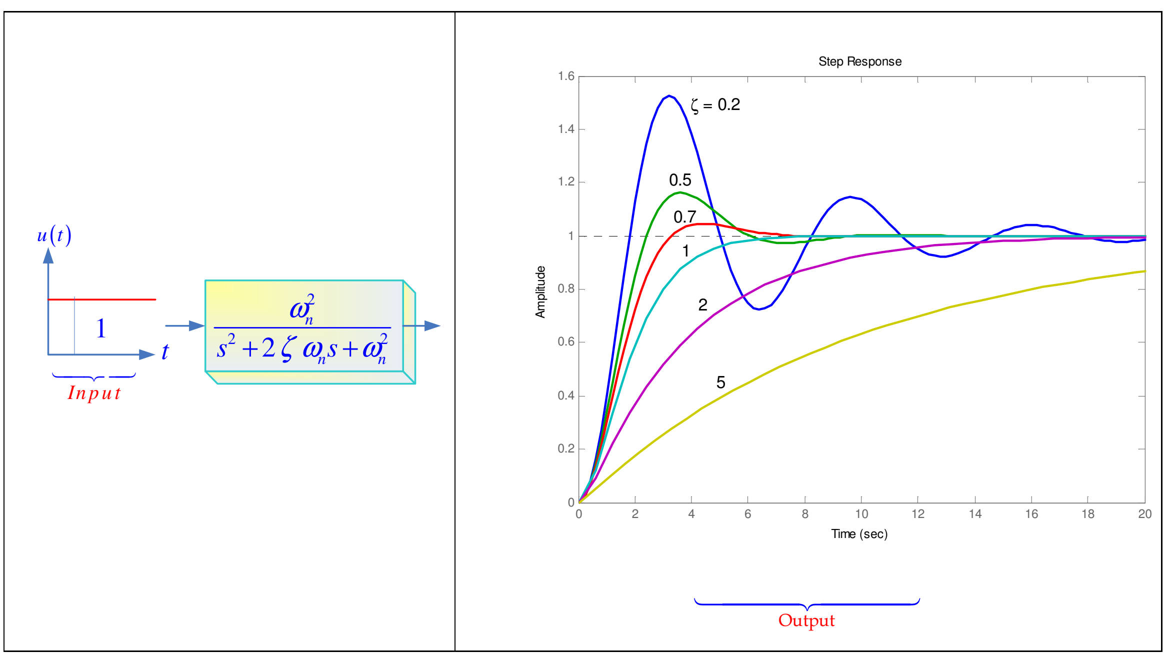


Figure 10-24 Unit step response curves for a second order system.

Delay Time. We define the delay time by the following approximate formula:

$$T_d = \frac{1+0.7\zeta}{\omega_n}$$

Rise Time. We find the rise time T_r by letting $c(T_r)=1$ in Equation (10-13), or

$$c(T_r) = 1 = 1 - e^{-\zeta\omega_n T_r}\left\{\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d T_r + \cos\omega_d T_r\right\} \qquad (10\text{-}15)$$

Since $e^{-\zeta\omega_n t} \neq 0$, Equation (10-15) yields

$$\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d T_r + \cos\omega_d T_r = 0$$

or

$$\tan\omega_d T_r = -\frac{\sqrt{1-\zeta^2}}{\zeta}$$

Thus, the rise T_r is

$$T_r = \frac{1}{\omega_d}\tan^{-1}\left(-\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = \frac{\pi-\beta}{\omega_d} \qquad (10\text{-}16)$$

where β is defined in Figure 10-25. Clearly to obtain a large value of T_r we must have a large value of β.

