Section 11.2 *Infinite series*

Learning outcomes

After completing this section, you will inshaAllah be able to

1. know what is meant by infinite series & its convergence
2. learn methods for knowing convergence/divergence of some basis series.
3. apply divergence test to determine divergence of an infinite series

Infinite series

An infinite series is an expression of the form

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \cdots\cdots u_k + \cdots$$

Most important question

To know the convergence of infinite series

Meaning of convergence of an infinite series

- Given a series $\sum_{k=1}^{\infty} u_k$.
- Define the kth partial sum of the series as

  $$S_k = u_1 + u_2 + \cdots + u_k$$ {Sum of k terms}

- This associates a sequence with the series $\sum_{k=1}^{\infty} u_k$ as

  $$S_1, S_2, S_3, \cdots, S_n, \cdots$$ {Sequence of partial sums}

- The series $\sum_{k=1}^{\infty} u_k$ converges to S if the associated sequence of partial sums converges to S (i.e. $\lim_{n\to\infty} S_n = S$) and S is called sum of the series.
- The series diverges if the associated sequence of partial sums diverges.

Examples of determining convergence of infinite series (using definition)

See examples 1, 2, 3 done in class

Examples of some important basic infinite series

- Telescoping series

Convergent

$$\sum_{k=1}^{\infty}\frac{1}{k(k+1)}=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots\cdots$$

and the sum is $S=1$ {See example 1 done above}

- The series $1-1+1-1+1-1+\cdots\cdots$

Divergent

- The Geometric series

$$\sum_{k=0}^{\infty}ar^k=a+ar+ar^2+\cdots ar^k+\cdots \qquad (a\neq 0)$$

(i) **Converges** if $|r|<1$ and has sum $S=\dfrac{a}{1-r}$

(ii) **Diverges** if $|r|\geq 1$.

Exercise: Is the series $5-\frac{10}{3}+\frac{20}{9}-\frac{40}{27}+\cdots\cdots$ convergent? If yes, find its sum.

Answer: $S=3$

- Harmonic series

$$\sum_{k=1}^{\infty}\frac{1}{k}=1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{k}+\cdots$$

Divergent

Basic properties of infinite series

- If $\sum_{k=1}^{\infty} u_k$ and $\sum_{k=1}^{\infty} v_k$ are convergent series then the series $\sum_{k=1}^{\infty} (u_k \pm v_k)$ is convergent and

$$\sum_{k=1}^{\infty} (u_k \pm v_k) = \sum_{k=1}^{\infty} u_k \pm \sum_{k=1}^{\infty} v_k \; .$$

- For $c \neq 0$, the series $\sum_{1}^{\infty} u_k$ and $\sum_{1}^{\infty} cu_k$ both converge or both diverge.

  In case of convergence,

$$\sum_{1}^{\infty} cu_k = c \sum_{1}^{\infty} u_k$$

See example 4 done in class

- Deleting finite number of terms from a series has no effect on its convergence or divergence.

  e.g. the series

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \cdots\cdots$$

  and

$$\sum_{k=20}^{\infty} u_k = u_{20} + u_{21} + u_{22} + \cdots\cdots$$

  both converge or both diverge.

Final comments before we study tests for convergence/divergence

- Studying convergence of infinite series using partial sums is very impractical
  - Since finding a formula for S_n is very difficult
- From this point forward, we will learn many efficient techniques of determining convergence/divergence of a series
- The first test we will study is "Divergence test" which can be tried on any series.

Divergence test

a. If $\lim_{n\to\infty} a_n \neq 0$ then the series $\sum_{n=a}^{\infty} a_n$ diverges

b. If $\lim_{n\to\infty} a_n = 0$ then the series $\sum_{n=a}^{\infty} a_n$ may converge or diverge and we need to check by some other test. i.e. the test fails.

See examples 5, 6, 7, 8, 9 done in class

End of 11.2