Section 5.1 *Areas and Distances*

Learning outcomes

This section is mainly a motivation for
learning integration and integral calculus.

After completing this section, you will inshaAllah be able to

1. know what is meant by area problem
2. explain "rectangle method" to solve area problem
3. estimate area by rectangle method
   a. using right endpoints
   b. using left endpoints
   c. using midpoints
4. understand area problem as a limit

What is Area Problem?

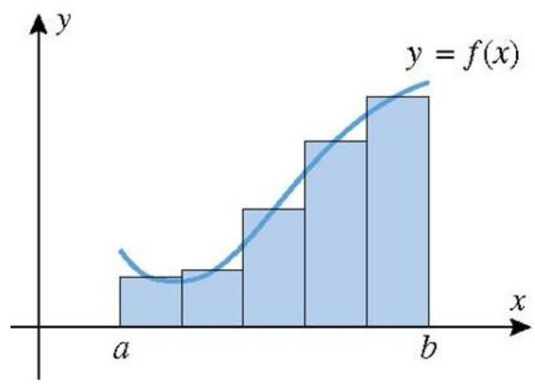
To find the area of the region that lies under the curve ***y=f(x)*** from ***a*** to ***b***.

Two approaches

- Rectangle Method (done here)
- Using Integration (done later)

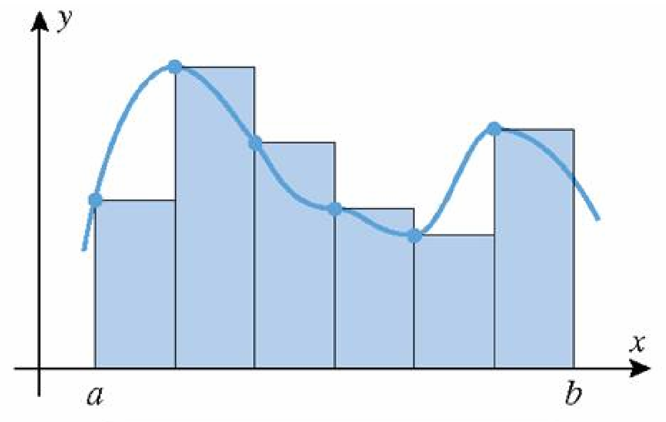
The Rectangle Method (estimating area)

- Divide the interval [a,b] into *n* equal subintervals
- Construct *n* rectangles as explained on next page

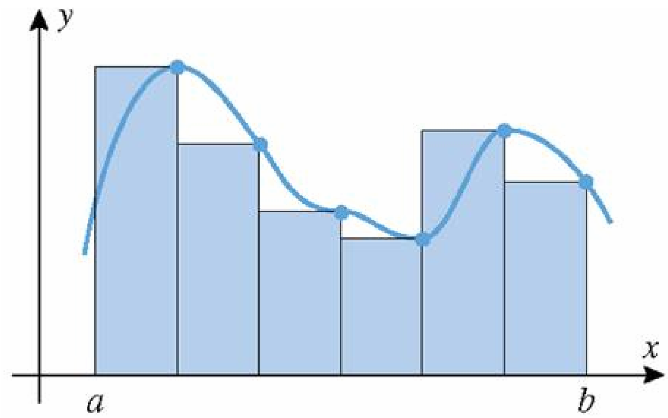


- Find the areas of rectangles
- The total area A_n of all rectangles approximately gives the required area.

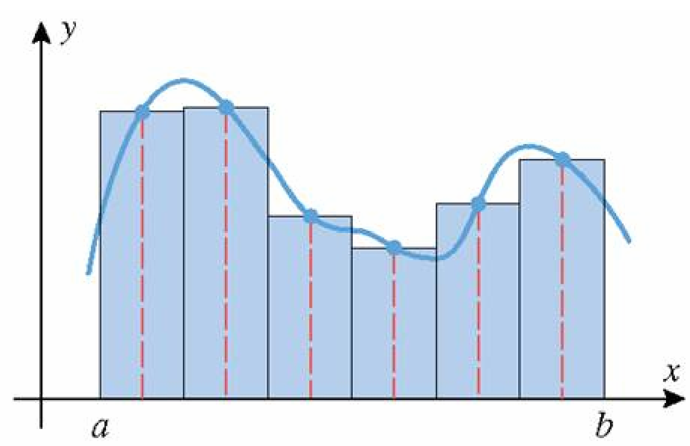
Different ways of constructing rectangles



Using left end points of subintervals



Using right end points of subintervals



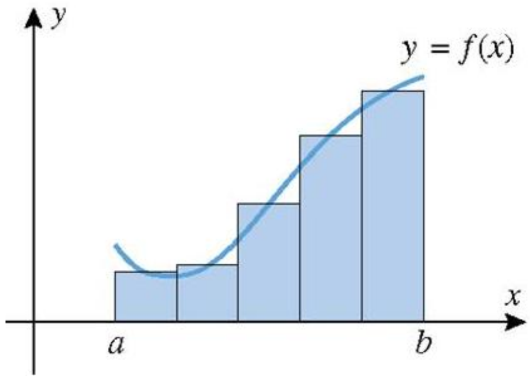
Using mid points of subintervals

See examples 1, 2, 3 done in class
(for estimating area using rectangles)

