CHAPTER 5
DEFLECTION AND STIFFNESS

Beam deflection can be found by 3 methods: 1) Integration, 2) Superposition and 3) Castigliano's Theorem. These are detailed as follows:

1) Beam deflection by Integration (Section 5-3, pg. 188 in the textbook):

If the deflection of a beam is mainly due to the bending moment, then the following formula is applied to find the deflection of the beam:

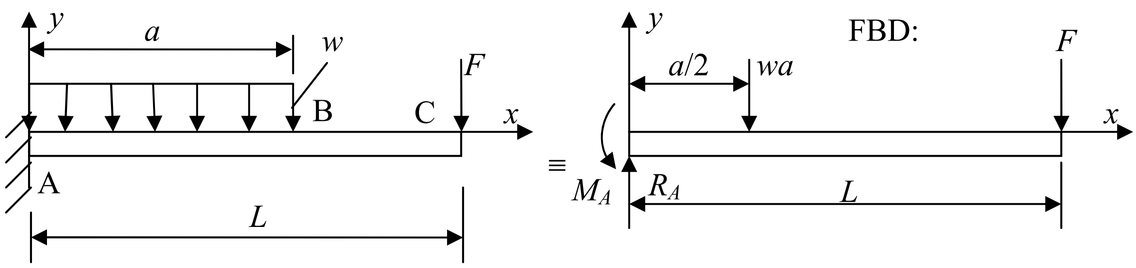
$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

where y is the vertical deflection, x is the horizontal distance, E is the modulus elasticity and I is the area moment of inertia as before. We also find the slope as:

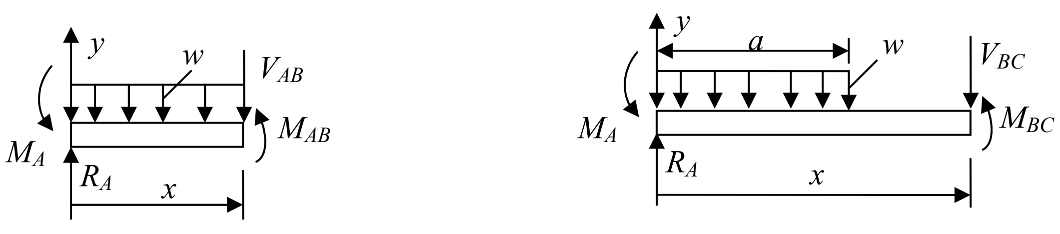
$$\frac{dy}{dx} = \theta = \int \frac{M}{EI}dx + c_1$$

where c_1 is a constant found from the end or boundary conditions of the beam.

Example: Find general expressions for the deflection and slope of the following beam.



First we draw a FBD for the system to find the reactions, which are found from Newton's equations as $R_A = F+wa$ and $M_A = FL+wa^2/2$. We then cut the beam between A-B and B-C to find the bending moments of M_{AB} and M_{BC} as:



We sum the moments at the cut sections and let them to be zero to find:

$M_{AB} = R_A x - wx^2/2 - M_A = -wx^2/2 + (F+wa)x - (FL + wa^2/2)$ and
$M_{BC} = R_A x - wa(x-a/2) - M_A = (F+wa)x - wa(x-a/2) - (FL + wa^2/2) = F(x-L)$.

<u>Deflection and Slope between A-B, i.e. $0 \le x \le a$:</u>

$$\text{Slope } =\theta_{AB} = \frac{dy}{dx} = \int \frac{M_{AB}}{EI}\,dx + c_1 = \frac{1}{EI}\int \left[\frac{-w}{2}x^2 + (F+wa)x - (FL + \frac{wa^2}{2}) \right] dx + c_1$$

$$= \frac{1}{EI}\left[\frac{-w}{6}x^3 + \frac{(F+wa)}{2}x^2 - (FL + \frac{wa^2}{2})x \right] + c_1$$

Since $\theta_{AB} = 0$ at $x = 0$ then $c_1 = 0$. Hence:

$$\theta_{AB} = \frac{1}{EI}\left[\frac{-w}{6}x^3 + \frac{(F+wa)}{2}x^2 - (FL + \frac{wa^2}{2})x \right].$$

$$\text{Deflection}=y_{AB} = \int \theta_{AB}\,dx + c_2 = \frac{1}{EI}\int \left[\frac{-w}{6}x^3 + \frac{(F+wa)}{2}x^2 - (FL + \frac{wa^2}{2})x \right] dx + c_2$$

$$= \frac{1}{EI}\left[\frac{-w}{24}x^4 + \frac{(F+wa)}{6}x^3 - \frac{(2FL + wa^2)}{4}x^2 \right] + c_2$$