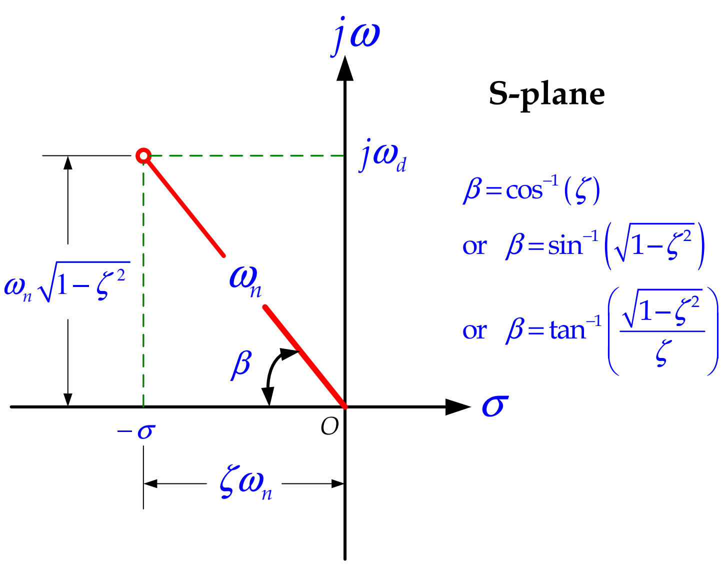


Figure 10-25 Definition of angle β

Peak Time. We obtain the peak time T_p by differentiating $c(t)$ in Equation (10-13), with respect to time and letting this derivative equal zero. That is,

$$\frac{dc(t)}{dt} = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_d t = 0$$

It follows that

$$\sin \omega_d t = 0$$

or

$$\omega_d t = 0, \pi, 2\pi, 3\pi, \ldots = n\pi, \quad n = 0, 1, 2 \ldots\ldots$$

Since the peak time T_p corresponds to the first peak overshoot $(n = 1)$, we have $\omega_d T_p = \pi$. Then

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} \tag{10-17}$$

The peak time T_p corresponds to one half-cycle of the frequency damped oscillations.

Maximum Overshoot M_p The maximum overshoot M_p occurs at the peak $T_p = \pi/\omega_d$. Thus, from Equation (10-13),

$$M_p = c(T_p) - 1 = -e^{-\zeta\omega_n(\pi/\omega_d)} \left\{ \underbrace{\frac{\zeta}{\sqrt{1-\zeta^2}} \sin \pi}_{=0} + \underbrace{\cos \pi}_{=-1} \right\}$$

or

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \tag{10-18}$$

Since $c(\infty) = 1$, the maximum percent overshoot is

$$M_p\% = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \times 100\%$$

The relationship between the damping ratio ζ and the maximum percent overshoot is shown in Figure 10-26. Notice that no overshoot for $\zeta \geq 1$ and overshoot becomes negligible for $\zeta > 0.7$.

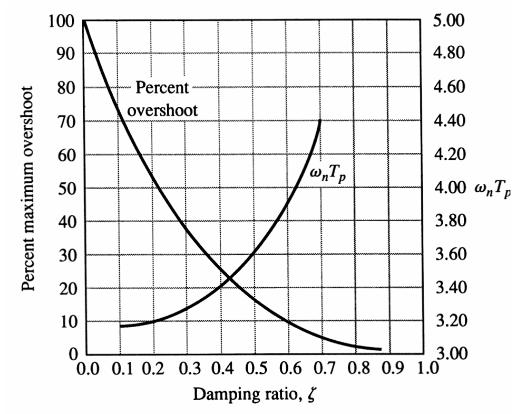


Figure 10-26 Relationship between the maximum percent overshoot $M_p\%$ and damping ratio ζ

Settling Time T_s Based on 2% criterion the settling time T_s is defined as:

$$e^{-\zeta\omega_n T_s} = 0.02$$

$$-\zeta\omega_n T_s = \ln(0.02) \quad \Rightarrow \quad T_s = \frac{\ln(0.02)}{-\zeta\omega_n} \approx \frac{4}{\zeta\omega_n}$$

$$T_s = \frac{4}{\zeta\omega_n} \qquad (2\% \text{ Criterion}) \tag{10-19}$$

Similarly for 5% we can get

$$T_s = \frac{3}{\zeta\omega_n} \qquad (5\% \text{ Criterion}) \tag{10-20}$$

REVIEW AND SUMMARY

TRANSIENT RESPONSE SPECIFICATIONS OF A SECOND ORDER SYSTEM

TABLE 1. Useful Formulas and Step Response Specifications for the Linear Second-Order Model $m\,\ddot{x} + c\,\dot{x} + k\,x = f(t)$ where m, c, k constants

