KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

Department of Mathematical Sciences
Dhahran, Saudi Arabia

Math 202 Final Examination. Wednesday, June 8, 2005.

Time Allowed: 7pm-9pm (2 hrs).

Instructor: Y. A. Fiagbedzi

Student Name: ______________________________ Sect. __________

Student ID. No. ______________________________

1. Either FILL IN the gaps or choose TRUE or FALSE as appropriate

   - Every boundary value problem has at least one solution.
     TRUE FALSE

   - Consider the initial value problem $x(x+2)y'' + 3y' + 4y = 0, \quad y'(1) = y(1) = 0.$

     The largest interval over which it is guaranteed to have unique solution is ....

   - A first order differential equation of the form $\dfrac{dy}{dx} = f(x)g(y)$ is always exact.
     TRUE FALSE

   - It can be shown that
     $$y = cx + a\sqrt{1+c^2}$$
     is a one-parameter family of solutions to the differential equation
     $$y = x\frac{dy}{dx} + a\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$
     It can also be shown that
     $$y = \sqrt{a^2 - x^2}$$
     is also a particular solution. Such a particular solution is called a .... solution.

2. Find the eigenvalues of $A = \begin{pmatrix} 0 & 8 & 0 \\ 0 & 0 & -2 \\ 2 & 8 & -2 \end{pmatrix}$. Compute one complex eigenvector.

3. Put the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = \cos t$$

in the state space form $\underline{x}'(t) = A\underline{x}(t) + \underline{F}(t)$. A, $\underline{F}(t)$ and $\underline{x}(t)$ must be clearly defined.

4. Given that the eigenvalues of $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ are $\lambda_1 = 1 + i,\ \lambda_2 = 1 - i$,

a) Show that $\underline{v}_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$ is an eigenvector.

b) Obtain the general solution to the system $\dot{x}(t) = Ax(t)$ in the form $x(t) = \Phi(t)\underline{c}$ where $\underline{c}$ is an arbitrary constant vector.

5. Use the following words: *exact*, *homogeneous*, *Bernoulli*, *Cauchy-Euler*, *separable* to label the differential equations given below. For each labelling, if available, give a substitution that faciliatates the solution of the corresponding differential equation.

Differential Equation	*Label*	*Substitution*
Example: $xy^2\frac{dy}{dx} = y^3 - x^3$	homogeneous	$y = ux$
$(e^x + y)dx + (2 + x + ye^y)dy = 0$		
$(x + ye^{\frac{x}{y}})dx + xe^{\frac{y}{x}}dy = 0$		
$x\frac{dy}{dx} - (\sin x)y = e^x y^2$		
$4x^2\frac{d^2y}{dx^2} + y = 0$		

6. Determine whether $x = 0$ is an ordinary point, regular singular point or irregular singular point of the differential equation

$$xy'' + (\sin x)\, y' + xy = 0.$$

Do the same problem again but with the differential equation

$$xy'' + (\cos x)\, y' + xy = 0.$$

7. a) Determine a differential operator, $p(D)$, (of the least order) which annihilates

$$y(x) = c_0 + (c_1 + c_2 x)\cos x + (d_1 + d_2 x)\sin x - x$$

where c_0, c_1, c_2, d_1, d_2 are arbitrary constants.

b) Determine a differential equation of the *lowest order* of which $y(x)$ is the general solution.

8. Specify a method you would use to solve

$$xy'' + 2y' = 0?$$

Given that $y_1 = 1$, $y_2 = \frac{1}{x}$ constitute a fundamental set of solutions for the above differential equation, obtain the general solution to

$$xy'' + 2y' = \sqrt{x}, \quad x > 0$$

9. A tank is partially filled with 200 liters of fluid in which 10kg of salt is dissolved. Brine containing $1kg$ of salt per liter is pumped into the tank at a rate of 15 liters per minute. The well-mixed solution is then pumped out at a slower rate of 10 liters per minute. If $A(t)$ denotes the amount (in kilograms) of salt in the tank at time t, determine

   - The rate at which salt enters the tank from the brine solution
   - The volume of the liquid in the tank at time t
   - The rate at which salt is being pumped out of the tank
   - Set up a differential equation for $A(t)$. Do not solve the differential equation.

10. Consider the IVP:

$$y' = (y-1)^2(y+2)$$
$$y(1) = 0$$

   - Identify the equilibrium (critical) points
   - Classify each critical point as *stable*, *unstable* or *semi-stable.*
   - If $y(x)$ is the solution of the IVP, find $\lim_{x\to\infty} y(x)$