Section 5.2 *The definite integral*

Learning outcomes

After completing this section, you will inshaAllah be able to

1. know what is meant by a Riemann sum
2. know what is meant by definite integral
   a. learn how to express a definite integral as limit of Riemann sum
   b. learn how to express a limit of a Riemann sum as definite integral
3. know basic properties of definite integrals
4. evaluate definite integrals
   a. as limit of a Riemann sum
   b. using geometric formulas of area
   c. using basic properties of integrals
5. apply comparison properties of definite integrals

What is a Riemann Sum?

Recall from Ex 5.1, where area was given as

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Such sums are significant and appear in many other applications like Volumes, Arc length, Surface Area etc.

The sum of the form $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called a **Riemann sum** for *f(x)*

x_i^* from i^{th} subinterval

- Right end point or left end point or midpoint

Example 1: Evaluate the Riemann sum for $f(x) = 2 - x^2$ on $[0,2]$ using 4 subintervals and right end points.

Solution: Done in class.

Exercise: Do Example 1 using mid-points.

Definite integral

Recall again from Ex 5.1, where area was defined as following limit

$$A = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

Such limits arise in many applications and therefore are given special name

upper limit

lower limit

The definite integral of f from $x = a$ to $x = b$ is defined as

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_i^*)\Delta x$$

where x_i^* is point in the i^{th} subinterval (left end or right end or mid-point)

See examples 2, 3, 4 done in class

Geometric interpretation of definite integral

First see explanation given in class

if $f \geq 0$ this gives area.

The definite integral

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_i^*)\Delta x$$

gives the net signed area between $y = f(x)$ and $[a,b]$

The answer may be positive, negative or zero.

See examples 5, 6, 7 done in class

How to evaluate definite integrals

Different ways

Using definition (as limit of Riemann sum)

- See Page 5.2_6

Using known area formulas from geometry

- See Page 5.2_7

Using properties of definite integrals

- See Page 5.2_9

Using integration or fundamental theorem of calculus

- See Section 5.3
- Most Practical

Evaluating definite integrals

Using definition (as limit of Riemann sum)

- Before learning the method of evaluating definite integrals (as limit of Riemann sum) we need to review important summation formulas required to solve questions.

1. $$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

2. $$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

3. $$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

open form of summation

closed form of summation

- To solve questions of finding evaluating integrals as a limit we need to write summations in closed form.
- Writing a summation in closed form is difficult, in general.
- This is the main hurdle in evaluating definite integrals as a limit of Riemann sum.

See next page for more

Evaluating definite integrals

Using definition (as limit of Riemann sum)

- Question: Given a function $f(x)$. Find $\int_a^b f(x)dx$ using the formula

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_i^*)\Delta x$$

Step 1 Divide $[a,b]$ into n subintervals of width $\Delta x = \dfrac{b-a}{n}$.

Hence the end points of subintervals are

$x_0 = a$, $x_1 = a + \Delta x$,

$x_2 = a + 2\Delta x$, ...,

$x_i = a + i\Delta x$, ..., $x_n = b$

Step 2 Choose points x_i^* in each subinterval.

Right: $x_i^* = x_i = a + i\Delta x$

Left: $x_i^* = x_{i-1} = a + (i-1)\Delta x$

Mid: $x_i^* = \dfrac{1}{2}(x_{i-1} + x_i) = a + \left(i - \dfrac{1}{2}\right)\Delta x$

Step 3 Find $f(x_i^*)\Delta x$

Step 4 Write $\sum_{i=1}^{n} f(x_i^*)\Delta x$ in closed form

Useful formulas

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Step 5 Find $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$

See example 8 done in class

Evaluating definite integrals

Using known area formulas from geometry

To learn this method

see examples 9, 10 11 done in class

Basic properties of definite integrals

(Note explanations provided in the class)

1) $\int_a^a f(x)dx = 0$

2) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

3) $\int_a^b c\,dx = c(b-a)$

4) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ (a, b, c any numbers)

5) $\int_a^b c \cdot f(x)dx = c\int_a^b f(x)dx$

6) $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

Odd function property:

If $f(x)$ is an odd function i.e. $f(-x) = -f(x)$

then $\int_{-a}^a f(x)dx = 0$.

Even function property:

If $f(x)$ is an even function i.e. $f(-x) = f(x)$

then $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$

Evaluating definite integrals

Using basic properties

See examples 12, 13 done in class

Comparison properties of definite integrals

C1) If $f(x) \geq 0$ then $\int_a^b f(x)dx \geq 0$

C2) If $f(x) \geq g(x)$ then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$

C3) If $m \leq f(x) \leq M$ then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

For applications,

see examples 14, 15, 16 done in class

End of Section 5.2