KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
Department of Mathematical Sciences
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Math 202 Second Major Examination.

Tuesday Dec. 6, 2005. Time Allowed: 75 min.

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STUDENT ID ____________________

Student Name: ____ Sect. ______________

1. We want to find the particular solutions for the differential equation

$$y'' + 3y' + 2y = g(x).$$

for various $g(x)$. Answer Yes or No in the following table. If your answer is Yes to the annihilator method, give the annihilator, $p(D)$.

$g(x)$	Can we use the annihilator method?	Can we use the variation of parameters method?	Annihilator, $p(D)$
xe^x	Yes	Yes	$(D-1)^2$
$(\ln x)/x$	No	Yes	—
$x \sin^{-1} x$	No	Yes	–
$(1+\cos 2x)/2$	Yes	Yes	$D(D^2+4)$
$1/(1+x)$	No	Yes	—

2. Given that $y_1(x) = (\sin x)/x$ is a solution of the differential equation

$$xy'' + 2y' + xy = 0,$$

determine a fundamental set of solutions for the differential equation

Solution. Put the differential equation in standard form, $\boxed{y'' + \frac{2}{x}y' + y = 0,}$ so as to identify $P(x) = \frac{2}{x}$. Then $\int P(x)d\,x = 2\ln|x| = \ln x^2$. We are given $y_1(x) = (\sin x)/x$. Therefore a second solution

$$\begin{aligned} y_2(x) &= y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} d\,x \\ &= y_1 \int \frac{e^{-\ln x^2}}{(\sin x/x)^2} d\,x \\ &= y_1 \int \csc^2(x) d\,x = -y_1 \cot x \\ &= -(\cos x)/x \end{aligned}$$

Therefore a fundamental set of solutions is given by $\{(\sin x)/x, (\cos x)/x\}$.

3. Obtain a fundamental set of solutions for the differential equation $\boxed{x^2y'' + xy' - y = 0.}$

Solution. This is a Cauchy-Euler equation. Thus, putting $x = e^t$ gives

$$xy' = \frac{dy}{dt}, \qquad x^2y'' = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

Under this transformation, the given equation becomes

$$\begin{aligned} 0 &= \frac{d^2y}{dt^2} - \frac{dy}{dt} + \frac{dy}{dt} - y \\ &= \frac{d^2y}{dt^2} - y \end{aligned}$$

with characteristic equation $m^2 - 1 = 0$. That is, $m_1 = 1$, $m_2 = -1$. This gives $y_1 = e^t$ and $y_2 = e^{-t}$ as the fundamental set of solutions. In terms of x, , this is

$$\left\{x, \ \frac{1}{x}\right\}.$$

4. Find the form of the particular solution of the differential equation

$$y'' + y = x\cos x$$

Solution. The general solution is given by $y = y_c(x) + y_p(x)$ where y_c denotes the complementary function and y_p represents the particular integral. The complementary function satisfies $(D^2 + 1)y_c = 0$ whose characteristic equation is given by

$$0 = (m^2 + 1) \Rightarrow m_1 = i, \quad m_2 = -i$$

This gives the fundamental set

$$y_{c1} = \cos x, \qquad y_{c2} = \sin x.$$

The particular solution, y_p, satisfies

$$(D^2 + 1)y_p = x\cos x$$

To obtain y_p, annihilate the right hand side of the above equation with

$$p(D) = (D^2 + 1)^2.$$

The result is

$$(D^2+1)^2(D^2+1)y_p = (D^2+1)^2[x\cos x]$$
$$= 0$$

whose characteristic equation is given by

$$0 = (m^2+1)^3 \Rightarrow m_1 = m_3 = m_5 = i, \quad m_2 = m_4 = m_6 = -i$$

Now, the fundamental set is given by

$$y_{p1} = \cos x, \qquad y_{p2} = x\cos x, \qquad y_{p3} = x^2\cos x$$
$$y_{p2} = \sin x, \qquad y_{p2} = x\sin x, \qquad y_{p3} = x^2\sin x$$

Ignoring the contribution due to $\cos x$, $\sin x$, the particular solution must be of the form

$$y_p = Ay_{p3} + By_{p4} + Cy_{p5} + Dy_{p6}$$
$$= x\cos x\,[A + Cx] + x\sin x\,[B + Dx]$$

5. If $y_1(x) = 1, \quad y_2(x) = \cos x, \quad y_3(x) = \sin x$ form a fundamental set of solutions for a constant coefficient linear ordinary differential equation, then the order of the differential equation is $\boxed{3}$. Also, find the differential equation.

Solution.

- D annihilates y_1
- (D^2+1) annihilates $y_2, \ y_3$
- Therefore $D(D^2+1)$ annihilates $y_1, \ y_2, \ y_3$.

Therefore, the differential equation is $\boxed{D(D^2+1)y = 0.}$

6. Given that $y_1(x) = e^{2x}$ is a solution of $\boxed{y''' - y'' - y' - 2y = 0}$, determine the general solution.

SOLUTION. We infer that $(m-2)$ is a factor of the system characteristic polynomial, $m^3 - m^2 - m - 2$. Therefore

$$\begin{aligned} 0 &= m^3 - m^2 - m - 2 \\ &= (m-2)(m^2+m+1) \\ &= (m-2)\left[(m+1/2)^2 + 3/4\right] \Rightarrow \end{aligned}$$

$$m_1 = 2, \qquad m_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \qquad m_3 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

Therefore,

$$y_1 = e^{2x}, \qquad y_2 = e^{-x/2}\cos\frac{\sqrt{3}}{2}x, \qquad y_3 = e^{-x/2}\sin\frac{\sqrt{3}}{2}x$$

constitute a fundamental set of solutions. As a consequence, the general solution is given by

$$\begin{aligned} y(x) &= c_1 y_1(x) + c_2 y_2(x) + c_3 y_3(x) \\ &= c_1 e^{2x} + e^{-x/2}\left[c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right] \end{aligned}$$