Since $y_{AB} = 0$ at $x = 0$ then $c_2 = 0$. Hence:

$$y_{AB} = \frac{1}{EI}\left[\frac{-w}{24}x^4 + \frac{(F+wa)}{6}x^3 - \frac{(2FL + wa^2)}{4}x^2 \right].$$

<u>Deflection and Slope between B-C, i.e. $a \le x \le L$:</u>

$$\text{Slope } = \theta_{BC} = \frac{dy}{dx} = \int \frac{M_{BC}}{EI}\,dx + c_3 = \frac{1}{EI}\int F(x-L)\,dx + c_3$$

$$= \frac{F}{EI}(\frac{x^2}{2} - Lx) + c_3 .$$

Since $\theta_{AB} = \theta_{BC}$ at $x = a$ where from the above equations:

$$\theta_{AB}\big|_{x=a} = \frac{1}{EI}\left[\frac{-w}{6}a^3 + \frac{(F+wa)}{2}a^2 - (FL + \frac{wa^2}{2})a \right] = \frac{1}{EI}\left[\frac{-w}{6}a^3 + Fa(\frac{a}{2} - L) \right]$$

$\theta_{BC}\big|_{x=a} = \frac{F}{EI}(\frac{a^2}{2} - La) + c_3$ and hence $c_3 = \frac{-wa^3}{6EI}$. Therefore:

$$\theta_{BC} = \frac{1}{EI}\left[F(\frac{x^2}{2} - Lx) - \frac{wa^3}{6} \right].$$

Deflection = y_{BC} = $\int \theta_{BC}\, dx + c_4 = \frac{1}{EI}\int\left[F(\frac{x^2}{2} - Lx) - \frac{wa^3}{6}\right]dx + c_4$.

$$= \frac{1}{EI}\left[F(\frac{x^3}{6} - L\frac{x^2}{2}) - \frac{wa^3}{6}x\right] + c_4$$

Since $y_{AB} = y_{BC}$ at $x = a$ where from the above equations:

$$y_{AB}\big|_{x=a} = \frac{1}{EI}\left[\frac{-w}{24}a^4 + \frac{(F+wa)}{6}a^3 - \frac{(2FL+wa^2)}{4}a^2\right] =$$

$$= \frac{1}{EI}\left[\frac{-wa^4}{8} + \frac{Fa^3}{6} - \frac{FLa^2}{2}\right].$$

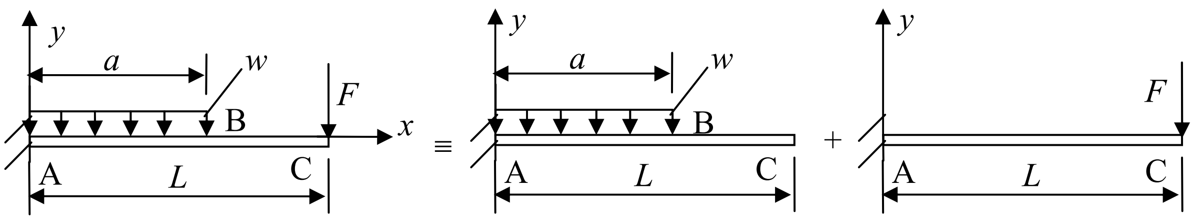
$y_{BC}\big|_{x=a} = \frac{1}{EI}\left[F(\frac{a^3}{6} - L\frac{a^2}{2}) - \frac{wa^3}{6}a\right] + c_4$ and hence $c_4 = \frac{wa^4}{24EI}$. Therefore:

$$y_{BC} = \frac{1}{EI}\left[F(\frac{x^3}{6} - L\frac{x^2}{2}) - \frac{wa^3}{6}x + \frac{wa^4}{24}\right].$$

Note: Review example 5-1, pg. 190 in the textbook.

2) Beam Deflection by Superposition (Section 5-5, pg. 192 in the textbook):

Refer to 16 cases given in the Appendix A-9, from pg. 969 to 976 in the textbook. The superposition for linear systems means that we can find the reactions, bending moments, shear forces, slope and deflection by summing up these entities for each loading case. So let's solve the above example by the superposition technique. We have,



<u>Deflection and Slope between A-B, i.e. $0 \le x \le a$:</u>

Slope = $\theta_{AB} = \theta_{1AB} + \theta_{2AB}$, where from Table A-9-3, pg. 970 in the textbook,