1. **Roots** $s_{1,2} = \dfrac{-c \pm \sqrt{c^2 - 4mk}}{2m}$
2. **Damping ratio or** $\zeta = c / 2\sqrt{mk}$
3. **Undamped natural frequency** $\omega_n = \sqrt{\dfrac{k}{m}}$
4. **Damped natural frequency** $\omega_d = \omega_n \sqrt{1-\zeta^2}$
5. **Time constant** $\tau = 2m/c = 1/\zeta\omega_n$ if $\zeta \le 1$
6. **Logarithmic decrement** $\delta = \dfrac{2\pi\zeta}{\sqrt{1-\zeta^2}}$ or $\zeta = \dfrac{\delta}{\sqrt{4\pi^2 + \delta^2}}$
7. **Stability Property** Stable if, and only if, both roots have negative real parts, this occurs if and only if , m, c, and k have the same sign.
8. **Maximum Percent Overshoot:** The maximum % overshoot M_p is the maximum peak value of the response curve. $M_p = 100 e^{-\pi\zeta/\sqrt{1-\zeta^2}}$
9. **Peak time**: Time needed for the response to reach the first peak of the overshoot $T_p = \pi / \omega_n \sqrt{1-\zeta^2}$
10. **Delay time**: Time needed for the response to reach 50% of its final value the first time $T_d \approx \dfrac{1+0.7\zeta}{\omega_n}$
11. **Settling time**: Time needed for the response curve to reach and stay within 2% of the final value $T_s = \dfrac{4}{\zeta\omega_n}$
12. **Rise time**: Time needed for the response to rise from (10% to 90%) or (0% to 100%) or (5% to 95%) of its final value $T_r = \dfrac{\pi - \beta}{\omega_d}$ (See Figure 10-25)

SOLVED PROBLEMS

Example 1

Problem Given the pole plot shown in Figure 4.20, find ζ, ω_n, T_p, $\%OS$, and T_s.

Solution The damping ratio is given by $\zeta = \cos\theta = \cos[\arctan(7/3)] = 0.394$. The natural frequency, ω_n, is the radial distance from the origin to the pole, or $\omega_n = \sqrt{7^2 + 3^2} = 7.616$. The peak time is

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{7} = 0.449 \text{ second} \tag{4.46}$$

The percent overshoot is

$$\%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100 = 26.018\% \tag{4.47}$$

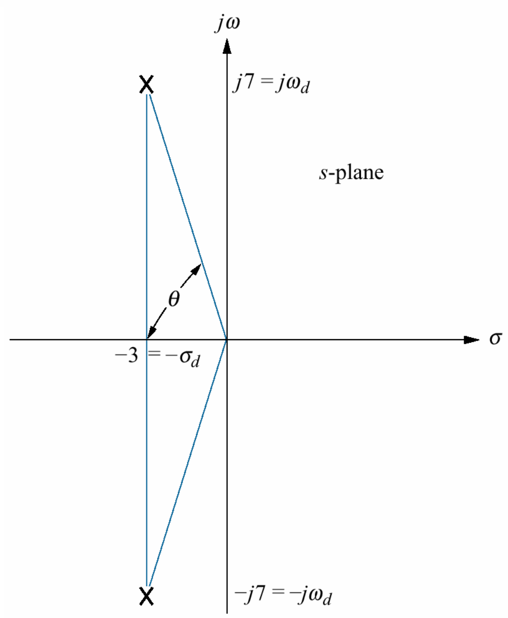


Figure 4-20 (for Example 1)

The approximate settling time is

$$T_s = \frac{4}{\sigma_d} = \frac{4}{3} = 1.333 \text{ seconds} \tag{4.48}$$

Students who are using MATLAB should now run ch4p1 in Appendix C. You will learn how to generate a second-order polynomial from two complex poles as well as extract and use the coefficients of the polynomial to calculate T_p, $\%OS$, and T_s. This exercise uses MATLAB to solve the problem in Example 4.6.

Example 2

Problem Given the system shown in Figure 4.21, find J and D to yield 20% overshoot and a settling time of 2 seconds for a step input of torque $T(t)$.

