Section 11.6 *Absolute convergence and the ratio & root tests*

Learning outcomes

After completing this section, you will inshaAllah be able to

1. understand what is meant by absolute convergence
2. use absolute convergence to determine convergence of a given series
3. apply ratio test to determine convergence/divergence of a series
4. apply root test to determine convergence/divergence of a series
5. understand what is meant by conditional convergence
6. determine whether a series is absolutely convergent, conditionally convergent or divergent

What is absolute convergence?

Given a series $\sum_{n=1}^{\infty} a_n$ (having both positive and negative terms)

- We can associate, with it, a series $\sum_{n=1}^{\infty} |a_n|$ of absolute values

  i.e. $\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + \cdots\cdots |a_n| + \cdots$

- **The series $\sum_{n=1}^{\infty} a_n$ is said to be absolutely convergent** if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Question: How to find out about convergence of absolute series?

Answer: All tests for positive term series (done earlier) can be applied.

Our aim

To use convergence of absolute series

to find out about the convergence of given series

(see below)

Knowing convergence of given series from convergence of series of absolute values

Question

To know about the convergence or divergence of a series $\sum_{n=1}^{\infty} a_n$ using the knowledge about the convergence or divergence of the series $\sum_{n=1}^{\infty} |a_n|$ of absolute values?

The answer lies in

- the following result 1

and

- the Ratio and Root test done below

Result 1

If the series $\sum_{n=1}^{\infty} |a_n|$ converges then the series $\sum_{n=1}^{\infty} a_n$ converges.

i.e. absolute convergence $\Rightarrow$ convergence of the given series.

See examples 1, 2 done in class

We next do two tests that can be used to determine convergence or divergence of series $\sum_{n=1}^{\infty} a_n$ which may contain negative terms.

Ratio test

Root test

Ratio test

Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series and suppose that $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$.

1) If $L < 1$ then the series $\sum_{n=1}^{\infty} a_n$ absolutely converges (and hence converges)

2) If $L > 1$ or $L = \infty$ then the series $\sum_{n=1}^{\infty} a_n$ diverges

3) If $L = 1$, test fails and we cannot say anything about the convergence or divergence.

Obviously can be applied to know convergence or divergence of positive term series

Usually useful when a_n involves

- factorial

or

- n as power

or

- rational functions

See examples 3, 4, 5, 6 done in class

Root test

Root Test

Let $\sum_{n=1}^{\infty} a_n$ be a series and suppose that $\lim_{n\to\infty} \left(|a_n|\right)^{1/n} = L$.

4) If $L < 1$ then the series $\sum_{n=1}^{\infty} a_n$ absolutely converges (and hence converges)

5) If $L > 1$ or $L = \infty$ then the series $\sum_{n=1}^{\infty} a_n$ diverges

6) If $L = 1$, test fails and we cannot say anything about the convergence or divergence.

Obviously can be applied to know convergence or divergence of positive term series

Usually useful when a_n involves

- n as power

See examples 7, 8, 9 done in class

What is the meaning of conditional convergence

- We have seen that absolute convergence $\Rightarrow$ convergence of given series
- What about the reverse.
- Let's look at an example

Example

Consider the series $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$

- By alternating series test this series converges
- But the associated absolute series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges

So there are series $\sum_{n=1}^{\infty} a_n$ which are convergent

but not absolutely convergent.

Such series are called conditionally convergent

Questions for a series with negative terms

- Is it absolutely convergent
- Is it conditionally convergent
- Is it divergent

Strategy for knowing whether a given series (with negative terms) is absolutely convergent, divergent or conditionally convergent

Given a series $\sum_{n=1}^{\infty} a_n$ with negative terms.

- If you try ratio or root tests you may be able to know about its absolute convergence or divergence (if the test doesn't fail).
- If you try divergence test you may be able to know about its divergence.
- If you study $\sum_{n=1}^{\infty} |a_n|$ then
  - if $\sum_{n=1}^{\infty} |a_n|$ is convergent then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent
  - if $\sum_{n=1}^{\infty} |a_n|$ is divergent we have to analyze $\sum_{n=1}^{\infty} a_n$ (as it may be convergent or divergent). In this situation
    - if $\sum_{n=1}^{\infty} a_n$ converges then we call it conditionally convergent
    - if $\sum_{n=1}^{\infty} a_n$ diverges then (obviously) we call it divergent.

See examples 10, 11, 12, 13, 14 done in class

End of Section 11.6