$\theta_{1AB} = \frac{dy}{dx} = \frac{w}{6EI}(3ax^2 - x^3 - 3a^2x)$ where l is replaced by a in the formula. Also,

from Table A-9-1, pg. 969 in the textbook, $\theta_{2AB} = \frac{dy}{dx} = \frac{F}{2EI}(x^2 - 2Lx)$. Hence,

$$\theta_{AB} = \frac{w}{6EI}(3ax^2 - x^3 - 3a^2x) + \frac{F}{2EI}(x^2 - 2Lx) = \frac{1}{EI}\left[\frac{-w}{6}x^3 + \frac{(F+wa)}{2}x^2 - (FL + \frac{wa^2}{2})x\right],$$

which is the same result obtained before using the integration method.
Deflection $= y_{AB} = y_{1AB} + y_{2AB}$, where from Table A-9-3, pg. 970 in the textbook, $y_{1AB} = \frac{wx^2}{24EI}(4ax - x^2 - 6a^2)$ where again l is replaced by a in the formula. Also, from Table A-9-1, pg. 969 in the textbook, $y_{2AB} = \frac{Fx^2}{6EI}(x - 3L)$. Hence,

$$y_{AB} = \frac{wx^2}{24EI}(4ax - x^2 - 6a^2) + \frac{Fx^2}{6EI}(x - 3L) = \frac{1}{EI}\left[\frac{-w}{24}x^4 + \frac{(F+wa)}{6}x^3 - \frac{(2FL + wa^2)}{4}x^2\right],$$

which is again the same result obtained before using the integration method.

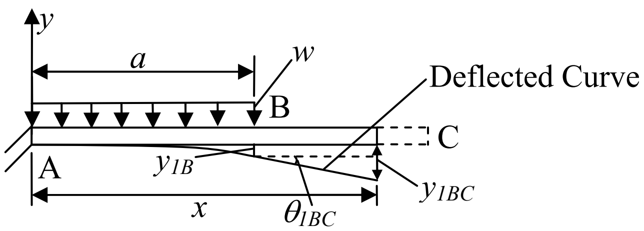
Deflection and Slope between A-B, i.e. $a \le x \le L$:

Slope $= \theta_{BC} = \theta_{1BC} + \theta_{2BC}$. Since there is no bending moment between B-C for the case of distributed load w, we can assume that the slope remains the same as θ_{1AB} at $x=a$. Hence, $\theta_{1BC} = \theta_{1AB}\big|_{x=a} = \frac{-wa^3}{6EI}$. Also, from Table A-9-1, pg. 969 in the textbook, $\theta_{2BC} = \theta_{2AB} = \frac{F}{2EI}(x^2 - 2Lx)$. Hence,

$$\theta_{BC} = \frac{-wa^3}{6EI} + \frac{F}{2EI}(x^2 - 2Lx) = \frac{1}{EI}\left[F(\frac{x^2}{2} - Lx) - \frac{wa^3}{6}\right],$$

which is the same result obtained before using the integration method.

Deflection $= y_{BC} = y_{1BC} + y_{2BC}$.
For the case of distributed load w, the angle θ_{1BC} remains the same between B-C, and hence the deflected curve between B-C is a linear one.



From the figure, $y_{1BC} = \theta_{1BC}(x-a) + y_{1B}$, where $y_{1B} = y_{1AB}\big|_{x=a} = \frac{-wa^4}{8EI}$. Therefore,

$$y_{1BC} = \frac{-wa^3(x-a)}{6EI} + \frac{-wa^4}{8EI} = \frac{wa^3}{24EI}(-4x + a).$$

For the end load of F, again referring to Table A-9-1, pg. 969 in the textbook, y_{2BC} = $y_{2AB} = \frac{Fx^2}{6EI}(x-3L)$. Hence,

$$y_{BC} = = y_{1BC} + y_{2BC} = \frac{wa^3}{24EI}(-4x+a) + \frac{Fx^2}{6EI}(x-3L) = \frac{1}{EI}\left[F(\frac{x^3}{6} - L\frac{x^2}{2}) - \frac{wa^3}{6}x + \frac{wa^4}{24}\right],$$

which is again the same result obtained before using the integration method.

Note: Review examples 5-2 and 5-3, pgs. 192 and 193 in the textbook.

3) Beam Deflection by Castigliano's Theorem (Sections 5-7 and 5-8, pgs. 198 and 201 in the textbook):

When a machine element is deformed, it stores a *potential* or *strain energy U*. This energy can be calculated for different loading cases, as given below:

Loading	Strain Energy (U)
Tension or Compression	$\frac{F^2L}{2EA}$
Direct Shear	$\frac{F^2L}{2GA}$
Torsion	$\frac{T^2L}{2GJ}$
Bending Moment	$\int\frac{M^2dx}{2EI}$
Bending Shear	$\int\frac{cV^2dx}{2GA}$

where for bending shear c is a factor that depends on the cross-section of the beam and is taken from Table 5-1, pg. 200, in the textbook. The Castigliano's theorem then states that *the displacement for any point on a machine element is the partial derivative of the strain energy with respective to a force which is on the same point and in the same direction of the displacement*. In short:

$$\delta_i = \frac{\partial U}{\partial F_i}$$

where δ_i is the displacement for an i^{th} point in the machine element and F_i is the force acting at the same point and in the same direction of the displacement asked for. If there is no force at the point for which the displacement is required, then we have to assume an imaginary force Q and let it be zero at the end.