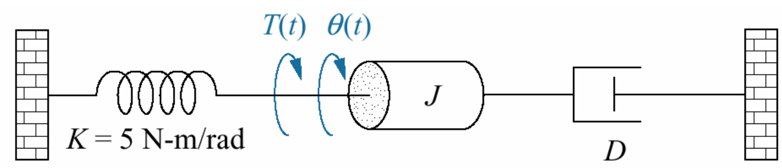


Figure 4-21 (for Example 2)

Solution

First The transfer function of the system is

$$G(s) = \frac{1/J}{s^2 + \frac{D}{J}s + \frac{K}{J}} \quad (4.49)$$

From the transfer function,

$$\omega_n = \sqrt{\frac{K}{J}} \quad (4.50)$$

and

$$2\zeta\omega_n = \frac{D}{J} \quad (4.51)$$

But, from the problem statement,

$$T_s = 2 = \frac{4}{\zeta\omega_n} \quad (4.52)$$

or $\zeta\omega_n = 2$. Hence,

$$2\zeta\omega_n = 4 = \frac{D}{J} \quad (4.53)$$

Also, from Eqs. (4.50) and (4.52),

$$\zeta = \frac{4}{2\omega_n} = 2\sqrt{\frac{J}{K}} \quad (4.54)$$

From Eq. (4.39), a 20% overshoot implies $\zeta = 0.456$. Therefore, from Eq. (4.54),

$$\zeta = 2\sqrt{\frac{J}{K}} = 0.456 \quad (4.55)$$

Hence,

$$\frac{J}{K} = 0.052 \quad (4.56)$$

From the problem statement, $K = 5$ N-m/rad. Combining this value with Eqs. (4.53) and (4.56), $D = 1.04$ N-m s/rad, and $J = 0.26$ kg-m^2.

Example 3 (Example 10-2in the Textbook Page 520-521)

Determine the values of T_d, T_r, T_p, T_s when the control system shown in Figure 10-28 is subject to a unit step input

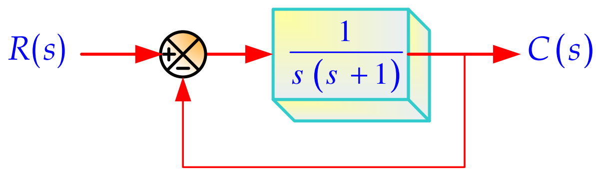


Figure 10-28 Control System

Solution

The closed-loop transfer function of the system is

$$\frac{C(s)}{R(s)} = \frac{\frac{1}{s(s+1)}}{1+\frac{1}{s(s+1)}} = \frac{1}{s^2+s+1}$$

Notice that $\omega_n = 1$ rad/s and $\zeta = 0.5$ for this system. So $\omega_d = \omega_n\sqrt{1-\zeta^2} = \sqrt{1-0.5^2} = 0.866$

Rise Time. $T_r = \dfrac{\pi - \beta}{\omega_d}$

where

$\beta = \sin^{-1}(\omega_d/\omega_n) = \sin^{-1}(0.866/1) = 1.05$ rad

or

$\beta = \cos^{-1}(\zeta\omega_n/\omega_n) = \cos^{-1}(\zeta) = \cos^{-1}(0.5)$

$= 1.05$ rad

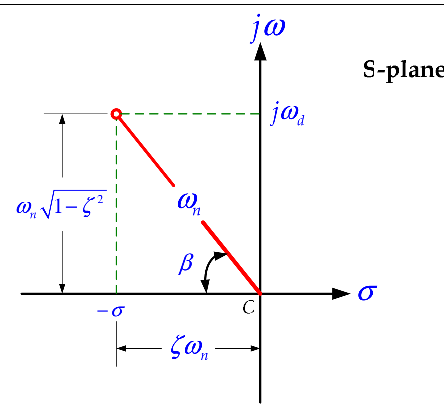
Therfore,

$$T_r = \frac{\pi - 1.05}{0.866} = 2.41 \text{ s}$$

Peak Time. $T_p = \dfrac{\pi}{\omega_d} = \dfrac{\pi}{0.866} = 3.63$ s

Delay Time.

$$T_d = \frac{1+0.7\zeta}{\omega_n} = \frac{1+0.7(0.5)}{1} = 1.35 \text{ s}$$



Maximum Overshoot : $M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} = e^{-\pi\times 0.5/\sqrt{1-0.5^2}} = e^{-1.81} = 0.163 = 16.3\%$

Settling time: $T_s = \dfrac{4}{\zeta\omega_n} = \dfrac{4}{0.5\times 1} = 8$ s