Section 7.4 *Integration of rational functions by partial fractions*

Learning outcomes

After completing this section, you will inshaAllah be able to

1. learn to decompose a proper rational function into partial fractions
2. use partial fractions to integrate rational functions

Rational Functions: $\dfrac{P(x)}{Q(x)}$

(P, Q polynomials)

Proper rational functions

- degree of $P(x)$ < degree of $Q(x)$

Improper rational functions

- degree of $P(x) \geq$ degree of $Q(x)$

- The method is based on partial fractions.
- So we first learn "How to make partial fractions"

Partial fraction decomposition

Any proper rational function $\frac{P(x)}{Q(x)}$ can be expressed as

$$\frac{P(x)}{Q(x)} = F_1(x) + F_2(x) + \cdots\cdots + F_n(x) \qquad (*)$$

where $F_1(x), F_2(x), \cdots\cdots, F_n(x)$ are functions of the form

$$\frac{A}{(ax+b)^k} \quad \text{or} \quad \frac{Bx+C}{(ax^2+bx+c)^k}$$

such that

- the terms $(ax+b)^k$ and $(ax^2+bx+c)^k$ are factors of $Q(x)$
- the factor ax^2+bx+c is irreducible.

The right hand side of equation (*) is called ***partial fraction decomposition*** *of* $\frac{P(x)}{Q(x)}$.

Guide line for finding partial fractions

Given any **proper** rational function $\dfrac{P(x)}{Q(x)}$

- Factorize $Q(x)$ completely into linear and irreducible quadratic factors.
- Express as a product of factors $(ax+b)^m$ and $(ax^2+bx+c)^m$.
- For each factor $(ax+b)^m$, the partial fraction decomposition is

$$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots\cdots+\frac{A_m}{(ax+b)^m}$$

- For each factor, $(ax^2+bx+c)^m$ the partial fraction decomposition is

$$\frac{B_1x+C_1}{ax^2+bx+c}+\frac{B_2x+C_2}{(ax^2+bx+c)^2}+\cdots\cdots+\frac{B_mx+C_m}{(ax^2+bx+c)^m}$$

- Find the unknown constants (How? see examples done in class)

Integrating **proper** rational functions

- Decompose into partial fractions
- Integrate by methods learnt earlier

See examples 1, 2, 3 done in class

How to integrate improper rational functions

- We understand with the help of an example

See examples 4 done in class

Before going further look at example 5, 6, 7, 8, 9 discussed in the class. This will help clearing any confusion regarding partial fraction decompositions

Solving

integrals involving rational functions

of $\sin x$ and $\cos x$

using

universal trigonometric substitution

Universal trigonometric substitution

Used for integrals of rational functions of $\sin x$ and $\cos x$

First we look at the following facts

- If $u = \tan\left(\frac{x}{2}\right)$ then
  - $\sin x = \frac{2u}{1+u^2}$ **(R1)**
  - $\cos x = \frac{1-u^2}{1+u^2}$ **(R2)**
  - $dx = \frac{2du}{1+u^2}$ **(R3)**

See class notes for proof of these relations

The integrals of rational functions of $\sin x$ and $\cos x$ can be evaluated by following procedure:

1. Use the substitution $u = \tan\left(\frac{x}{2}\right)$
2. Use relations (R1), (R2), (R3) to convert the integral into a form that can be solved by previous methods

See example 10 done in class

A trick to solve

integrals involving fractional powers of

x **and** $ax+b$

Integrals involving fractional powers of x or $ax+b$

Main Idea

Make a substitution to get rid of fractional power(s)

See example 11, 12 done in class

End of 7.4