Example: Let's solve the above problem for its vertical displacement at the free end, i.e. at point C, using the Castigliano's theorem. So the question is to find δ_C in the vertical ($-y$) direction. We have,

$$\delta_C = \frac{\partial U}{\partial F}$$

where the strain energy U is contributed by the bending shear force V and bending moment M, but we'll only consider the effect of M in U. Thus, as given above,

$$U = \int \frac{M^2 dx}{2EI} = \int_0^a \frac{M_{AB}^2 dx}{2EI} + \int_a^L \frac{M_{BC}^2 dx}{2EI}$$

and

$$\delta_C = \int_0^a \frac{M_{AB} \frac{\partial M_{AB}}{\partial F} dx}{EI} + \int_a^L \frac{M_{BC} \frac{\partial M_{BC}}{\partial F} dx}{EI}$$

where as found before: $M_{AB} = -wx^2/2 + (F + wa)x - (FL + wa^2/2)$ and $M_{BC} = F(x - L)$. Therefore: $\frac{\partial M_{AB}}{\partial F} = x - L$ and $\frac{\partial M_{BC}}{\partial F} = x - L$. Substituting these in the formula above yields:

$$\delta_C = \int_0^a \frac{\left[-wx^2/2 + (F + wa)x - (FL + wa^2/2)\right](x - L)}{EI} dx + \int_a^L \frac{F(x - L)^2}{EI} dx \,.$$

which upon calculation results in

$$\delta_C = \frac{wa^3}{24EI}(4L - a) + \frac{FL^3}{3EI} \,.$$

This result is in agreement with the result that can be found from the two methods above by substituting $x=L$ in y_{BC}. But remember that $y_{BC}\big|_{x=L} = -\delta_C$. Why?

Note: Review examples 5-10, 5-11 and 5-12, pgs. 202 to 206 in the textbook.

5.10 Statically Indeterminate Problems

When a static problem has more unknowns than the Newton's static equations, then this problem is known as the statically indeterminate problem. In these cases, we need extra displacement equations to solve the problem. There are two procedures to handle such problems, among which we'll follow the *Procedure 1* in the textbook, whose steps are given as follows:

1) Choose the redundant (extra) reaction(s). This redundant reaction could be a force or a moment.
2) Write the Newton's equations of static equilibrium in terms of the applied loads and the redundant reaction(s) of step 1.
3) Write the deflection or slope equation(s) for the point(s) of the redundant reaction(s) of step 1. Normally this deflection or slope will be zero.
4) Solve the equations in steps 2 and 3 for the reactions.

Note: Review example 5-14, pgs. 212 and 213 in the textbook.

5.12 Long Columns with Central Loading

The long columns with central **compressive** loading may exhibit a phenomenon called "buckling", which needs to be checked for the safe use of these columns. Such columns with 4 different end conditions are shown in the textbook in Figure 5-18, pg. 217. The formula used for the long columns is named *Euler formula*, given as:

$$P_{cr} = \frac{c\pi^2 EI}{L^2} \quad \text{or} \quad \frac{P_{cr}}{A} = \frac{c\pi^2 E}{(L/k)^2}$$

where P_{cr} is the critical or maximum central load for the column, c is the end condition constant that should be taken from Table 5-2, pg. 220, in the textbook, and k is the radius of gyration found from $I = Ak^2$, which is equal to $d/4$ for a column having a solid round cross-section with a diameter of d. L/k in the formula is known as the *slenderness ratio*. There are 2 scenarios here:

1) If the critical load P_{cr} is known, then one should use the above equations to find an adequate cross-section for the column.
2) If the cross-section of the column is known, one can use the above equations to solve for the P_{cr}.

How do we know a column is "long enough" or it is of intermediate length? First we define:

$$\left(\frac{L}{k}\right)_1 = \left(\frac{2\pi^2 cE}{S_y}\right)^{1/2}$$

where S_y is the yield strength of the column material. We follow the following criterion:

1) If $\left(\frac{L}{k}\right) > \left(\frac{L}{k}\right)_1$ then use the above *Euler formula*.
2) If $\left(\frac{L}{k}\right) \leq \left(\frac{L}{k}\right)_1$ then use the *Johnson formula* given below in section 5-13 for the intermediate-length columns.

5.13 Intermediate-Length Columns with Central Loading

Use the *Johnson formula*:

$$\frac{P_{cr}}{A} = S_y - \left(\frac{S_y}{2\pi}\frac{L}{k}\right)^2 \frac{1}{cE}$$

Note: Review examples 5-16, 5-17, 5-18 and 5-19, pgs. 223 to 225 in the textbook.