The Rectangle Method (exact area)
Area as a limit

Recall: To estimate the area we did

- Divide the interval [a,b] into *n* equal subintervals
- Construct *n* rectangles

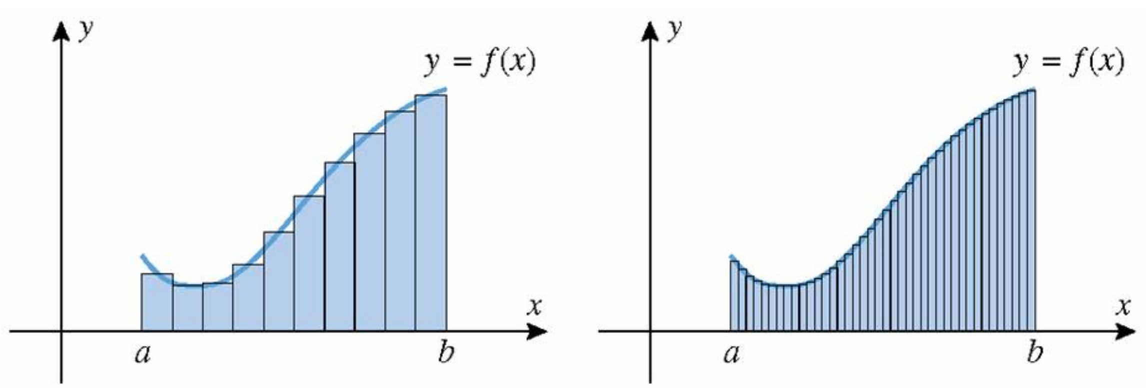


- Find the areas of rectangles
- The total area A_n of all rectangles approximately gives the required area.

Ho to get exact area?

Idea 

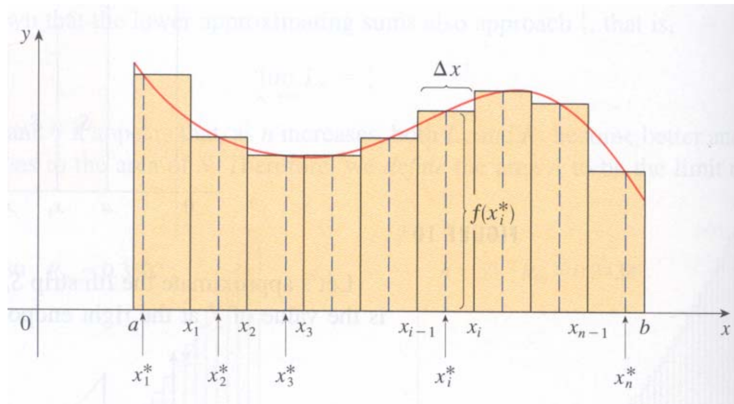
- As n increases, the approximation A_n of the area improves
- $\lim_{n \to \infty} A_n = exact \ \ area$



This leads us to the following formulation for area as a limit

Area as a limit (continued)

- Take the function *f(x)* shown in the figure.
- To find exact area under *f(x)* from $x=a$ to $x=b$, we complete the following steps.



Step 1 Divide $[a,b]$ into n equal subintervals of width $\Delta x = \dfrac{b-a}{n}$.

Hence the points are

$$x_0 = a\,,\quad x_1 = a + \Delta x\,,\quad x_2 = a + 2\Delta x\,,\ \ldots\ldots,\ x_i = a + i\Delta x\,,\ \ldots\ldots,\ x_n = b$$

Step 2 Choose points $x_1^*, x_2^*, \cdots, x_n^*$ in each subinterval to make n-rectangles $A_1, A_2, \cdots, A_n$

See next page about choosing the point.

Step 3 Find area of i[th] rectangle: $f(x_i^*)\Delta x$

Step 4 Find area of n rectangles: $\sum_{i=1}^{n} f(x_i^*)\Delta x$

Step 5 Find exact area: $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$

Area as a limit (continued)

From above we have

If f is continuous on $[a,b]$ and if $f(x) \geq 0$ for all $x \in [a,b]$ then

area under $y = f(x)$ over $[a,b]$ is defined by

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

where x_i^* denotes the point chosen in i^{th} subinterval and $\Delta x = \dfrac{b-a}{n}$

Typically chosen points x_i^*

(for solving area as limit problems)

1. **Right end points** of interval

$$x_i^* = x_i = a + i\Delta x$$

2. **Left end points** of interval

$$x_i^* = x_{i-1} = a + (i-1)\Delta x$$

3. **midpoints** of interval

$$x_i^* = \frac{1}{2}(x_{i-1} + x_i) = \frac{1}{2}\big(a + (i-1)\Delta x + a + i\Delta x\big) = a + \left(i - \frac{1}{2}\right)\Delta x$$

- The calculation of above limit is a tricky matter. We will solve such questions in the next section.
- Here our aim was to formulate the area problem as a limit

See example 4 done in class

End of Section 5.1